# Modular Deformation of Curve Singularities

Bernd Martin

**Abstract**

We are investigating different concepts of modular deformations of germs of isolated singularities (infinitesimal, formal). An obstruction calculus based on the graded Lie algebra structure of the tangent cohomology for the enlargement of a modular deformation is introduced. The maximal formal modular stratum of a versal deformation of a space curve singularity coincide with the flattening stratum of the relative Tjurina module of the family extending the similar result for ICIS. Examples are computed for modular deformations of curve singularities which have a splitting of its singular locus inside its $\tau$-constant stratum.

## Introduction

The notion of a modular deformation has been introduced for complete complex varieties by Palamodov, cf. [P1], later on by Laudal in a more general context, cf. [L], and for analytic polyhedron in [P3].

The deformation functor of an isolated singularity is usually not universal because deformations often contain trivial subfamilies in any representing family as, for instance, in the case of isolated complete intersection singularities. One approach to the construction of moduli for singularities may be the study of those deformations that do not contain trivial subfamilies. This can be done by restricting the versal family to subgerms which have an universal property at least for all families induced from it.

The associated infinitesimal notion corresponds to injectivity of the relative Kodaira-Spencer map of the deformation. A formal version is given in terms of a lifting property of vector fields of $\mathbf{X_0}$.

The obstruction for enlarging a modular subgerm is induced from the Lie bracket

$$T^0(\mathbf{X_0}) \times T^1(\mathbf{X_0}) \longrightarrow T^1(\mathbf{X_0}) \qquad (1)$$

¿From the definition follows that a modular deformation is formal modular, and a formal modular family is infinitesimal modular. We shall show that infinitesimal modular is equivalent to formal modular. But, we only can prove so far that modularity follows from formal modularity under an additional assumption on teh critical locus.

In [M2] the author has characterised formal modular families of ICIS as flattening of the relative Tjurina module of the deformation. Based on implementations of the computation of versal deformations, cf. [M1], and of the flattening algorithm in Singular, cf. [S], explicit computations of interesting examples are possible.

Moreover, we prove an extension of the characterisation of formal modular deformations as flattening of the first relative tangent cohomology of the versal deformation to the case of space curve singularities. This is caused by the freeness of the relative normal bundle of the total space of an embedded versal deformation in this case, cf. [St].

# 1 Modular deformations and obstructions

Throughout the paper we are dealing with analytic germs. Let $\mathbf{X_0}$ be an isolated singularity. Choose a miniversal deformation $F : \mathbf{X} \longrightarrow \mathbf{S}$. By definition of versality any other deformation of $\mathbf{X_0}$ over $\mathbf{T}$ is induced from the family $F$, i.e. the functorial map $\xi_{\mathbf{T}} : Hom(\mathbf{T}, \mathbf{S}) \longrightarrow Def_{\mathbf{X_0}}(\mathbf{T})$, $g \mapsto g^*(F)$, is surjective.

**Definition 1.1.** A subgerm $\mathbf{M} \subset \mathbf{S}$ of a miniversal deformation $F : \mathbf{X} \longrightarrow \mathbf{S}$ is called modular if for all germs $\mathbf{T}$ the induced maps $\xi_{\mathbf{T}}$ restricted to $\xi_{\mathbf{T}}^{-1}(\xi_{\mathbf{T}}(Hom(\mathbf{T}, \mathbf{M}))$ are injective.

Any two modular strata of $\mathbf{X_0}$ are uniquely isomorphic and independent of the miniversal family by definition. Only few examples of modular strata have been computed so far. For instance, the modular stratum of a quasi homogeneous isolated complete intersection singularity (ICIS) consists of its reduced $\tau$-constant stratum, cf. [A].

