CANONICAL DECOMPOSITION OF POLYNOMIAL IDEALS

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Abstract. V.Ortiz established in [10] the existence of a canonical decomposition of ideals in a commutative noetherian ring. In this paper we study the canonical decomposition of ideals in a polynomial ring and we give an algorithmic procedure to compute canonical decompositions.

1. Preliminaries

Let $S := k[x_1, \dots , x_n]$ be the polynomial ring in n variables over a field k.

Definition 1.1. Given an ideal I in S, there exists an integer e such that

$$\left(\sqrt{I}\right)^e \subseteq I.$$

The *index* or *degree of nilpotency* $\operatorname{nil}(I)$ of an ideal I in S is the smallest such integer e. In some texts the index of nilpotency is also called the *exponent* of I.

It is well known that the primary decomposition of an ideal is not uniquely determined. However, it is possible to give a primary decomposition uniqueness theorem.

Theorem 1.2. [10] *Every ideal I in a commutative noetherian ring admits a unique minimal primary decomposition:*

$$I = Q_1^* \cap Q_2^* \cap \dots \cap Q_t^*,$$

such that if

$$I = Q_1 \cap Q_2 \cap \dots \cap Q_t$$

is another minimal primary decomposition of I, then we have

(a) $\operatorname{nil}(Q_i^*) \leq \operatorname{nil}(Q_i),\ i = 1, \dots , t;$
(b) *If* $\operatorname{nil}(Q_i^*) = \operatorname{nil}(Q_i)$, *then* $Q_i^* \subseteq Q_i,\ i = 1, \dots , t.$

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The primary ideals Q_i^* are called *canonical components* and $Q_1^* \cap Q_2^* \cap \ldots \cap Q_t^*$, is called the *canonical decomposition* of I.

The following theorem provides a characterization of the canonical components of a given ideal in a commutative noetherian ring.

Notation 1.3. We write $\mathrm{Hull}(I)$ for the intersection of the minimal primary components of an ideal I.

Theorem 1.4. [10] *Let I be an ideal in a commutative noetherian ring. If P is an associated prime of I and Q^* is the canonical P-component of I, then*

$$Q^* = \mathrm{Hull}\left(I + P^{\mathrm{nil}(Q^*)}\right).$$

In J.Gago's PhD Thesis [6] it can find an application of the canonical decomposition in the classification of finitely generated modules.

2. Canonical Decomposition of Polynomial Ideals

Next theorem is an effective version from the primary decomposition given in [3].

Theorem 2.1. *If I is an in S, then it can be computed effectively a set of integers $\{q_P \mid P \in \mathrm{Ass}(S/I)\}$ such that*

$$I = \bigcap_{P \in \mathrm{Ass}(S/I)} \mathrm{Hull}\left(I + P^{q_P}\right)$$

is a minimal primary decomposition.

Proof. Let $I = \bigcap_{P \in \mathrm{Ass}(S/I)} Q_P$ be minimal primary decomposition and let $P = \sqrt{Q_P}$. Since there are bounds for the index of nilpotency of polynomial ideals (see [1, 7, 5, 9], we can compute effectively a positive integer q_P such that $q_P \geq \mathrm{nil}(Q_P)$, then we have $I + P^{q_P} \subseteq Q_P$, for each $P \in \mathrm{Ass}(S/I)$. Thus, by localization in P, we obtain that

$$I \subseteq \mathrm{Hull}\left(I + P^{q_P}\right) \subseteq \mathrm{Hull}(Q_P) = Q_P$$

and that $\mathrm{Hull}\left(I + P^{q_P}\right)$ is primary, for every $P \in \mathrm{Ass}(S/I)$.

Putting this together, it can be easily deduced that

$$I = \bigcap_{P \in \mathrm{Ass}(S/I)} \mathrm{Hull}\left(I + P^{q_P}\right)$$

is a minimal primary decomposition. $\square$

D.Eisenbud and B.Sturmfels showed in [4] that, over an algebraically closed field, there exist binomial primary decompositions of binomial ideals. The algorithms in [4] has been completed in [8].

The main results in [8] are summarized in the following theorem.

Notation 2.2. If q is a positive integer, then we write $I^{[q]}$ for the q-th Fröbenius power of I.

Theorem 2.3. *Assume k algebraically closed. If I is a binomial ideal in S, then it can be computed effectively a set of integers $\{q'_P \mid P \in \operatorname{Ass}(S/I)\}$ such that*

$$I = \bigcap_{P \in \operatorname{Ass}(S/I)} \operatorname{Hull}\left(I + P^{[q'_P]}\right) \tag{1}$$

is a minimal primary decomposition into binomial ideals.

