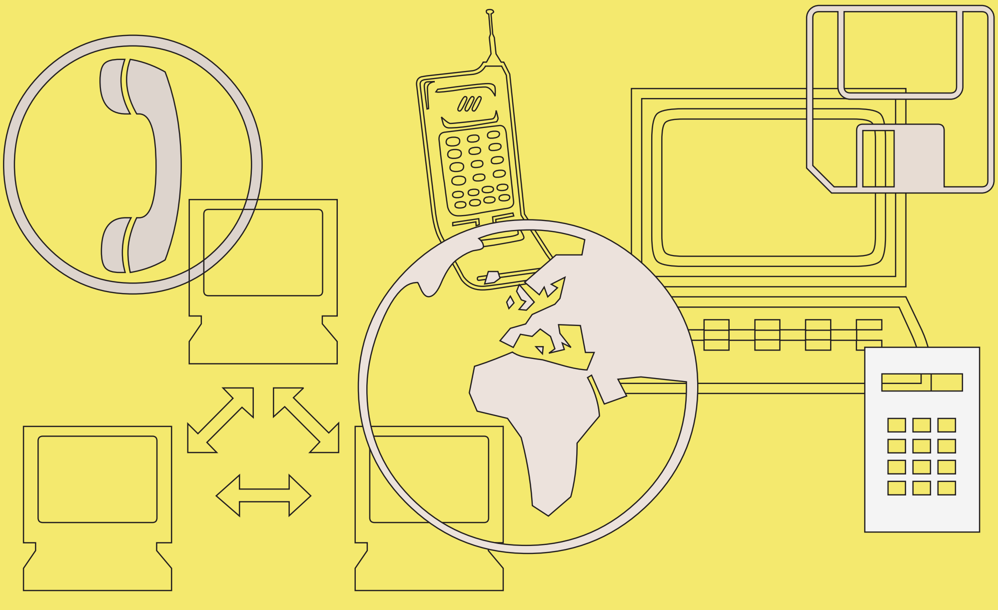
FUNDAMENTALS OF PHYSICS FOR INFORMATICS

E.T.S. de Ingeniería Informática

UNIVERSIDAD de SEVILLA



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Preface

This collection of notes on Electromagnetism and Circuits seeks to help the students in the semester course *Fundamentals of Physics for Informatics* at the ETS of Informatics Engineering, University of Seville. Although these notes have been inspired by various sources (let me emphasize and acknowledge the important contribution of the teachers at the School of Informatics Engineering, Department of Applied Physics 1 of the University of Seville and, especially, to Prof. Gonzalo Plaza), any defect or mistake should be attributable solely to the author of these notes. Importantly, these notes cannot and should not replace other more elaborate texts (or textbooks) on this subject.

The main purpose of the material here presented is to provide some of the physical principles that constitute the basics of the performance of the electric and electronic devices used in computer systems. A great part of modern computer technology is based on Electronics and, since Electronics is basically the control of the flow of electrons inside conductor and semiconductor materials, it is apparent the need to study firstly the general behavior of charges and electrical currents. This study will be carried out through a series of topics dedicated to basic electromagnetism and theory of circuits (direct and alternating currents). In addition, given the importance of electromagnetic waves in communications current (particularly in data transmission networks computers), a general survey of waves will be presented and continued with an elementary study of electromagnetic waves.

Finally, I would like to end these lines reminding a very old saying attributed to Confucius, which accurately summarizes the nature of the learning process:

> I hear, and I forget.
> I see, and I remember.
> I do, and I understand.

Or this very similar one:

> Tell me and I'll listen.
> Show me and I'll understand.
> Involve me and I'll learn.

With these words I just wish to encourage the potential readers of these notes to be aware that only their personal involvement and hard work can properly guide them on the way of a fruitful learning. ("Learn to do by doing")

Francisco L. Mesa Ledesma
Sevilla, september 2015

Part I

MATH REVIEW

Lesson 1

Math Tutorial for FFI

1.1 Algebra

Some basic rules

When algebraic operations are carried out, we are applying the rules of Arithmetic. Symbols as x, y and z are usually employed to represent unspecified quantities, denoted as **unknowns**.

Consider the following equation:

$$8x = 32 \, .$$

To find x, we can divide (or multiply) each side (or member) of the equation by the same factor without breaking the identity. Thus, we can proceed as follows:

$$\frac{8x}{8} = \frac{32}{8} \qquad \Rightarrow \qquad x = 4 \, .$$

Let us consider now this equation:

$$x + 2 = 8 \, .$$

In this type of equation we can sum or subtract the same quantity in each side of the equation. If we subtract 2 to each member, we obtain

$$x + 2 - 2 = 8 - 2 \qquad \Rightarrow \qquad x = 6 \, .$$

In general, if $x + a = b$, then $x = b - a$. To find x in the equation

$$\frac{x}{5} = 9$$

we can multiply each member by 5 to get

$$\left(\frac{x}{5}\right)(5) = 9 \cdot 5 \qquad \Rightarrow \qquad x = 45 \, .$$

In all the previous cases, any operation carried out in the left side of the equation must also be performed in the right side.

Next, some basic rules are presented when operating with fractions:

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd} \tag{1.1}$$

$$\frac{(a/b)}{(c/d)} = \frac{ad}{bc} \tag{1.2}$$

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} \, . \tag{1.3}$$

Powers

When raising to a power, the following rules should be followed:

$$x^n x^m = x^{n+m} \tag{1.4}$$

$$\frac{x^n}{x^m} = x^{n-m} \tag{1.5}$$

$$x^{1/n} = \sqrt[n]{x} \tag{1.6}$$

$$(x^n)^m = x^{nm} \, . \tag{1.7}$$

Logarithms

Any positive number x can be expressed as the power y of a given quantity a, called **base**; namely,

$$x = a^y \, . \tag{1.8}$$

Then, the number y is said to be the logarithm of x to the base a:

$$y = \log_a(x) \, . \tag{1.9}$$

Inversely, the **antilogarithm** of y is the number x:

$$x = \text{antilog}_a(y) \, .$$

It is worth noting the following identity coming from (1.8) and (1.9):

$$x = a^{\log_a(x)} \, . \tag{1.10}$$

In practice, the two most common bases are 10, the base of the **common logarithms**, and the number e = 2.71828 ..., known as Euler constant or base of **natural logarithms**. When we refer to common logarithms, they will be denoted as

$$y = \log_{10}(x) \qquad [\text{or } x = 10^y]$$

and for natural logarithms

$$y = \ln(x) \qquad [\text{or } x = \mathrm{e}^y] \, .$$

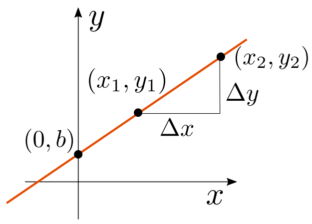
Some useful properties of logarithms are

$$\log(ab) = \log(a) + \log(b) \tag{1.11}$$
$$\log(a/b) = \log(a) - \log(b) \tag{1.12}$$
$$\log(a^n) = n\log(a) \tag{1.13}$$
$$\log(1/a) = -\log(a) \tag{1.14}$$
$$\log_b(x) = \log_b(a)\log_a(x) \tag{1.15}$$
$$\log(1) = 0 \tag{1.16}$$
$$\ln(\mathrm{e}) = 1 \tag{1.17}$$
$$\ln(\mathrm{e}^a) = a\,. \tag{1.18}$$

Activity 1.1:

- Is $\log(a+b) = \log(a) + \log(b)$?
- Can you write $\log(a)$ in terms of $\ln(a)$?
- What is the value of $\ln(-3)$?
- Plot the functions e^x, e^{x^2}, $\ln(x)$, $\ln(x^2)$, and $x\ln(x)$.

