The hull Resolution of a monomial curve in $\mathbf{A}^3(k)$

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Abstract

We characterize the hull resolution of a monomial curve in the three dimensional affine space and compare it with its minimal free resolution. Concretely, we give a necessary and sufficient condition for which the hull resolution is minimal in terms of the semigroup associated with.

Introduction

Let $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k. Throughout this paper $\mathbf{x}^{\mathbf{u}}$ will denote the monomial $x_1^{u_1} \cdots x_n^{u_n}$ with $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{Z}_0^n$.

The hull resolution of the $\mathbf{Z}^n/\mathcal{L}$-graded lattice ideal

$$I_{\mathcal{L}} := \langle\, \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u} - \mathbf{v} \in \mathcal{L} \text{ with } \mathbf{u}, \mathbf{v} \in \mathbf{Z}_0^n \rangle,$$

where $\mathcal{L} \subseteq \mathbf{Z}^n$ is a $\mathbf{Z}$-module such that $\mathcal{L} \cap \mathbf{Z}_0^n = \{\mathbf{0}\}$, was introduced by D. Bayer and B. Sturmfels in [2]. In that work, the authors construct a new canonical free resolution of $I_{\mathcal{L}}$ from an unbounded convex polyhedron P_t (originally introduced by I. Barany, R. Howe and H. Scarf in [1]) and a regular cell complex X (cf. [4] pp. 253–255).

The hull resolution of a lattice ideal is far from being minimal, but, unlike minimal resolutions, it respects symmetry and preserves the action on $I_{\mathcal{L}}$ by the lattice $\mathcal{L}$. Furthermore, the involved free modules are of finite rank over $k[\mathbf{x}]$ and there are finitely many of them. This makes interesting the comparison of minimal and hull resolutions of lattice ideals in order to decide when they agree.

In this paper, we center our attention in a particular class of lattice ideals. We only consider the ideals defining monomial curves in the 3-dimensional affine space. From a new and explicit description of the minimal resolution in terms of combinatorial arguments (Theorem 1.2), we obtain a complete characterization of the hull resolution of a monomial curve in $\mathbf{A}^3(k)$, our main Theorem (2.6). As a corollary we give a necessary and sufficient condition for which the hull resolution of a monomial curve in $\mathbf{A}^3(k)$ is minimal in terms of the semigroup associated with.

Finally, we would like to emphasize that the study of the connections between semigroups and lattice ideals is an active research field as it can be seen through the abundant literature about it (for more details see [5]).

1 The minimal resolution of a monomial curve in $\mathbf{A}^3(k)$.

Let S be a semigroup of positive integers generated by $\{n_1, n_2, n_3\}$, with $n_i \in \mathbf{Z}_+,\ i = 1,2,3$, and $\gcd(n_1, n_2, n_3) = 1$, and let $G(S) \subset \mathbf{Z}$ be the group generated by S.

We consider $\mathbf{u}_1 = (1,0,0),\ \mathbf{u}_2 = (0,1,0)$ and $\mathbf{u}_3 = (0,0,1)$ in $\mathbf{Z}^3$, and the $\mathbf{Z}$-linear surjective map $\pi : \mathbf{Z}^3 \longrightarrow G(S)$, where $\pi(\mathbf{u}_i) = n_i,\ i = 1,2,3$. We write $\mathcal{L}$ for the kernel of π,

$$\mathcal{L} := \ker \pi = \{\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{Z}^3 \ \mid\ \sum_{i=1}^{3} v_i n_i = 0\}$$

Obviously $\mathcal{L} \subseteq \mathbf{Z}^3$ is a lattice such that $\mathcal{L} \cap \mathbf{Z}_0^3 = \{\mathbf{0}\}$. Thus, we have that the ideal of the affine monomial curve $\{(\lambda^{n_1}, \lambda^{n_2}, \lambda^{n_3}) \ \mid\ \lambda \in k\}$ is the lattice ideal $I_{\mathcal{L}}$ (cf. [6]). Since the lattice $\mathcal{L}$ is defined from the semigroup S, in the following we will write I_S for $I_{\mathcal{L}}$.

We define $\alpha_1 \in \mathbf{Z}_+$ to be the least positive integer such that $\alpha_1 n_1 \in \mathbf{Z}_0 n_2 + \mathbf{Z}_0 n_3$ and α_2 and α_3 analogously. That choice of α_1, α_2 and α_3 implies the existence of γ_{ij} and $\gamma_{ik} \in \mathbf{Z}_0$ (not uniquely defined) such that $\alpha_i n_i = \gamma_{ij} n_j + \gamma_{ik} n_k$, for each threesome $\{i,j,k\} = \{1,2,3\}$.