In [9] it is given an algorithmic procedure to compute implicit bounds for the index of nilpotency for binomial ideals which in many cases are much better than the older ones.

Thus, in Theorem 2.1, when I is a binomial ideal in S with k algebraically closed we can use theorem above to obtain a binomial primary decomposition to start with and compute the index of nilpotency of the primary components using the bounds in [9].

Unfortunately, the canonical decomposition of binomial ideals is not always binomial.

Example 2.4. Consider the binomial ideal $I = (x^2, x(y-1))$ in $k[x, y]$. It is easy to see that $(x) \cap (x^2, x(y-1), (y-1)^2)$ is the canonical decomposition of I which is not binomial when $\operatorname{char}(k) \neq 2$.

Returning to the general case, the next theorem assures the correctness of our algorithm.

Theorem 2.5. *Let I be an ideal in S and let $I = \bigcap_{i=1}^{t} \operatorname{Hull}(I + P_i^{q_i})$ be a minimal primary decomposition. If I is not equal to*

$$\left(\bigcap_{i=1}^{j-1} \operatorname{Hull}(I + P_i^{q_i})\right) \cap \left(I + P_j^{q_j - 1}\right) \cap \left(\bigcap_{i=j+1}^{t} \operatorname{Hull}(I + P_i^{q_i})\right)$$

then $\operatorname{Hull}\left(I + P_j^{q_j}\right)$ is the canonical P_j-component of I, for each $j = 1, \dots, t$.

Proof. First of all it is convenient to recall that, by Theorems 1.2 and 1.4, if $\operatorname{Hull}\left(I + P_i^{q_i^*}\right)$ is the canonical P_i-component, then $q_i^* \leq q_i$ and, consequently,

$$I \subseteq \operatorname{Hull}(I + P_i^{q_i}) \subseteq \operatorname{Hull}\left(I + P_i^{q_i^*}\right), \tag{2}$$

for every $i = 1, \dots, t$.

And now, we will prove the theorem for $j = 1$. If $q_1^* = q_1$ there is nothing to prove. Otherwise $q_1^* < q_1$ and, in this case,

$$I \subseteq \mathrm{Hull}\left(I + P_1^{q_1-1}\right) \subseteq \mathrm{Hull}\left(I + P_1^{q_1'}\right). \tag{3}$$

From formulas (2) and (3) it follows

$$I = \mathrm{Hull}\left(I + P_1^{q_1-1}\right) \cap \left(\bigcap_{i=2}^{t} \mathrm{Hull}\left(I + P_i^{q_i}\right)\right)$$

and then

$$I = \left(I + P_1^{q_1-1}\right) \cap \left(\bigcap_{i=2}^{t} \mathrm{Hull}\left(I + P_i^{q_i}\right)\right),$$

because $I \subseteq I + P_1^{q_1-1} \subseteq \mathrm{Hull}\left(I + P_1^{q_1-1}\right)$. □

Finally, we present the algorithm to compute the canonical decomposition of polynomial ideals.

Algorithm 2.6. Canonical decomposition.
Input: An ideal $I \neq (1)$ in S.
Output: The canonical decomposition of I.

1. If I is primary then output I.
2. Otherwise, compute a set of positive integers $\{q_P \mid P \in \mathrm{Ass}(S/I)\}$ such that $I = \bigcap_{P \in \mathrm{Ass}(S/I)} \mathrm{Hull}(I + P^{q_P})$ is a minimal primary decomposition.
3. For each $P' \in \mathrm{Ass}(S/I)$:
   3.1 Set $e := q_{P'}$.
   3.2 Set $J := I + (P')^{e-1}$.
   3.3 If $I \neq J \cap \left(\bigcap_{P \in \mathrm{Ass}(S/I) \setminus P'} \mathrm{Hull}(I + P^{q_P})\right)$, then define $Q_{P'}^* := \mathrm{Hull}(I + (P')^e)$.
   3.4 Otherwise, return to Step 3.2 with $e = e - 1$.
4. Output $\{Q_P^* \mid P \in \mathrm{Ass}(S/I)\}$.

Comments. In order to check Step 1, one can use the results in [11] or Algorithm 9.4 in [4] in the binomial case. The set of integers in Step 2 can be computed running the algorithms in [11] (in [8] when I is binomial and k algebraically closed, resp.) and using the bounds in [1, 7, 5] (in [9], resp.). The computation of Hull($-$), in Step 3.3, it can be done using localization or primary decomposition procedures (see [11]); for binomial ideals, Algorithm 9.6 in [4] (completed with the algorithms in [8]) can be used.

3. An example

The family of ideals used in the following example is taken from [2]. Primary Decompositions of these ideals give useful descriptions of components of a graph arising in problems from combinatorics, statistics, and operations research.