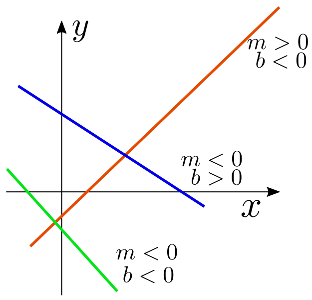
Linear equations



A linear equation of variables x and y (for instance, $4x+7y-9=0$) can always be written as

$$y = mx + b \tag{1.19}$$

where m and b are constants. This equation is called “linear” since the plot of variable y as a function of x is just a straight line, as shown in the attached figure.. The constant b is called the **y intercept** and is the value of y at $x = 0$. The constant m, called the **slope** of the line, is equal to the tangent of the angle that the line forms with the x axis. If any two points of the line are given by coordinates (x_1,y_1) and (x_2,y_2), as shown in the figure, the slope of the line can be expressed as



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}\,. \tag{1.20}$$

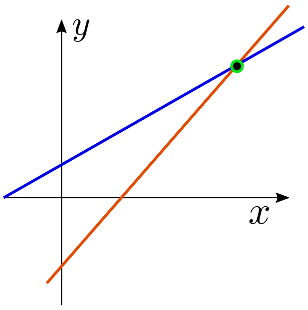
Note that m and b can be either positive or negative numbers. In the previous figure, both b and m were positive. In the attached figure, other three possibilities are shown.

1.1.1 Solution of systems of linear equations

Let us consider the equation $3x + 5y = 15$ with two unknowns x and y. This equation does not have a unique solution; for instance, it can be checked that ($x = 0, y = 3$), ($x = 5, y = 0$) and ($x = 2, y = 9/5$) are solutions of this equation. If a problem has two unknowns, we need at least two equations to find the solution. In general, if we have N unknowns, N equations are required.

To solve for systems of equations with two unknowns, x and y, one possibility is to find an expression of x in terms of y in one of the equations and then substitute the obtained expression into the other equation.

A system of two linear equations can also be solve graphically. If the straight lines corresponding to the two equations are plotted in a system of Cartesian axes, the intersection of the two lines is just the solution. For example, if we consider the following two equations:



$$\begin{cases} x - y = 2 \\ x - 2y = -1 \end{cases}$$

the intersection of the two corresponding lines is the point ($x = 5, y = 3$), which will be the solution of this system of equations. (This system should be solved numerically by the student.)

Factoring

Some useful identities to break apart an equation into factors are the following:

$$\begin{aligned} ax + ay + az &= a(x + y + z) && \text{(common factor)} \\ a^2 \pm 2ab + b^2 &= (a \pm b)^2 && \text{(perfect square)} \\ a^2 - b^2 &= (a + b)(a - b) && \text{(difference of squares)} \,. \end{aligned} \tag{1.21}$$

Quadratic equations

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{1.22}$$

where x is a **variable** and a, b and c are constants known as the **coefficients** of the equation. This equation has two solutions given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \,. \tag{1.23}$$

If $b^2 \geq 4ac$, both roots (solutions of the equation) are real numbers.

Activity 1.2: Solve the equation

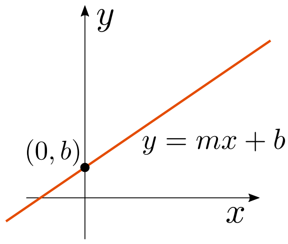
$$\begin{cases} x^2 - y^2 = 7 \\ 2(x + y) = 9 \end{cases}$$

[**Sol.**: $x = 109/36$, $y = 53/36$]

1.2 Geometry

- The **distance** d between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ . \qquad (1.24)$$



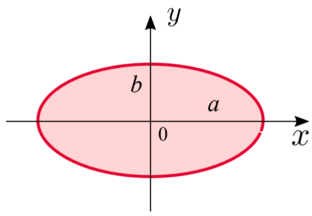
- The equation of a **straight line** (see figure) is

$$y = mx + b$$

where b is the point where the line intercepts the y axis and m is the slope of the line.

- The equation of a **circumference** of radius R centered in the origin is given by

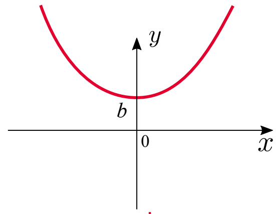
$$x^2 + y^2 = R^2\ . \qquad (1.25)$$



- The equation of an **ellipse** whose center is in the origin (see figure) is

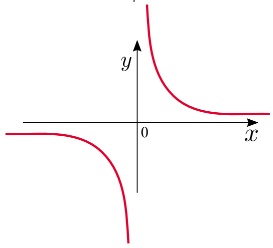
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1.26)$$

where a is the length of the major semiaxis and b the length of the minor semiaxis.



- The equation of a **parabola** whose vertex is located at $y = b$ (see figure) is given by

$$y = ax^2 + b\ . \qquad (1.27)$$



- The equation of a **rectangular hyperbola** (see figure) is

$$xy = \text{constant} \qquad (1.28)$$

Activity 1.3:

- Write the equation of a circumference of radius 5 and centered at $(3, 2)$.
- Prove that the equation $4x^2 + 9y^2 - 8x - 18y - 23 = 0$ stands for an ellipse centered at $(1, 1)$ and semiaxes 3 and 2 respectively.

Fig. 1.1 shows the areas and volumes of different geometric figures often employed in this course.

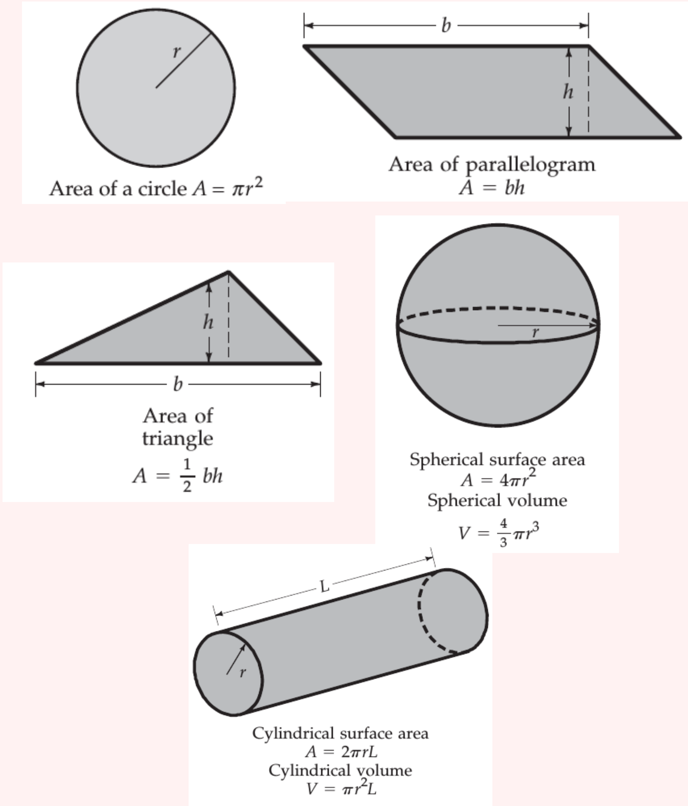
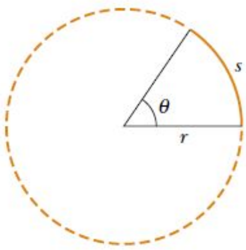


Figure 1.1: Areas and volumes of different geometric figures.