Any pull back of an automorphism of $\mathbf{M}$ to the modular family induces an isomorphism of the modular family by definition. Hence, infinitesimally, it corresponds to the statement that an vector field on $\mathbf{M}$ causes a trivial infinitesimal deformation of the modular family, or equivalently, that the generalised Kodaira-Spencer map is injective. Therefor we can define an infinitesimal version:

**Definition 1.2.** Let $F : \mathbf{X} \to \mathbf{S}$ be a miniversal deformation of $\mathbf{X_0}$. Its restriction to a subgerm $\mathbf{M} \subset \mathbf{S}$ is called infinitesimal modular iff the restriction to $\mathbf{M}$ of the Kodaira-Spencer map $\theta_F$ is injective,

$$\theta_F : T^0(\mathbf{S}) \longrightarrow T^1(\mathbf{X}, \mathbf{S}), \quad \delta \mapsto cl(\delta(F)).$$

Take a lift $\phi$ to the ambient space $\mathbb{C}^n$ of an isomorphism of $\mathbf{X_0}$ and apply $\phi \times id_{\mathbf{M}}$ to the modular family. We obtain another equivalent deformation over $\mathbf{M}$. We formulate this fact on the level of vector fields in the following notion.

**Definition 1.3.** Let $F : \mathbf{X} \to \mathbf{S}$ be a miniversal deformation of $\mathbf{X_0}$. Its restriction to a subgerm $\mathbf{M} \subset \mathbf{S}$ is called formal modular iff for any Artinean subgerm $\mathbf{A} \subset \mathbf{M}$ the associated restriction maps $\eta_{\mathbf{A}}$ of vertical vector fields are surjective:

$$\eta_{\mathbf{A}} : T^0(\mathbf{X_A}, \mathbf{A}) \longrightarrow T^0(\mathbf{X_0}).$$

*Remark 1.4.* Note, that this lifting condition for vector fields is fulfilled iff $\eta_{\mathbf{M}}$ itself is surjective.

The notation 'formal' is motivated by the following

**Lemma 1.5.** *A subgerm* $\mathbf{M}$ *of a miniversal deformation* $F$ *is infinitesimal modular iff it is modular over any Artinean subgerm of* $\mathbf{M}$.

In order to prove the lemma we construct an obstruction map as follows: Let $\mathbf{A} \subset \mathbf{A}'$ be a small extension of Artinean subgerms of $\mathbf{S}$ by an ideal $I_{\mathbf{A}} \subset \mathcal{O}_{\mathbf{A}'}$. The Lie bracket (1) of the tangent cohomology of $\mathbf{X_0}$ in degree 0 and 1 induces a map

$$o_{\mathbf{A}',\mathbf{A}} : T^0(\mathbf{X_0}) \otimes I_{\mathbf{A}} \longrightarrow T^1(\mathbf{X_0})$$

which has the following obstruction property: Assume $F$ is modular over $\mathbf{A}$ then $F$ is modular over $\mathbf{A}'$ iff $o_{\mathbf{A}',\mathbf{A}}$ is the zero map.

**Proposition 1.6.** *A subgerm* $\mathbf{M}$ *of a miniversal deformation* $F$ *is infinitesimal modular iff it is formal modular.*

The idea of the prove consists in a careful analysis of the so called Kodaira-Spencer sequence of the family $F$ and its evaluation at the special fibre. We use arguments similar to that from [P2, 1.8] or [M2, 6]:

$$\begin{array}{ccccccccc}
0 & \to & T^0(\mathbf{X},\mathbf{S}) & \longrightarrow & T^0(F) & \longrightarrow & T^0(\mathbf{S}) & \xrightarrow{\theta_F} & T^1(\mathbf{X},\mathbf{S}) & \to & \cdots \\
 & & \eta_{\mathbf{S}} \downarrow & & \sigma_0 \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T^0(\mathbf{X_0}) & \longrightarrow & T^0(F,\mathbf{X_0}) & \longrightarrow & \mathsf{T}(\mathbf{S},\mathbf{0}) & \xrightarrow{\theta_0} & T^1(\mathbf{X_0}) & \to & \cdots .
\end{array}$$

The equivalence of modularity and formal modularity, as it holds for deformations of complete varieties or analytic polyhedron, cf. [P1], [P3], is open in a pure local context. We can only show it under an additional assumption.

