Koszul free divisors

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Abstract

We introduce some material concerning locally quasi-homogeneous free divisors and Koszul free divisors.

1 Locally quasi-homogeneous and Koszul free divisors

Let X be a n-dimensional complex analytic manifold. We denote by $\pi : T^*X \to X$ the cotangent bundle, $\mathcal{O}_X$ the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of linear differential operators on X (with holomorphic coefficients), $\mathcal{G}r_{F^\bullet}(\mathcal{D}_X)$ the graded ring associated with the filtration $F^\bullet$ by the order, $\sigma(P)$ the principal symbol of a differential operator P and $\{-,-\}$ the Poisson bracket on $\mathcal{O}_{T^*X}$ or $\mathcal{G}r_{F^\bullet}(\mathcal{D}_X)$. We will note $\mathcal{O} = \mathcal{O}_{X,p}$, $\mathcal{D} = \mathcal{D}_{X,p}$ and $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}r_{F^\bullet}(\mathcal{D}_X)_p$ the respective stalks at p, with p a point of X. If $J \subset \mathcal{D}$ is a left ideal, we denote by $\sigma(J)$ the corresponding graded ideal of $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$. Given a divisor $D \subset X$, we denote by $\mathcal{D}\mathrm{er}(\log D)$ the $\mathcal{O}_X$-module of the logarithmic vector fields with respect to D [14]. If f is a local equation of D at p, we denote by $\mathrm{Der}(\log f)$ the stalk at p of $\mathcal{D}\mathrm{er}(\log D)$, whose elements are germs at p of vector fields δ such that $\delta(f) \in (f)$.

Definition 1.1. A divisor D is Euler-homogeneous at $p \in D$ if there is a local equation h for D around p, and a germ of (logarithmic) vector field δ such that $\delta(f) = f$. A such δ is called a local Euler vector field for f.

The set of points where a divisor is Euler-homogeneous is open.

Definition 1.2. (cf. [7]) A germ of divisor $(D,p) \subset (X,p)$ is quasi-homogeneous if there are local coordinates $x_1, \ldots, x_n \in \mathcal{O}_{X,p}$ with respect to which (D,p) has a weighted homogeneous defining equation (with strictly positive weights). A divisor D in a n-dimensional complex manifold X is locally quasi-homogeneous if the germ (D,p) is quasi-homogeneous for each point $p \in D$. A germ of divisor $(D,p) \subset (X,p)$ is locally quasi-homogeneous if the divisor D is locally quasi-homogeneous in a neighborhood of p.

Obviously a locally quasi-homogeneous divisor is Euler-homogeneous at every point.

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Definition 1.3. We say that a reduced germ $f \in \mathcal{O}_{X,p}$ is locally quasi-homogeneous if the germ of divisor $(\{f=0\}, p)$ is.

Remark 1.4. A reduced germ $f \in \mathcal{O}_{X,p}$ is locally quasi-homogeneous if and only if for every $q \in \{f=0\}$ near p there are local coordinates $z_1, \dots, z_n \in \mathcal{O}_{X,q}$ and a quasi-homogeneous polynomial $P(t_1, \dots, t_n)$ (with strictly positive weights) such that $f_q = P(z_1, \dots, z_n)$.

Definition 1.5. ([14], [3], def. 4.1.1) Let $D \subset X$ be a divisor. We say that D is free at $p \in X$ if $\mathcal{D}\mathrm{er}(\log D)_p$ is a free $\mathcal{O}$-module (of rank n). We say that D is a Koszul free divisor at $p \in X$ if it is free at p and there exists a basis $\{\delta_1, \dots, \delta_n\}$ of $\mathcal{D}\mathrm{er}(\log D)_p$ such that the sequence of symbols $\{\sigma(\delta_1), \dots, \sigma(\delta_n)\}$ is regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)_p$. If D is a free (resp. Koszul free) divisor at each point of X, we simply say that it is free (resp. Koszul free). We say that a reduced germ $f \in \mathcal{O}_{X,p}$ is free if the divisor $f^{-1}(0)$ is free at p.

Let us remark that a divisor D is automatically Koszul free at every $p \in X \setminus D$.

Remark 1.6. The ideal $I_{D,p} = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\, \mathcal{D}\mathrm{er}(\log D)_p$ is generated by the elements of any basis of $\mathcal{D}\mathrm{er}(\log D)_p$. As D is Koszul free at p if and only if $\mathrm{depth}(I_{D,p}, \mathrm{Gr}_{F^\bullet}(\mathcal{D})) = n$ (cf. [11], cor. 16.8), it is clear that the definition of Koszul free divisor does not depend on the election of a particular basis. By the coherence of $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$, if a divisor is Koszul free at a point, then it is Koszul free near that point.

We have not found a reference for the following well known proposition (see [11], th. 17.4 for the local case).