Angles



The arc length s of a circular arc (see figure) is proportional to the radius r for a fixed value of θ (in radians):

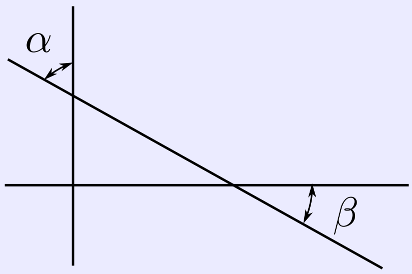
$$s = r\theta \qquad \Rightarrow \qquad \theta = \frac{s}{r}\,. \tag{1.29}$$

The relation between radians and sexagesimal degrees is given by

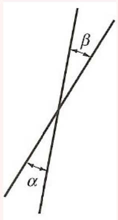
$$360^{\mathrm{o}} = 2\pi \text{ rad}\,.$$

In Fig. 1.2 several useful relations between angles are shown.

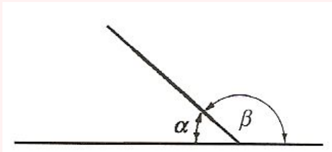
Activity 1.4:



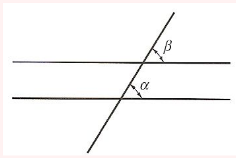
Write β in terms of α.



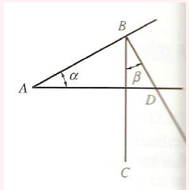
$\alpha = \beta$



$\alpha + \beta = 180^o$



$\alpha = \beta$

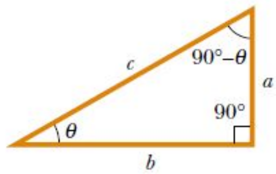


$AB \perp BD, AD \perp BC \rightarrow \alpha = \beta$

Figure 1.2: Relations between angles.

1.3 Trigonometry

The part of Mathematics that deals with the properties of a right triangle is called **Trigonometry**. By definition, a right triangle is the one with an angle of $\pi/2$ (90°). In the triangle shown in the figure, side a is the opposite side to angle θ, side b is the adjacent side to angle θ, and side c is the hypotenuse. The three basic trigonometric functions that are defined for this triangle are the sine (sin), cosine (cos) and tangent (tan), expressed in terms of the angle θ as



$$\sin\theta \equiv \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c} \tag{1.30}$$

$$\cos\theta \equiv \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c} \tag{1.31}$$

$$\tan\theta \equiv \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} = \frac{\sin\theta}{\cos\theta} \tag{1.32}$$

Pythagorean theorem provides the following relation between the side of a right triangle:

$$c^2 = a^2 + b^2 \; . \tag{1.33}$$

From the definition of the above trigonometric functions and Pythagorean theorem, it can be deduced that

$$\sin^2\theta + \cos^2\theta = 1 \; . \tag{1.34}$$

Other trigonometric functions that can be defined are the secant (sec), cosecant (csc), and cotangent (cot):

$$\csc\theta \equiv \frac{1}{\sin\theta} = \frac{c}{a} \tag{1.35}$$

$$\sec\theta \equiv \frac{1}{\cos\theta} = \frac{c}{b} \tag{1.36}$$

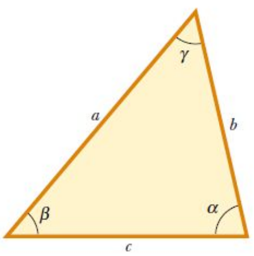
$$\cot\theta \equiv \frac{1}{\tan\theta} = \frac{b}{a}\,. \tag{1.37}$$

The angle θ whose sine is x is called arcsine of x, written as $\arcsin x$. Analogously, the functions arccosine and arctangent are given by

$$\begin{aligned} \sin\theta = x &\quad \Rightarrow \quad \theta = \arcsin x \\ \cos\theta = x &\quad \Rightarrow \quad \theta = \arccos x \\ \tan\theta = x &\quad \Rightarrow \quad \theta = \arctan x\,. \end{aligned} \tag{1.38}$$

The following relations are directly obtained for the right triangle previously shown:

$$\begin{aligned} \sin\theta &= \cos(\pi/2-\theta) \\ \cos\theta &= \sin(\pi/2-\theta) \\ \cot\theta &= \tan(\pi/2-\theta)\,. \end{aligned}$$



Other properties of the trigonometric functions are

$$\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \tan\theta \\ \tan(-\theta) &= -\tan\theta\,. \end{aligned}$$

The following relations apply for *any* triangle (as the one shown in the figure):

$$\alpha+\beta+\gamma = 180^{\circ} \tag{1.39}$$

Cosine theorem

$$a^2 = b^2+c^2-2bc\cos\alpha \tag{1.40}$$

$$b^2 = a^2+c^2-2ac\cos\beta \tag{1.41}$$

$$c^2 = a^2+b^2-2ab\cos\gamma \tag{1.42}$$

Sine theorem

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \tag{1.43}$$

Next, some useful trigonometric identities are shown:

$$\sin^2\theta + \cos^2\theta = 1 \tag{1.44}$$
$$\sec^2\theta = 1 + \tan^2\theta \tag{1.45}$$
$$\csc^2\theta = 1 + \cot^2\theta \tag{1.46}$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{1.47}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{1.48}$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{1.49}$$
$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2} \tag{1.50}$$
$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2} \tag{1.51}$$
$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}\,. \tag{1.52}$$

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B \tag{1.53}$$
$$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B \tag{1.54}$$
$$\tan(A\pm B) = \frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \tag{1.55}$$
$$\sin A\pm\sin B = 2\sin\frac{A\pm B}{2}\cos\frac{A\mp B}{2} \tag{1.56}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \tag{1.57}$$
$$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2} \tag{1.58}$$
$$\tan A\pm\tan B = \frac{\sin(A\pm B)}{\cos A\cos B}\,. \tag{1.59}$$

In Fig. 1.3 several trigonometric functions are plotted in terms of θ. Functions sine and cosine have a period of 2π rad (or 360°). It means that, for any value of θ, $\sin(\theta+2\pi) = \sin\theta$. Function tangent has a period of π rad (or 180°). Also it should be noted that the sine and cosine functions are bounded (between -1 and 1) whereas the tangent function is unbounded.