Theorem 1.1. *([6, 3]) With the notation introduced above:*

(a) I_S is complete intersection (equivalently S is symmetric) if and only if there exist $i,j \in \{1,2,3\},\ i \neq j$ such that $\alpha_i n_i = \alpha_j n_j$. In this case, the only minimal binomial systems of generators (except unity in $k[\mathbf{x}]$) is

$$F_1 = x_i^{\alpha_i} - x_j^{\alpha_j},\ F_2 = x_k^{\alpha_k} - x_i^{\gamma_{ki}} x_j^{\gamma_{kj}},$$

for some threesome $\{i,j,k\} = \{1,2,3\}$. Moreover, if $\alpha_k n_k \neq \alpha_i n_i$, then such a threesome is unique.

(b) I_S is not complete intersection (equivalently S is not symmetric) if and only if γ_{ki}, γ_{kj} are both not zero for every threesome $\{i,j,k\} = \{1,2,3\}$. In this case, one has that the pairs $\{\gamma_{ki}, \gamma_{kj}\}$ are unique. Moreover, the only minimal binomial system of generators (except unity in $k[\mathbf{x}]$) is

$$F_1 = x_1^{\alpha_1} - x_2^{\gamma_{12}} x_3^{\gamma_{13}},\ F_2 = x_2^{\alpha_2} - x_1^{\gamma_{21}} x_3^{\gamma_{23}},\ F_3 = x_3^{\alpha_3} - x_1^{\gamma_{31}} x_2^{\gamma_{32}},$$

where $0 < \gamma_{ki} < \alpha_i,\ i = 1,2,3$ and $k \neq i$.

The explicit description of the minimal generating sets of I_S in above theorem can be found in [6], and the uniqueness can be deduced from the combinatorial description of these sets (cf. [3]) by means of some simplicial complexes associated with the elements in the semigroup.

Let $k[S] \simeq k[\mathbf{x}]/I_S$ be the k-algebra associated with the semigroup, and

$$\Phi_0 : k[\mathbf{x}] \longrightarrow k[S],$$

the presentation map.

Theorem 1.2. *With the same notation as above*

(a) *If I_S is complete intersection (equivalently S is symmetric) the minimal free resolution of I_S is:*

$$0 \longrightarrow k[\mathbf{x}] \xrightarrow{\Phi_2} k[\mathbf{x}]^2 \xrightarrow{\Phi_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Moreover, Φ_1 and Φ_2 can be represented respectively by the matrices

$$A_1 = (F_1 \;\; F_2) \quad \textit{and} \quad A_2 = \begin{pmatrix} F_1 \\ -F_2 \end{pmatrix},$$

where the F_1 and F_2 denote the binomials defined in Theorem 1.1(a).

(b) *If I_S is not complete intersection (equivalently S is not symmetric) the minimal free resolution of I_S is:*

$$0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{\Phi_2} k[\mathbf{x}]^3 \xrightarrow{\Phi_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Moreover, Φ_1 and Φ_2 can be represented respectively by the matrices

$$A_1 = (F_1 \;\; F_2 \;\; F_3) \quad \textit{and} \quad A_2 = \begin{pmatrix} x_2^{\gamma_{32}} & x_3^{\gamma_{23}} \\ x_3^{\gamma_{13}} & x_1^{\gamma_{31}} \\ x_1^{\gamma_{21}} & x_2^{\gamma_{12}} \end{pmatrix}$$

where the F_1, F_2 and F_3 denote the binomials defined in Theorem 1.1(b).

Remark 1.3. There exist commutative algebra results (cf. [8]) which assure that a free resolution of I_S is $0 \longrightarrow k[\mathbf{x}]^2 \longrightarrow k[\mathbf{x}]^3 \longrightarrow k[\mathbf{x}] \longrightarrow k[S] \longrightarrow 0$. These arguments are used in [9] in order to get a similar explicit description of the minimal free resolution of I_S when S is not symmetric.

Corollary 1.4. *([7]) $k[S]$ is Gorenstein if and only if I_S is complete intersection (equivalently S is symmetric).*

2 The hull resolution of a monomial curve in $\mathbf{A}^3(k)$.

Our aim in this section consist of characterize the hull resolution of I_S in terms of the semigroup S.

First of all we will study the structure of the hull complex $X = \mathrm{hull}(M_{\mathcal{L}})$ ([2]). Since $\mathcal{L} \cong \mathbf{Z}^2$ we will start with a characterization of all $\mathbf{Z}^2$-invariant triangulations of $\mathbf{R}^2$ whose set of vertices is $\mathbf{Z}^2$.