**Theorem 1.7.** *An formal modular subgerm* $\mathbf{M}$ *of a miniversal deformation* $F$ *of* $\mathbf{X_0}$ *is modular if the critical locus of* $\mathbf{X}$ *is umramified over* $\mathbf{M}$.

As examples from the next section will show, that examples of formal modular families exist that have a splitting singular locus.

*Remark 1.8.* The unramification of the critical locus can be interpreted as a kind of finiteness condition: It implies the coherence of $T^1(\mathbf{X_M},\mathbf{M})$ and that the family is simultaneously finitely determined. Note, that we lose coherence of the tangent cohomology in a strict local context. This can be avoided in the category of multi germs. But, when doing computation in the next section we strictly need the category of local analytic (or formal) algebras.

# 2 Flatness of the Tjurina module

The condition of formal modularity may be reformulated as a flatness condition. Let $k$ be the number of generators of the ideal $\mathbf{X_0} \subset \mathbb{C}^n$: $I_0 = (f_1, \dots, f_k) \subset \mathcal{O}_n = \mathbb{C}\{X\}$. The miniversal family $F$ may be chosen as embedded deformation. The total deformation space is then defined by an ideal $I = (F_1, \dots, F_k) \subset \mathcal{O}_{\mathbf{S}}\{X\}$. Denote by $J(F)$ the relative Jacobian matrix of $F$ over $\mathbf{S}$ modulo $I$.

**Proposition 2.1.** *A subgerm* $\mathbf{M}$ *of a miniversal deformation* $F$ *is formal modular iff* $\mathcal{O}^k_{\mathbf{X}}/J(F)$ *is flat over* $\mathbf{M}$.

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & T^0(\mathbf{X},\mathbf{S}) & \longrightarrow & \mathcal{O}^n_{\mathbf{X}} & \overset{J(F)}{\longrightarrow} & \mathcal{O}^p_{\mathbf{X}} & \longrightarrow & \mathcal{O}^k_{\mathbf{X}}/J(F) & \to & 0 \\ & & \eta_{\mathbf{S}}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & T^0(\mathbf{X_0}) & \longrightarrow & \mathcal{O}^n_{\mathbf{X_0}} & \overset{J(f)}{\longrightarrow} & \mathcal{O}^p_{\mathbf{X_0}} & \longrightarrow & \mathcal{O}^k_{\mathbf{X_0}}/J(f) & \to & 0. \end{array} \qquad (2)$$

$J(f)$ is a representation of $\mathcal{O}^k_{\mathbf{X}}/J(F)$ as $\mathcal{O}_{\mathbf{X}}$-module lifting the representation of $\mathcal{O}^k_{\mathbf{X_0}}/J(f)$. The module of vector fields can be considered as syzygies of the columns of the representation matrices. The surjectivity of $\eta_{\mathbf{M}}$ means that any syzygy over the special fibre lifts to a syzygy over $\mathbf{M}$. This is exactly a characterisation of flatness, cf. [E, 6.].

If $\mathbf{X_0}$ is an ICIS the left side modules in (2) are just the Tjurina modules, hence we get:

**Corollary 2.2.** *If* $\mathbf{X_0}$ *is an ICIS then a subgerm* $\mathbf{M} \subset \mathbf{S}$ *of the base space of a miniversal deformation is formal modular iff the relative Tjurina module* $T^1(\mathbf{X},\mathbf{S})$ *is flat over* $\mathbf{M}$.