Proposition 1.7. *Let $\mathbb{C}\{x\}$ be the ring of convergent power series in the variables $x = x_1, \dots, x_n$ and let G be the graded ring of polynomials in the variables $\xi_1, \dots, \xi_t$ with coefficients in $\mathbb{C}\{x\}$. A sequence $\sigma_1, \dots, \sigma_s$ of homogeneous polynomials in G is regular if and only if the set of zeros $V(I)$ of the ideal I generated by $\sigma_1, \dots, \sigma_s$ has dimension $n+t-s$ in $U \times \mathbb{C}^t$, for some open neighborhood U of 0 (then each irreducible component has dimension $n+t-s$).*

Proof: Let $\mathbb{C}\{x, \xi\}$ be the ring of convergent power series in the variables $x_1, \dots, x_n$ and $\xi_1, \dots, \xi_t$. As the σ_i are homogeneous and the ring $\mathbb{C}\{x, \xi\}$ is a flat extension of G, the σ_i are a regular sequence in G if and only if they are a regular sequence in $\mathbb{C}\{x, \xi\}$. But the last condition is equivalent to the equality (*loc. cit.*):

$$\dim_{(0,0)}(V(I)) = \dim\left(\mathbb{C}\{x, \xi\}/I\right) = n+t-s.$$

Finally, using the fact that all the σ_i are homogeneous in the variables ξ, the local dimension of $V(I)$ at $(0,0)$ coincides with its dimension in $U \times \mathbb{C}^t$ for some neighborhood U of 0. C.Q.D.

Corollary 1.8. *Let $D \subset X$ be a free divisor. Let J be the ideal in $\mathcal{O}_{T^*X}$ generated by $\pi^{-1}\,\mathcal{D}\mathrm{er}(\log D)$. Then, D is Koszul free if and only if the set $V(J)$ of zeros of J has dimension n (in this case, each irreducible component of $V(J)$ has dimension n).*

Proposition 1.9. *Let X be a complex manifold of dimension n and let $D \subset X$ be a divisor. Then:*

1. *Let $X' = X \times \mathbb{C}$ and $D' = D \times \mathbb{C}$. The divisor $D \subset X$ is Koszul free if and only if $D' \subset X'$ is Koszul free.*

2. *Let Y be another complex manifold of dimension r and let $E \subset Y$ be a divisor. Then:*
   a) The divisor $(D \times Y) \cup (X \times E)$ is free if $D \subset X$ and $E \subset Y$ are free.
   b) The divisor $(D \times Y) \cup (X \times E)$ is Koszul free if $D \subset X$ and $E \subset Y$ are Koszul free.

Proof:

1. It is a consequence of [7], lemma 2.2, (iv) and the fact that $\sigma_1, \ldots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X,p}[\xi_1, \ldots, \xi_n]$ if and only if $\xi_{n+1}, \sigma_1, \ldots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X',(p,t)}[\xi_1, \ldots, \xi_n, \xi_{n+1}]$.

2. a) It is an immediate consequence of Saito's criterion (cf. [7], lemma 2.2, (v)).
   b) It is a consequence of a) and Corollary 1.8.

C.Q.D.

Example 1.10. Examples of Koszul free divisors are:
1) Nonsingular divisors.
2) Normal crossing divisors.
3) Plane curves: If $\dim_{\mathbb{C}} X = 2$, we know that every divisor $D \subset X$ is free [14], cor. 1.7. Let $\{\delta_1, \delta_2\}$ be a basis of $\mathcal{D}\mathrm{er}(\log D)_x$. Their symbols $\{\sigma_1, \sigma_2\}$ are obviously linearly independent over $\mathcal{O}$, and by Saito's criterion [14], 1.8, they are relatively primes in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \xi_2]$. So they form a regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, and D is Koszul free (see [3], cor. 4.2.2).
4) Proposition 1.9 gives a way to obtain Koszul free divisors in any dimension.
5) There are irreducible Koszul free divisors in dimensions greater than 2, which are not constructed from divisors in lower dimension [13]: $X = \mathbb{C}^3$ and $D \equiv \{f = 0\}$, with

$$f = 2^8 z^3 - 2^7 x^2 z^2 + 2^4 x^4 z + 2^4 3^2 x y^2 z - 2^2 x^3 y^2 - 3^3 y^4.$$

A basis of $\mathcal{D}\mathrm{er}(\log f)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\begin{array}{rclcccccc} \delta_1 & = & 6y & \partial_x & + & (8z - 2x^2) & \partial_y & - & xy & \partial_z, \\ \delta_2 & = & (4x^2 - 48z) & \partial_x & + & 12xy & \partial_y & + & (9y^2 - 16xz) & \partial_z, \\ \delta_3 & = & 2x & \partial_x & + & 3y & \partial_y & + & 4z & \partial_z, \end{array}$$

and the sequence $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular.

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