Given a finite set of vertices $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_r\}$, we write $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle$ for

$$\left\{ \sum_{i=0}^{r} \lambda_i \mathbf{v}_i \in \mathbf{R}^n \;\middle|\; \sum_{i=0}^{r} \lambda_i = 1 \text{ with } \lambda_i > 0 \right\}.$$

We start seeing that he only infinite $\mathcal{L}$-invariant triangulations of $\mathcal{L} \otimes_{\mathbf{Z}} \mathbf{R}$ whose set of vertices is $\mathcal{L}$ are determined by a basis of $\mathcal{L}$. That is, any triangulation K of $\mathcal{L} \otimes_{\mathbf{Z}} \mathbf{R}$ of this kind has got as facets $\langle 0, \mathbf{e}_1, \mathbf{e}_2 \rangle + \mathbf{b}$ and $\langle 0, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_1 \rangle + \mathbf{b}$, for every $\mathbf{b} \in \mathcal{L}$ and some basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathcal{L}$. Then, we obtain the following result:

Theorem 2.1. *There exists a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathcal{L}$ such that the 2-cells of X consists of, one and only one, the following configurations:*

(1) squares $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle + \mathbf{b}$ such that $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2} \rangle + \mathbf{b}$ is not a 1-cell of X, for each $\mathbf{b} \in \mathcal{L}$;

(2) triangles $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2} \rangle + \mathbf{b}$ and $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle + \mathbf{b}$, for each $\mathbf{b} \in \mathcal{L}$.

Once we have limited the suitable forms of the hull complex X, we can restrict the hull resolution of I_S to the two following cases.

Corollary 2.2. *The hull resolution of I_S admits exclusively two possibilities:*

$$0 \longrightarrow k[\mathbf{x}] \xrightarrow{f_2} k[\mathbf{x}]^2 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0$$

or

$$0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

In the view of result above, we have only to determine when it happens one or another resolution. This fact will only depend on the semigroup S. The non symmetric case can be reduced to well-known results (generic case in the sense of [2], Example 3.12).

Proposition 2.3. *If S is not symmetric, then the hull resolution of I_S is minimal. So the hull resolution is $0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0$.*

Assume now that I_S is complete intersection. By Theorem 1.1(a) we have that a minimal system of generators of I_S is $F_1 = x_i^{\alpha_i} - x_j^{\alpha_j}$ and $F_2 = x_k^{\alpha_k} - x_i^{\gamma_{ki}} x_j^{\gamma_{kj}}$ for some threesome $\{i, j, k\} = \{1, 2, 3\}$. Without loss of generality, we can suppose $i = 1,\ k = 2$ and $j = 3$, so $F_1 = x_1^{\alpha_1} - x_3^{\alpha_3}$ and $F_2 = x_2^{\alpha_2} - x_1^{\gamma_{21}} x_3^{\gamma_{23}}$.

Lemma 2.4. *If I_S is complete intersection, with the notation above, $\gamma_{21} = \gamma_{23}$ if and only if $\Gamma := \langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{v}_1}, \mathrm{t}^{\mathbf{v}_2}, \mathrm{t}^{\mathbf{v}_1 + \mathbf{v}_2} \rangle$ is a 2-cell of X, where $\mathbf{v}_1 = (\alpha_1, 0, -\alpha_3)$ and $\mathbf{v}_2 = (-\gamma_{21}, \alpha_2, -\gamma_{23})$.*

Lemma 2.5. *If I_S is complete intersection, with the notation above, $\gamma_{21} \neq \gamma_{23}$ (for every possible choice of them) if and only if the 2-cells of X are triangles.*

All these results are the proof of our main theorem.

Theorem 2.6. *Let S be a semigroup of positive integers generated by $\{n_1, n_2, n_3\}$, with $n_i \in \mathbf{Z}_+,\ i = 1, 2, 3,$ and $\gcd(n_1, n_2, n_3) = 1$. The hull resolution of I_S is*

$$0 \longrightarrow k[\mathbf{x}] \xrightarrow{f_2} k[\mathbf{x}]^2 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0,$$

when S is symmetric with $\alpha_i n_i = \alpha_j n_j$ and there exist $\gamma_{ki} = \gamma_{kj}$ such that $\alpha_k n_k = \gamma_{ki} n_i + \gamma_{kj} n_j$, for some threesome $\{i, j, k\} = \{1, 2, 3\}$. Otherwise the hull resolution of I_S is

$$0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Corollary 2.7. *Let S be a semigroup of positive integers generated by $\{n_1, n_2, n_3\}$, with $n_i \in \mathbf{Z}_+,\ i = 1, 2, 3,$ and $\gcd(n_1, n_2, n_3) = 1$. The hull resolution of I_S is minimal if and only if*

- *S is not symmetric, or*
- *S is symmetric with $\alpha_i n_i = \alpha_j n_j$ and there exist $\gamma_{ki} = \gamma_{kj}$ such that $\alpha_k n_k = \gamma_{ki} n_i + \gamma_{kj} n_j$, for some threesome $\{i, j, k\} = \{1, 2, 3\}$.*

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