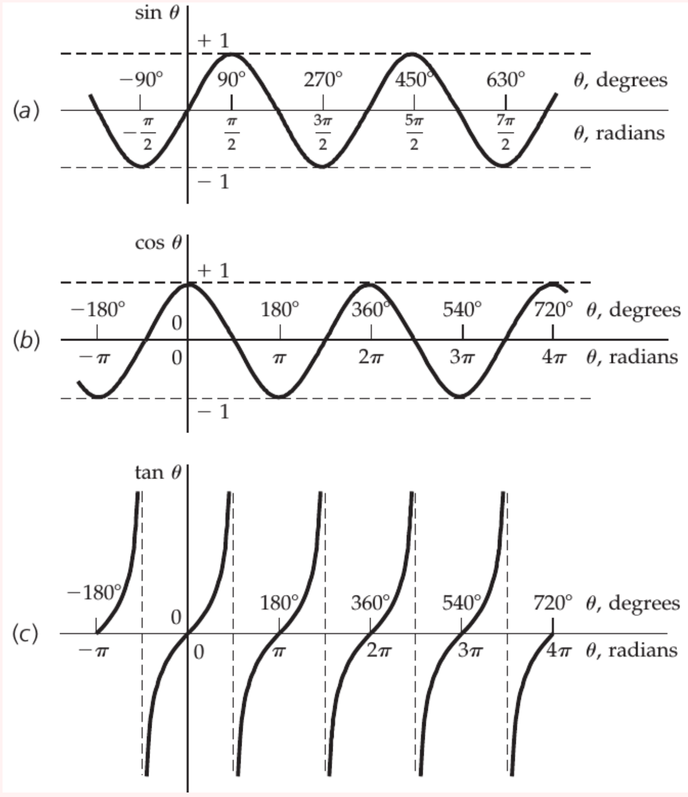


Figure 1.3: Trigonometric functions.

Activity 1.5:

- Write $\sin^2\alpha$, $\cos^2\alpha$, $\sin^3\alpha$, and $\cos^4\alpha$ in terms of sines and cosines of α and 2α.
- Find the value of x which makes $\sec^2 x + 2\tan x = 16$.
  [**Sol.**:$x = \arctan(3)$]
- Explain why the equation $\sin^2 x + 2\cos x = 16$ does hot have real solutions.
- Find the period of the functions $\sin(3x)$, $\tan(x/2)$, and $\cos^2(x)$.
- Plot the function $\arctan(x)$. Note that this function is a multivalued function.

1.4 Series expansion

A list of useful series expansions is shown next:

$$(a+b)^n = \sum_k^n \binom{n}{k} x^{n-k} y^k , \quad \text{with } \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.60}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \tag{1.61}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1.62}$$

$$\ln(1 \pm x) = \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \cdots \tag{1.63}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{1.64}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{1.65}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \qquad (|x| < \pi/2) . \tag{1.66}$$

For $x \ll 1$, the following approximations can be employed:

$$(1+x)^n \approx 1 + nx \tag{1.67}$$

$$e^x \approx 1 + x \tag{1.68}$$

$$\ln(1 \pm x) \approx \pm x \tag{1.69}$$

$$\sin x \approx x \tag{1.70}$$

$$\cos x \approx 1 \tag{1.71}$$

$$\tan x \approx x . \tag{1.72}$$

Activity 1.6: If the imaginary unit is defined as $\mathrm{j} = \sqrt{-1}$ and its powers are given by $\mathrm{j}^2 = -1, \mathrm{j}^3 = -\mathrm{j}, \mathrm{j}^4 = 1, \ldots$, prove the so-called Euler identity: $e^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$.

1.5 Differential Calculus

In several branches of Science, it is sometimes necessary to use the basic tools of infinitesimal calculus invented by Newton to describe the mathematic aspects of physical phenomena. The use of infinitesimal calculus is fundamental in the treatment of different problems of Newtonian mechanics, electricity and magnetism. In this section we will summarize some basic properties and practical rules.

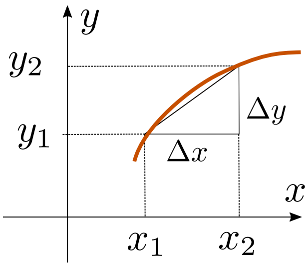
First, we need to specify a **function** that describes how a variable relates to another variable (for example, a space coordinate as a function of time).

Suppose we call y to one of the variables (the dependent variable), and x to the other variable (the independent variable). For these two variables, we could have the following functional relation:

$$y(x) = ax^3 + bx^2 + cx + d \,.$$

If a, b, c and d are known constants, then it is possible to obtain y for any value of x. In general we will work with **continuous** functions; namely, those functions where y varies smoothly as x does it (the plot of y in terms of x is a "continuous", unbroken line). The **derivative** of y with respect to x is defined as the limit, as Δx tends to zero, of the slope of the line that is tangent to the curve at the point (x, y). Mathematically, the derivative can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \tag{1.73}$$



where Δy and Δx are defined as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (see the attached figure). As $\Delta x \to 0$, the curve between the two points turns into the line that is tangent to the curve at (x_1, y_1) in such a way that the derivative coincides with the slope of that tangent line. It should be noted that $\mathrm{d}y/\mathrm{d}x$ does not mean "dy divided by dx" but rather it is a simple notation for the procedure involved in the calculation of the previous limit to obtain the derivative.

Basic properties of derivatives

Derivative of the sum of two functions. If $f(x) = g(x) + h(x)$, then the derivative of the sum of functions equals the sum of the derivatives of the functions:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}[g(x)+h(x)]}{\mathrm{d}x} = \frac{\mathrm{d}g(x)}{\mathrm{d}x} + \frac{\mathrm{d}h(x)}{\mathrm{d}x} \,. \tag{1.74}$$

Derivative of the product of two functions. If $f(x) = g(x)h(x)$, then the derivative of $f(x)$ is given by

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}[g(x)h(x)]}{\mathrm{d}x} = \frac{\mathrm{d}g(x)}{\mathrm{d}x}h(x) + g(x)\frac{\mathrm{d}h(x)}{\mathrm{d}x} \,. \tag{1.75}$$

Derivative of the quotient of two functions. If $f(x) = g(x)/h(x)$, the derivative of $f(x)$ is defined as

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}[g(x)/h(x)]}{\mathrm{d}x} = \frac{1}{h^2(x)}\left[\frac{\mathrm{d}g(x)}{\mathrm{d}x}h(x) - g(x)\frac{\mathrm{d}h(x)}{\mathrm{d}x}\right] \,. \tag{1.76}$$

Chain rule. If $y = f(x)$ and $x = g(z)$, then $\mathrm{d}y/\mathrm{d}z$ is given by the product of the two following derivatives:

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}z} \,. \tag{1.77}$$

Second derivative. The second derivative of y with respect to x is defined as the derivative of function $\mathrm{d}y/\mathrm{d}x$ (the derivative of the derivative). This second derivative is usually expressed as

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) . \tag{1.78}$$

Reciprocal derivative. The derivative of x with respect to y is the reciprocal of the derivative of y with respect to x, provided that none of them is zero:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1} . \tag{1.79}$$

Some specific derivatives of functions. Next, some useful and common derivatives of functions are shown (a and n are constants):

$$\frac{\mathrm{d}}{\mathrm{d}x}(a) = 0 \tag{1.80}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(ax^n) = nax^{n-1} \tag{1.81}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{ax}) = a\mathrm{e}^{ax} \tag{1.82}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x} \tag{1.83}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(ax)) = a\cos(ax) \tag{1.84}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(ax)) = -a\sin(ax) \tag{1.85}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(ax)) = a\sec^2(ax) \tag{1.86}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x)) = \frac{1}{1+x^2} \tag{1.87}$$

Activity 1.7:

- Plot some examples of functions whose derivative at $x = 0$ is not continuous.
- Why is the concept of derivative so important in Engineering?
- Prove expressions (1.75) and (1.76) starting from the definition of derivative.
- Express in words the meaning of the following equation: $f'(x) = f(x)$. Is there any function that satisfies the above equation?
- Find the derivative of the following functions: $(\sin x)/x$, $\sin(1/x)$, x^{3x}, and $\ln(\sec x)$.