Using the algorithm from [M2] we are able to compute modular deformations via flattening of $T^1$. The simplest so far found example of a formal modular deformation not being $\tau$-constant is a deformation of the following (degenerated with respect to its Newton boundary, hence not semi-quasi homogeneous) curve singularity: $\mathbf{X_0}$ defined by $f_0 = (x-y^3)^2(x+2y^3)+y^{11}$ with $\tau = 16$ and $\mu = 18$. Consider the family $\mathbf{X} \to \mathbf{T}$ defined by $f(x,y,t) = f_t := f_0 + t^2y^9 + 2ty^{10}$ over $\mathbf{T} = \mathbb{C}^1$ with $\tau(f_t) = 15$ and $\mu(f_t) = 16$ for $t \neq 0$. But its Tjurina algebra $T(f_t) := \mathbb{C}[t]\{x,y\}/(f_t, \partial_x f_t, \partial_y f_t)$ is flat over $\mathbb{C}[t]$. We find two interesting observations: The family $T^1(\mathbf{X},\mathbf{T})$ is not coherent over $\mathbf{T}$, because it is flat, but not free. Moreover, $f_t$ has another critical value for $t \neq 0$ at $(-t^3,-t)$, hence its global Tjurina number is constant and the critical locus splits. This does not occur for a $\mu$-constant deformation.

# 3 Space curve singularities

It is not clear at all, whether the statemanet (2.2) holds in general or under slightly weaker assumption as $T^2(\mathbf{X_0}) = 0$, i.e. $\mathbf{X_0}$ unobstructed and $\mathbf{S}$ smooth. But, for space curves, which are unobstructed, we have many other informations, cf. [St]:

**Lemma 3.1.** *The relative normal bundle* $\mathcal{N}_{\mathbf{X}|\mathbf{S}}$ *of the total space of an embedded versal deformation* $\mathbf{X} \to \mathbf{S}$ *of a space curve singularity is free as* $\mathcal{O}_{\mathbf{S}}$*-module.*

**Corollary 3.2.** *A deformation of space curve singularities is formal modular iff its relative first tangent cohomology is flat over the base space.*

Here we may use the same arguments as in (2.1) applied to a similar diagram (2) replacing the rows by the exact sequence

$$0 \to T^0(\mathbf{X},\mathbf{S}) \longrightarrow \theta(\mathbb{C}^n \times \mathbf{S},\mathbf{S}) \otimes \mathcal{O}_{\mathbf{X}} \overset{J(F)}{\longrightarrow} \mathcal{N}_{\mathbf{X}|\mathbf{S}} \longrightarrow T^1(\mathbf{X},\mathbf{S}) \to 0$$

and its specialisation over the special fibre.

## References


[A] A. G. Aleksandrov, *Cohomology of quasi homogeneous complete intersections*; Iz. AN SSSR, ser. mat. **49** (1985), 467-510.

[E] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Springer-Verlag, Berlin, New York, (1996).

[L] O.A. Laudal, *Formal moduli of algebraic structures,* Lecture Notes in Math. **754**, Springer-Verlag, Berlin-New York, (1979).

[M1] B. Martin, *Computing Versal Deformations with Singular*, in: Algorithmic Algebra and Number Theory, Springer-Verlag, Berlin-New York, (1998), 283-294.

[M2] B. Martin, *Algorithmic Computation of Flattenings and of Modular Deformations*, Prep. Reihe Mathematik der BTU Cottbus, **M-04** (2001) (to appear)

[P1] V.P. Palamodov, *Moduli and versal deformations of complex spaces*, in: Varietes analytiques compactes, LNM **683** Springer-Verlag, Berlin-New York, (1978), 74-115.

[P2] V.P. Palamodov, *Tangent fields on deformations of complex spaces*, Math. USSR Sbornik Vol. **71/1** (1992), 163-182.

[P3] V.P. Palamodov, *Deformation of analytic polyhedrons*, J. Algebraic Geometry **2**, (1993), 263-294. (1992), 163-182.

[S] G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 2.0. *A Computer Algebra System for Polynomial Computation,* Centre for Computer Algebra, Univ. of Kaiserslautern, (2001), `http://www.singular.uni-kl.de`.

[St] D. van Straten *A note on a discriminat of a space curve*, manuscr.math. **87**, (1995), 167-177.



Bernd Martin; BTU Cottbus, Inst. f. Mathematik, PF 101324, D-03046 Cottbus
e-mail:martin@math.tu-cottbus.de