Graphs associated with nilpotent Lie algebras of maximal rank

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Abstract

In this paper, we use the graphs as a tool to study nilpotent Lie algebras. It implies to set up a link between graph theory and Lie theory. To do this, it is already known that every nilpotent Lie algebra of maximal rank is associated with an generalized Cartan matrix A and it is isomorphic to a quotient of the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$. Then, if A is affine, we can associate $\mathfrak{n}_+$ with a directed graph (from now on, we use the term digraph) and we can also associate a subgraph of this digraph with every isomorphism class of nilpotent Lie algebras of maximal rank and of type A. Finally, we show an algorithm which obtains these subgraphs and also groups them in isomorphism classes.

Introduction

The rank of a nilpotent Lie algebra $\mathfrak{L}$ is the common dimension of all the maximal tori on $\mathfrak{L}$. We say that $\mathfrak{L}$ is *of maximal rank* if its rank r is equal to the dimension ℓ of $\mathfrak{L}/[\mathfrak{L},\mathfrak{L}]$.

If $\mathfrak{L}$ is a nilpotent Lie algebra of maximal rank and T is a maximal torus on $\mathfrak{L}$, then $\mathfrak{L}$ is decomposed in a direct sum of root spaces for T:

$$\mathfrak{L} = \bigoplus_{\beta \in T^*} \mathfrak{L}^{\beta},$$

where T^* is the dual of the vector space T and $\mathfrak{L}^{\beta} = \{X \in \mathfrak{L} : tX = \beta(t)X, \forall t \in T\}$. The set $R(T) = \{\beta \in T^* : \mathfrak{L}^{\beta} \neq \{0\}\}$ is called *root system* of $\mathfrak{L}$ associated with T. We can obtain a minimal system $\{X_1, X_2, \ldots, X_\ell\}$ of generators of $\mathfrak{L}$ verifying that for each X_i there exists $\beta_i \in R(T)$ such that $X_i \in \mathfrak{L}^{\beta_i}$. The $\{\beta_1, \beta_2, \ldots, \beta_\ell\}$ is a basis of T^* and for each $\beta \in R(T)$ there exists $d_1, \ldots, d_\ell \in \mathbb{N}$ such that $\beta = \sum_{i=1}^{\ell} d_i \beta_i$.

In [4] an generalized Cartan matrix $A = (a_{i,j})_{i,j=1}^{\ell}$ is associated with every nilpotent Lie algebra $\mathfrak{L}$ of maximal rank ℓ and thus, one says that $\mathfrak{L}$ is of type A.

Then, by using generalized Cartan matrices we link the nilpotent Lie algebras of maximal rank with the Kac-Moody algebras. Indeed, we have that every nilpotent Lie algebras $\mathfrak{L}$ of maximal rank and of type A is a quotient of the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$ associated with A.

For a general overview on Kac-Moody Lie algebras and Graphs Theory the reader can see [2] and [3], respectively.

1 Associated graphs

Definition 1.1. Let $\mathfrak{L}$ be a nilpotent Lie algebra of maximal rank ℓ, T a maximal torus on $\mathfrak{L}$ and $R(T)$ the root system associated with T. We define the following digraph:

- the set of vertices $V(G_{\mathfrak{L}})$ is $R(T)$.
- we draw a directed edge from γ to μ if there exists β_i such that $\mu = \gamma + \beta_i$, where μ, $\gamma \in R(T)$.

Theorem 1.2. *The digraph above mentioned is unique, up to isomorphism, for every nilpotent Lie algebra $\mathfrak{L}$ of maximal rank. We will denote it by $G_{\mathfrak{L}}$.*

Then, let consider an generalized Cartan matrix A of affine type. Let Δ_+ be the set of positive roots, $\mathfrak{g}_\alpha$ the root subspace associated with $\alpha \in \Delta_+$ and $\alpha_1, \alpha_2 \dots, \alpha_l$ the simple roots of the Kac-Moody algebra $\mathfrak{g}(A)$. We have the following decomposition for the positive part $\mathfrak{n}_+$ of $\mathfrak{g}(A)$:

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.$$

Definition 1.3. We define the following digraph G, associated with $\mathfrak{n}_+$:

- the set of vertices $V(G)$ is $\{0\} \cup \Delta_+$.
- we draw a directed edge from γ to μ if there exists α_i such that $\mu = \gamma + \alpha_i$, where μ, $\gamma \in \{0\} \cup \Delta_+$.

Due to properties of the positive root system Δ_+ when A is affine, the following lemma is verified:

Lemma 1.4. *The digraph G is infinite, it has a countable infinite set $\{n\delta \,/\, n \geq 1\}$ of cut points and there are countably infinite many finite subgraphs G_n $(n \geq 0)$ of G such that:*

1. $V(G_{n-1}) \cap V(G_n) = \{n\delta\}$
2. $V(G_n) = \{\alpha + n\delta \,/\, \alpha \in V(G_0)\}$

As a consequence of this lemma and the results related to the root systems, see [2], the digraph G has the following structure:

$$G_0 \vee G_1 \vee \ldots \vee G_n \vee G_{n+1} \vee \ldots = \bigvee_{n \geq 0} G_n$$

where we identify the vertices $n\delta$ of G_{n-1} and G_n for all $n \geq 1$.