1.6 Integral Calculus

Integration is the operation inverse to differentiation.[1] For instance, the following expression:

$$f(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = 3ax^2 + b$$

is just the result of differentiating the function

$$y(x) = ax^3 + bx + c\ .$$

If we wish to obtain the function $y(x)$ from $f(x)$, first we should introduce the concept of **differential** of y, denoted as $\mathrm{d}y$, to express the infinitesimal variation of function $y(x)$ between x and $x + \mathrm{d}x$. Formally we can write that

$$\mathrm{d}y = y(x + \mathrm{d}x) - y(x)\ . \tag{1.88}$$

In general it can be deduced that this differential can also be expressed as

$$\mathrm{d}y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x \tag{1.89}$$

namely, $\mathrm{d}y = f(x)\mathrm{d}x$, or $\mathrm{d}y = (3ax^2 + b)\mathrm{d}x$ for our previous example. It implies that $y(x)$ can be obtained by "summing" all the values of x. Mathematically, this inverse operation is expressed as

$$y(x) = \int f(x)\,\mathrm{d}x\ . \tag{1.90}$$

For the function $f(x)$ given above, we will find that

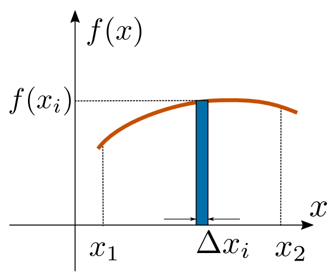
$$y(x) = \int (3ax^2 + b)\,\mathrm{d}x = ax^3 + bx + c$$

where c is an "integration constant". This type of integral is known as **indefinite integral**, since its value depends on the chosen arbitrary value for c.

An indefinite integral $I(x)$ is, in general, defined as

$$I(x) = \int f(x)\mathrm{d}x$$

where $f(x)$ is denoted as the **integrand** and $f(x) = \mathrm{d}y(x)/\mathrm{d}x$.



For any continuous function $f(x)$, its **definite integral** can be defined as the area under the curve $f(x)$ and the x-axis between two specific values of x; for instance, x_1 and x_2, as shown in the attached figure.

The area of the blue-colored element is approximately $f(x_i)\Delta x_i$. If similar elements as this one are summed from x_1 up to x_2 and the limit of this sum

[1] To differentiate means "to take the derivative". It shouldn't be said "to derive". "Differentiation" is the name derived from "to differentiate".

is calculated as $\Delta x_i \to 0$, the actual area under the curve bounded by $f(x)$ and the x-axis between x_1 and x_2 is obtained:

$$\text{Area} = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\Delta x_i = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x = I(x_2) - I(x_1)\ . \tag{1.91}$$

A very common indefinite integral that appears in many practical situations is the following:

$$I(x) = \int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c \qquad (x \neq -1)\ .$$

This result is rather apparent, since the derivative of the right side of the equation with respect to x is $f(x) = x^n$, as it can directly be verified. If the limits of integration are known, this integral turns into a definite integral, which will be written as

$$\int_{x_1}^{x_2} x^n \mathrm{d}x = \frac{x_2^{n+1} - x_1^{n+1}}{n+1} \qquad (n \neq -1)\ .$$

Integration by parts

Sometimes it is very useful to apply the method of *integration by parts* to obtain certain integrals. This method makes use of the following identity:

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u\ . \tag{1.92}$$

In the application examples, functions u and v have to be chosen *judiciously* so that a complex integral can be reduced to an easier one. In many cases it is necessary to carry out several reductions. For instance, the integral

$$I(x) = \int x\mathrm{e}^x \mathrm{d}x$$

can be solved by integrating by parts after choosing $u \equiv x$ and $\mathrm{d}v \equiv \mathrm{e}^x\mathrm{d}x$, so that $\mathrm{d}u = x\mathrm{d}x$ y $v = \mathrm{e}^x$. In this way,

$$I(x) = \int x\mathrm{e}^x \mathrm{d}x = x\mathrm{e}^x - \int \mathrm{e}^x \mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + c$$

namely,

$$I(x) = \int x\mathrm{e}^x \mathrm{d}x = \mathrm{e}^x(x-1) + c\ .$$

Integration by substitution

In this method we carry out a *change of variable*. Consider the following integral:

$$I(x) = \int \cos^2 x \sin x\, \mathrm{d}x$$

This integral is easy to compute if we express $u = \cos x$, which implies that $du = -\sin x dx$. The integral can then be written as

$$I(u) = -\int u^2 du = -\frac{u^3}{3} + C\,.$$

Undoing the change of variable it is finally obtained

$$I(x) = -\frac{\cos^3 x}{3} + c\,.$$

Activity 1.8: Find the solution of the following integrals:

- $\int_0^1 \cos^3 x \, dx$
- $\int_0^1 \frac{x^2}{3x^3+1} dx$.

Immediate integrals

In the following list it is shown some very useful indefinite integrals (the integration constant c has been set equal to 0):

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \qquad \text{if } n \neq -1 \tag{1.93}$$

$$\int \frac{1}{x} dx = \ln x \tag{1.94}$$

$$\int \frac{f'(x)}{f(x)} dx = \ln[f(x)] \tag{1.95}$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} \tag{1.96}$$

$$\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) \tag{1.97}$$

$$\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) \tag{1.98}$$

$$\int \tan(ax) \, dx = -\frac{1}{a} \ln[\cos(ax)] = \frac{1}{a} \ln[\sec(ax)] \tag{1.99}$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} \tag{1.100}$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \frac{a+x}{a-x} \qquad \text{if } a^2 - x^2 > 0 \tag{1.101}$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \frac{x-a}{x+a} \qquad \text{if } x^2 - a^2 > 0 \tag{1.102}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} = -\arccos \frac{x}{a} \quad \text{if } a^2 - x^2 > 0\,. \tag{1.103}$$

Activity 1.9: Find the solution of the following integrals:

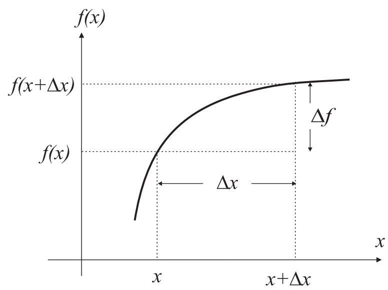
- $\int f'(x) f(x)\, \mathrm{d}x$
- $\int_a^b f''(x) f'(x)\, \mathrm{d}x$
- $\int_0^\pi x \sin x\, \mathrm{d}x$, $\int_0^\pi \cos^2(2t)\, \mathrm{d}t$, $\int_0^\infty \frac{\sin x}{x}\, \mathrm{d}x$.