Let $I_{\mathcal{L}}$ be the prime ideal generated by all 2×2-minors of

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1b} \\ x_{21} & x_{22} & \dots & x_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a1} & x_{a2} & \dots & x_{ab} \end{pmatrix}$$

in $S = k[\{x_{ij}\}]$, where $a, b \geq 3$. Let $R := (x_{11}, \dots, x_{1b})$ and $C := (x_{11}, \dots, x_{a1})$. In [2], it is shown that the ideal of *corner minors* $I_{\mathcal{B}_{\text{cor}}} := (\{x_{11}x_{ij} - x_{1j}x_{i1} \mid 2 \leq i \leq a,\ 2 \leq j \leq b\})$ has the minimal primary decomposition

$$I_{\mathcal{B}_{\text{cor}}} = I_{\mathcal{L}} \cap R \cap C \cap \left(I_{\mathcal{B}_{\text{cor}}} + R^2 + C^2\right).$$

In what follows we use the notation $Q := I_{\mathcal{B}_{\text{cor}}} + R^2 + C^2$.

The ideals $I_{\mathcal{L}}, R$ and C are prime, thus we can already assure that they are canonical components. On the other hand, the radical of the primary ideal Q is $R + C$.

We shall prove that

$$I_{\mathcal{B}_{\text{cor}}} = I_{\mathcal{L}} \cap R \cap C \cap \operatorname{Hull}\left(I_{\mathcal{B}_{\text{cor}}} + (R + C)^3\right) \tag{4}$$

is the canonical decomposition.

We first show that $\operatorname{nil}(Q) \leq 3$. To see that, it suffices to check that $(R + C)^3 \subseteq R^2 + C^2 \subseteq Q$. Moreover, this implies that (4) is a minimal primary decomposition.

Note that $x_{12}x_{21} \in (R + C)^2$ does not lie in Q. So we can assure that $\operatorname{nil}(Q) = 3$.

We next prove

$$I_{\mathcal{L}} \cap R \cap C \subseteq I_{\mathcal{B}_{\text{cor}}} + (R + C)^2. \tag{5}$$

Let $f \in I_{\mathcal{L}} \cap R \cap C$. Since $I_{\mathcal{L}}$ is a binomial ideal not containing any monomial, the by Corollary 1.5 in [4], we can suppose f homogeneous of degree at least 2, that is, $f = m_1 - m_2$ with $\deg(m_1) = \deg(m_2) \geq 2$. On the other hand, since C is a monomial ideal and $f \in C$, the terms m_1, m_2 lie in C, thus we can write $m_1 = x_{i_1 1}m_{11}$ and $m_2 = x_{i_2 1}m_{12}$, with $\deg(m_{11}), \deg(m_{12}) \geq 1$; by the same argument on R, we have $m_1 = x_{1j_1}m_{21}$ and $m_2 = x_{1j_2}m_{22}$, with $\deg(m_{21}), \deg(m_{22}) \geq 1$. Therefore, either $m_1 = x_{11}m_{11} = x_{11}m_{21}$ or $m_1 = x_{i_1 1}x_{1j_1}m_{31}$, with i_1 and j_1 not simultaneously equal 1. If $m_1 = x_{11}m_{11}$, then

$$x_{11}m_{11} = x_{11}x_{kl}m_{31} = (x_{11}x_{kl} - x_{k1}x_{1l})m_{31} + x_{k1}x_{1l}m_{31} \in I_{\mathcal{B}_{\rm cor}} + (R+C)^2,$$

otherwise $m_1 \in (R+C)^2$. In any case, $m_1 \in I_{\mathcal{B}_{\rm cor}} + (R+C)^2$. Similarly, one can prove $m_2 \in I_{\mathcal{B}_{\rm cor}} + (R+C)^2$. Therefore, $f = m_1 - m_2 \in I_{\mathcal{B}_{\rm cor}} + (R+C)^2$, as desired.

By (5), we can assure that I is strictly contained in $I_{\mathcal{L}} \cap R \cap C \cap \left(I_{\mathcal{B}_{\rm cor}} + (R+C)^2\right)$. Thus, by Theorem 2.5, it follows that (4) is the canonical decomposition, as claimed.

Using the improved version of Algorithm 9.6 in [4] given in [8], we have computed $\operatorname{Hull}\left(I_{\mathcal{B}_{\rm cor}} + (R+C)^3\right)$. In such a way we obtain that

$$I_{\mathcal{B}_{\rm cor}} = I_{\mathcal{L}} \cap R \cap C \cap \left(\left(I_{\mathcal{B}_{\rm cor}} + (R+C)^3\right) : \left(\prod_{i,j \neq 1} x_{ij} \right)^{\infty} \right)$$

is the canonical decomposition which, in this case, is binomial.

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