Theorem 1.5. *If $\mathfrak{L}$ is a nilpotent Lie algebra of maximal rank and of type A, then the corresponding associated digraph $G_{\mathfrak{L}}$ is isomorphic to a subgraph $G'_{\mathfrak{L}}$ of G.*

This result is an immediate consequence of $\mathfrak{L}$ being isomorphic to a quotient of $\mathfrak{n}_+$.

This digraph $G'_{\mathfrak{L}}$ is a subgraph of G verifying that there exists $j \geq 0$ such that $j\delta \in V(G'_{\mathfrak{L}})$ and $(j+1)\delta \notin V(G'_{\mathfrak{L}})$, and thus, we can consider that $G'_{\mathfrak{L}}$ is a subgraph of the finite digraph

$$G_0 \vee G_1 \vee \ldots \vee G_j = \bigvee_{n=0}^{j} G_n.$$

Let the digraph $\tilde{G}_{\mathfrak{L},j}$ be the subgraph of $\bigvee_{n=0}^{j} G_n$ whose set of vertices $V(\tilde{G}_{\mathfrak{L},j})$ is $V(G) - V(G_{\mathfrak{L}})$ and whose edges are all edges in $\bigvee_{n=0}^{j} G_n$ which connect two vertices in $V(\tilde{G}_{\mathfrak{L},j})$. This digraph $\tilde{G}_{\mathfrak{L},j}$ is a subgraph of the digraph G_j which verifies the following properties:

1. $(j+1)\delta \in V(\tilde{G}_{\mathfrak{L},j})$ and $\delta \notin V(\tilde{G}_{\mathfrak{L},j})$
2. if $\alpha \in V(\tilde{G}_{\mathfrak{L},j})$ and $\alpha + \alpha_i \in V(G_j)$, then $\alpha + \alpha_i \in V(\tilde{G}_{\mathfrak{L},j})$ and the edge from α to $\alpha + \alpha_i$ belongs to $\tilde{G}_{\mathfrak{L},j}$.

Theorem 1.6. *To classify nilpotent Lie algebras of maximal rank and of type A is needed to compute (up to isomorphism), as a first step, all the subgraphs of G_j verifying properties* 1 *and* 2 *above mentioned, for each $j \geq 0$.*

For a more general overview on this result the reader can see [1].

Moreover, as $V(G_j) = \{\alpha + j\delta \,/\, \alpha \in V(G_0)\}$, we have a bijection between the set of subgraphs of G_0 and the set of subgraphs of G_j, for $j \geq 1$. So, it is sufficient to obtain, up to isomorphism, all the subgraphs of G_0 which verify properties 1 and 2 (see [1]).

Finally, since that the digraph G_0 has a great number of vertices and edges (which increase with the generalized Cartan matrix order), we have designed an algorithm which allows, in the first place, to obtain all the subgraphs of G_0 verifying these properties and secondly, to group them in isomorphism classes.

The main steps of this algorithm, described in a short way, are the following:

Algorithm

Input:

- The digraph G_0 such that
$$G = \bigvee_{n \geq 0} G_n$$
is the digraph associated with the positive part $\mathfrak{n}_+$ of the Kac-Moody algebra $\mathfrak{g}(A)$. $V(G_0) = \{0, v_1, \ldots, v_p, v_{p+1} = \delta\}$
- The automorphism group of the matrix A, $Aut(A)$.

Output: For each isomorphism class of subgraphs of G_0 which verify the properties 1 and 2, we give a representative.

Method.

Step 1: We calculate the subgraphs of G_0 which verify the properties 1 and 2. (If I is a subgraph of G_0 which verifies the property 2, then I is generated by $v_{i_1}, \dots, v_{i_k} \in V(I)$ and , by the property 1, $v_{i_j} \neq 0$, $j = 1, \dots, k$.)

Step 1.1: We obtain the list of the subgraphs generated by one vertex $v \neq 0$.

Step 1.1.1: For each vertex v we determinate the subgraph generated by v, $\langle v \rangle$.

Step 1.2: We obtain the list of the ideals generated by two or more vertices non nulls.

Step 1.2.1: For each ideal $\langle v_{i_1}, \dots, v_{i_k} \rangle$ obtained in the preceding iteration (considering the step 1.1.1 as the first iteration) and for each vertex v_j of G_0 non null, we determine if $j > i_k$ and if $\langle v_{i_1}, \dots, v_{i_k}, v_j \rangle$ is a new subgraph with $k+1$ generators and in this case we add this subgraph to the list of subgraphs generated by $k+1$ roots.

Step 2: We determine the action of $Aut(A)$ on G_0. (The set of vertices $V(G)$ is $\{0\} \cup \Delta_+$ and G_0 is a subgraph of G.)

Step 3: We calculate the isomorphism classes of subgraphs of G_0 which verify 1 and 2. (We use $Aut(A)$, since I and I' are isomorphic subgraphs of G_0 which verify 1 and 2 if and only if it exists an automorphism $\sigma \in Aut(A)$ such that $\sigma(I) = I'$.)

Step 3.1: We obtain a representative of each isomorphism class, for recurrence in the number k of vertices which generate the subgraph.

References

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