1.7 Functions of multiple variables

1.7.1 Differential of functions of one variable

Given a function of one variable $f = f(x)$, the derivative of this function $f(x)$ with respect to x was already defined in (1.73) [$f(x) \equiv y(x)$], and is again defined here for convenience:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} . \tag{1.104}$$



The geometrical interpretation of this derivative was shown to be the value of the slope of the line that is tangent to the curve $f(x)$ at x. The concept of **differential** of $f(x)$, generically denoted by $\mathrm{d}f$, accounts for the infinitesimal change of function $f(x)$ between x and $x + \mathrm{d}x$, which symbolically can be expressed as

$$\mathrm{d}f(x) = f(x + \mathrm{d}x) - f(x) . \tag{1.105}$$

From a mathematical point of view, the differential of f is given by the following expression:

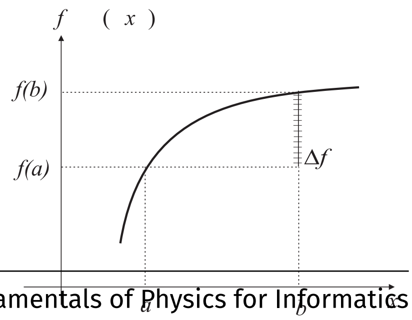
$$\mathrm{d}f(x) = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \mathrm{d}x . \tag{1.106}$$

It should be noted again that $(\mathrm{d}f/\mathrm{d}x)$ **does not** express the quotient of $\mathrm{d}f$ and $\mathrm{d}x$. On the contrary, it has to be taken as the action of the *operator* $\mathrm{d}/\mathrm{d}x$ on function $f(x)$. This fact is more apparent when using other possible notations to express the derivative of function $f(x)$ with respect to x; for instance, $\mathrm{D}_x f(x)$, with $\mathrm{D}_x \equiv \mathrm{d}/\mathrm{d}x$ standing here for the "derivative" operator.

1.7.2 Fundamental theorem of calculus

This theorem states the following relation between the operation of integration and differentiation of function $f(x)$:

$$\int_a^b \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \mathrm{d}x = f(b) - f(a) . \tag{1.107}$$



A plausible way of "deducing" the above expression may start by considering that $(\mathrm{d}f/\mathrm{d}x)\,\mathrm{d}x = \mathrm{d}f(x)$, and therefore the above integral simply means the "total variation of $f(x)$ from a to b"; namely,

$$\int_a^b \mathrm{d}f(x) = f(b) - f(a)\,. \tag{1.108}$$

1.7.3 Differential and partial derivative of a function of multiple variables

In general the physical quantities will depend on more than one variable. For instance, the temperature T of a room depends on the position of the point where it is measured; i.e., it depends on the three spatial coordinates of the point. This fact is mathematically expressed by saying that the temperature is a function of x, y and z, which is denoted as $T = T(x, y, z)$.

Similarly, the differentiation concept introduced in the previous section for functions of one variable could now be extended to functions of multiple variables. For this purpose we first have to define the concept of *partial derivative*. This derivative refers to the variation of a given function w.r.t. only *one* of the variables when the others remain constant. Thus, the partial derivative of function $f(x, y, z)$ with respect to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \tag{1.109}$$

and analogously for the remaining variables. From the concept of partial derivative, it can be deduced that the infinitesimal variation of $f(x, y, z)$ as such function varies from x to $x + \mathrm{d}x$ can be symbolically expressed as

$$\mathrm{d}f\big|_x = \left(\frac{\partial f}{\partial x}\right) \mathrm{d}x\,. \tag{1.110}$$

The infinitesimal variation of $f(x, y, z)$ as this function varies between point (x, y, z) to point $(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ could then be obtained by adding the partial infinitesimal variations along each of the coordinates. Thus it can be written that

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}\right) \mathrm{d}x + \left(\frac{\partial f}{\partial y}\right) \mathrm{d}y + \left(\frac{\partial f}{\partial z}\right) \mathrm{d}z\,. \tag{1.111}$$

Activity 1.10:

- If $x(t) = 3t^2 + 9t - 5$, write the differential of x.
  Is $\mathrm{d}x$ invariant with t?
  Express with words the meaning of $\mathrm{d}x$ if $x(t)$ represents the position of a particle.
- Find an integral whose solution is $\sin(\pi/3) - \cos(\pi/3) + 1$.
- Write the differential of the following function: $x(u, v) = u^2 v - v^2 + v \sin(u)$.
- If the temperature (in Celsius degrees) of a room is given by $T(x, y, z) = x^2 + y^2 + 20$, express in words the meaning of it.
  What does it mean that $\partial T/\partial x \propto x$?
  What does it mean that $\partial T/\partial z = 0$?

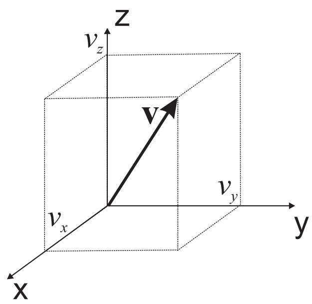
1.8 Vector Algebra

In Nature there are many physical quantities that are fully determined by giving their value and corresponding units. These quantities are called **scalar**. Examples of these quantities are mass, distance, temperature, etc. In contrast, other quantities have an additional property besides their value and units: direction. Such quantities are known as **vectors**, and examples of them are: position, velocity, force, electric field, etc. In the following, some basic operations with vectors will be studied.

1.8.1 Vector Notation

Vector quantities are usually denoted in textbooks by lower/upper-case **bold** types (**v**, **V**), leaving the notation with an arrow/line over these letters ($\vec{v}, \vec{V}$) for handwriting. Nevertheless, in the present notes and with the idea of showing more explicitly the vector nature of the corresponding quantities, we will use the notation involving an arrow over the variables. In many figures of these notes, vectors will appear however denoted in bold type.

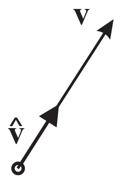
Vectors can be written in different forms. The most usual ones are the following:



- As an array of three numbers/symbols that are the components of the vector along the coordinate Cartesian axes x, y, z:

$$\vec{v} = (v_x, v_y, v_z) \ . \tag{1.112}$$

  Geometrically, the vector components are the projections of this vector onto the Cartesian axes.



Magnitude of vector $\vec{v}$

- In terms of its **magnitude** and its **unit vector**.
  The magnitude of vector $\vec{v}$ is usually denoted as v (just the variable employed for the vector but written in non-bold type and/or without arrow) as well as $|\vec{v}|$. According to Pythagorean theorem, the magnitude (or length of the vector) is given by

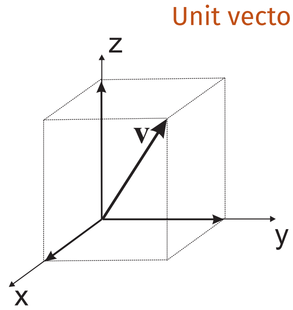
$$|\vec{v}| \equiv v = \sqrt{v_x^2 + v_y^2 + v_z^2} \ . \tag{1.113}$$

  The unit vector associated with a vector $\vec{v}$ is defined as the vector of magnitude 1 with the same direction as $\vec{v}$. Such unit vector will here be denoted as $\hat{\mathbf{v}}$ and can be expressed as

$$\hat{\mathbf{v}} = \frac{\vec{v}}{v} = \frac{(v_x, v_y, v_z)}{\sqrt{v_x^2 + v_y^2 + v_z^2}} \ . \tag{1.114}$$

  Certainly, vector $\vec{v}$ can also be written as $\vec{v} = v\,\hat{\mathbf{v}}$.

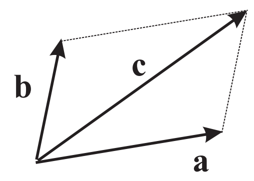
Unit vector of $\vec{v}$



- As the sum of each of the vector components times the unit vector along the corresponding coordinate axis. The unit vectors along axes x, y, z will be denoted as $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ respectively. Other usual notations for these unit vectors are **i**, **j**, **k** as well as $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$. Using this notation, vector $\vec{v}$ is written as

$$\vec{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}} \,. \tag{1.115}$$

1.8.2 Vector Addition



The so-called *parallelogram law* gives the rule for vector addition of two or more vectors. For two vectors $\vec{a}$ and $\vec{b}$, the vector sum $\vec{a} + \vec{b}$ is obtained by placing them head to tail and drawing the vector from the free tail to the free head. In Cartesian coordinates, vector addition can be performed simply by adding the corresponding components of the vectors. Thus, if

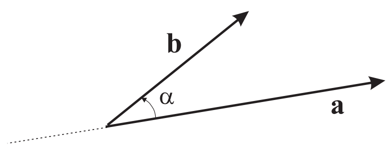
$$\vec{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$$

$$\vec{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}} \,,$$

the resulting vector sum $\vec{c}$ is given by

$$\begin{aligned} \vec{c} &= \vec{a} + \vec{b} \\ &= (a_x + b_x)\hat{\mathbf{x}} + (a_y + b_y)\hat{\mathbf{y}} + (a_z + b_z)\hat{\mathbf{z}} \,. \end{aligned} \tag{1.116}$$

1.8.3 Dot product

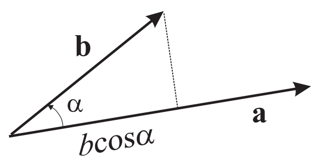


The dot product of two vectors $\vec{a}$ and $\vec{b}$, denoted as $\vec{a} \cdot \vec{b}$ is a *scalar* quantity defined as the following operation:

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \tag{1.117}$$

$$= ab\cos\alpha \tag{1.118}$$

with α being the angle between these two vectors [regardless this angle is taken clockwise or counter-clockwise, since $\cos(2\pi - \alpha) = \cos\alpha$].



The dot product $\vec{a} \cdot \vec{b}$ can be interpreted geometrically as the projection of a vector onto the other (apart from a numeric factor). This fact is apparent in the dot product of $\vec{a}$ by one of the unit vectors associated with the coordinate axes, that is,

$$\vec{a} \cdot \hat{\mathbf{x}} = a_x \,,$$

where it is clearly seen that $\vec{a} \cdot \hat{\mathbf{x}}$ is just the projection of vector $\vec{a}$ on axis x.

Some properties of the dot product are

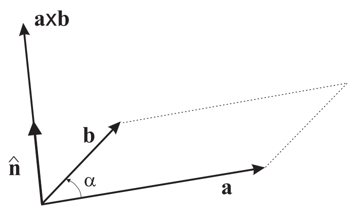
- It is commutative:
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \,. \tag{1.119}$$

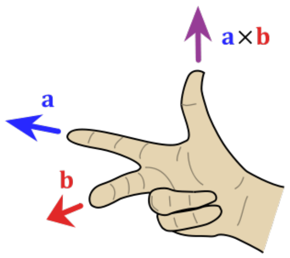
- It is distributive with respect to the vector addition:
$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \,. \tag{1.120}$$

- The dot product of two orthogonal vectors is zero:
$$\vec{a} \cdot \vec{b} = 0 \quad \Rightarrow \quad \vec{a} \perp \vec{b} \,. \tag{1.121}$$

- The dot product of a vector by itself is equal to the square of the magnitude of such vector:
$$\vec{a} \cdot \vec{a} = |\vec{a}|^2 \, (\equiv a^2) \, . \tag{1.122}$$

1.8.4 Cross Product





The cross (or vector) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is a *vector* that is perpendicular to both $\vec{a}$ and $\vec{b}$ (and therefore normal to the plane containing them) defined by

$$\vec{a} \times \vec{b} = ab \sin\alpha \, \hat{\mathbf{n}} \tag{1.123}$$

where α is the *smaller* angle between these two vectors and $\hat{\mathbf{n}}$ is the unit vector *normal* to the plane containing vectors $\vec{a}$ and $\vec{b}$ in the direction given by the right-hand rule (see attached figure).[2] From a geometrical point of view, the magnitude of the cross product, $|\vec{a} \times \vec{b}|$, is equal to the positive area of the parallelogram having $\vec{a}$ and $\vec{b}$ as sides.

From the definition of the cross product (1.123), the following properties can be derived:

- It is anti-commutative,
$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \, . \tag{1.124}$$

- It is distributive over the sum of vector,
$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \, . \tag{1.125}$$

- The cross product of two parallel vectors is null,
$$\vec{a} \times \vec{b} = 0 \quad \Rightarrow \quad \vec{a} \parallel \vec{b} \, . \tag{1.126}$$

- Cross product with a scalar α,
$$\alpha(\vec{a} \times \vec{b}) = \alpha\vec{a} \times \vec{b} = \vec{a} \times \alpha\vec{b} \, . \tag{1.127}$$

Taking into account definition (1.123) and properties (1.124)–(1.126), the cross product of $\vec{a}$ with $\vec{b}$ can be obtained as

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}) \times (b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}) = \\ &= (a_yb_z - a_zb_y)\hat{\mathbf{x}} + (a_zb_x - a_xb_z)\hat{\mathbf{y}} + (a_xb_y - a_yb_x)\hat{\mathbf{z}} \, . \end{aligned} \tag{1.128}$$

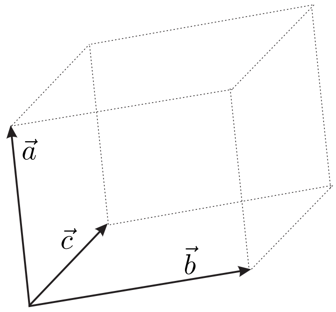
Using the definition of the *determinant* of a matrix, the cross product can also be expressed as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \, . \tag{1.129}$$

[2] This rule is also known as *right-hand screw rule* and it says that turning a screw from $\vec{a}$ to $\vec{b}$ along the smaller angle route, the direction of $\vec{a} \times \vec{b}$ is determined by the resulting direction of the screw.

1.8.5 Triple Products

Given that the cross product of two vectors is another vector, we can perform a further dot-product/cross-product of this resulting vector with another vector to form the so-called triple products.



- **Scalar triple product**: $\vec{a} \cdot (\vec{b} \times \vec{c})$.

  Geometrically, the magnitude of this triple product can be interpreted as the volume of the parallelepiped having $\vec{a}$, $\vec{b}$ and $\vec{c}$ as sides. As shown in the attached figure, $|\vec{b} \times \vec{c}|$ is the area of the basis and $|a \cos \alpha|$ is the height (α is the angle between $\vec{a}$ and $\vec{b} \times \vec{c}$). Using this geometric interpretation, it is easy to deduce that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \,. \tag{1.130}$$

  It is interesting to note that the "alphabetical order" is preserved in the above expression.

  The scalar triple product can also be obtained from the following determinant:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \,. \tag{1.131}$$

- **Vector triple product**: $\vec{a} \times (\vec{b} \times \vec{c})$.

  This triple product can be obtained as

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b}) \,. \tag{1.132}$$

  It should be noted that the vector

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) = -\vec{a}\,(\vec{b} \cdot \vec{c}) + \vec{b}\,(\vec{a} \cdot \vec{c}) \tag{1.133}$$

  is completely different as that defined in (1.132).

Activity 1.11:

- Given the following points: $A(0,0)$, $B(1,-1)$, $C(1,1)$ and $D(2,0)$, find the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ and draw them. Among the possible conclusions from the above operation, could you say that there exists something as an “starting point” in the definition of a vector?
- Find the magnitude and unit vector of $\vec{a} = 3\hat{\mathbf{x}} - 7\hat{\mathbf{y}} + \hat{\mathbf{z}}$ and also of $\vec{r}(t) = 3t^2\hat{\mathbf{x}} + \hat{\mathbf{y}} - t^3\hat{\mathbf{z}}$.
- Write in terms of its Cartesian components the vector $\vec{b}(t)$ whose magnitude is given by $b(t) = 9t^2$ and its unit vector $\hat{\mathbf{b}}(t) = (3t\hat{\mathbf{x}} + 5\hat{\mathbf{z}})/\sqrt{9t^2+25}$.
- Can the projection of vector $\hat{\mathbf{a}}$ onto $\hat{\mathbf{b}}$ be 1.7? Explain the reasons.
- Find the vectors that are orthogonal to $\vec{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$ and also to $\vec{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$.
- Using the properties of the addition and scalar product of vectors, deduce the relations given in (1.40)-(1.42).
- Find the vectors that are orthogonal to the plane defined by the following three points: $A(0,2,1)$, $B(1,-1,0)$, and $C(2,1,1)$.
- What is the minimum distance from point $D(3,3,3)$ to the plane described in the above item?

1.9 Vector Calculus

In this section the basic concepts of differentiation and integration will be revisited for the case of vector functions and operators.

1.9.1 Gradient

For a function of multiple variables, $f(x,y,z)$, it was already seen in Eq. (1.111) that the differential of this function,

$$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z}\right) dz$$

can alternatively be expressed as the following dot product:

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (dx, dy, dz)\,. \tag{1.134}$$

If we define the vector differential operator $\vec{\nabla}$ (nabla symbol) as

Operator $\vec{\nabla}$

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{1.135}$$
$$\equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\,, \tag{1.136}$$

when this operator is applied to a given function $f(x, y, z)$, it is obtained the **gradient** of the scalar function f, $\vec{\nabla} f$, which is given by the following vector quantity:

Definition of gradient of f

$$\vec{\nabla} f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \tag{1.137}$$
$$= \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}\,. \tag{1.138}$$

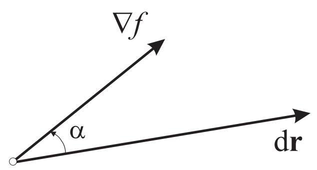
The above definition makes it possible to write the differential of function f as the following dot product:

$$\boxed{\mathrm{d}f = \vec{\nabla} f \cdot \mathrm{d}\vec{r}} \tag{1.139}$$

where $\mathrm{d}\vec{r} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$.

Making use of the definition of the dot product, $\mathrm{d}f$ can also be written as

$$\mathrm{d}f = |\vec{\nabla} f|\,|\mathrm{d}\vec{r}|\,\cos\alpha \tag{1.140}$$

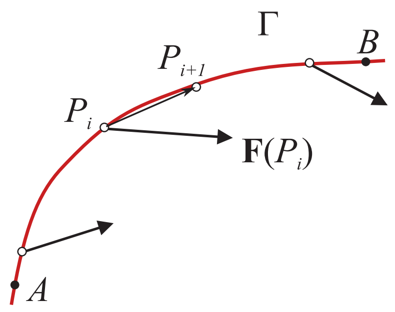


which allows us to deduce that the maximum variation of function $f(x, y, z)$ occurs when $\alpha = 0$; that is, when $\mathrm{d}\vec{r}$ is parallel to the gradient of f, $\vec{\nabla} f$. Consequently, *the direction of vector $\vec{\nabla} f$ points to the direction of maximum variation of the function at point (x, y, z).*

Activity 1.12:

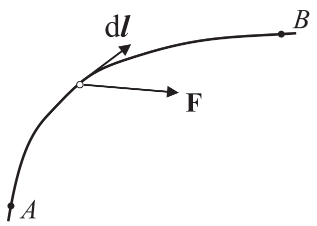
- Find the expression of $\vec{\nabla} f$ and $\mathrm{d}f$ for $f(x, y, z) = xy^2 + z^3$.
- Obtain the unit vector associated with the direction of maximum variation of the above function.

1.9.2 Line integral



Given a vector field $\vec{F}$ (that is, a *vector* quantity whose components depend on the spatial position), the line integral of $\vec{F}$ between two points A y B along the curve Γ is defined by the following integral:

$$C_A^B = \int_{A,\Gamma}^{B} \vec{F} \cdot \mathrm{d}\vec{l}$$
$$= \lim_{P_{i+1} \to P_i(\Gamma)} \sum_i \vec{F}(P_i) \cdot \overline{P_i P_{i+1}}\,. \tag{1.141}$$



The above integral may be interpreted as the infinitesimal superposition of the dot product $\vec{F}\cdot d\vec{l}$ for each differential element of the curve Γ between points A and B. Vector $d\vec{l}$ is a vector whose magnitude is a differential length of the curve and its direction is given by the tangent to the curve in that point; for instance, along the x-axis this vector is given by $d\vec{l} = dx\hat{\mathbf{x}}$. Line integrals are very useful and common in Physics; an important example is the work done by a given force between two points along certain trajectory. In general, *the line integral depends on the path chosen to go from A to B.*

Some important properties of the line integrals are

- $\displaystyle\int_{A,\Gamma}^{B}\vec{F}\cdot d\vec{l} = -\int_{B,\Gamma}^{A}\vec{F}\cdot d\vec{l}\,.$
- If A' is an intermediate point belonging to the curve Γ (between A and B), it can be written that
  $\displaystyle\int_{A,\Gamma}^{B}\vec{F}\cdot d\vec{l} = \int_{A,\Gamma}^{A'}\vec{F}\cdot d\vec{l} + \int_{A',\Gamma}^{B}\vec{F}\cdot d\vec{l}\,.$

Activity 1.13:

- A given force can be expressed as $\vec{F} = x^3y\hat{\mathbf{x}} - z^2\hat{\mathbf{y}}$. Find the work done by this force to translate a particle from point $A(0,1,0)$ to $B(3,1,0)$ following a trajectory parallel to the x-axis.

1.9.3 Fundamental theorem of gradient

Similarly as was done for functions of one variable in (1.107), the following very relevant identity can be proven:

$$\int_{A}^{B}\vec{\nabla}f\cdot d\vec{l} = f(B) - f(A) \tag{1.142}$$

where the integral of the above expression should be recognized as a *line integral*.

The identity (1.142) can be somehow “justified” by considering the definition of the differential of f given by (1.139). From this definition, the integral in (1.142) can be seen as the superposition of the infinitesimal variations of function f between points A and B, which is precisely $f(B) - f(A)$.

Two important corollaries deduced from expression (1.142) are

1.)

$$\int_{A,\Gamma}^{B}\vec{\nabla}f\cdot d\vec{l} = \int_{A,\gamma}^{B}\vec{\nabla}f\cdot d\vec{l} \tag{1.143}$$

namely, $\int_A^B\vec{\nabla}f\cdot d\vec{l}$ is *independent of the path* taken to go from point A to B. It should be noted that, in general, the line integral $\mathcal{C}_A^B = \int_{A,\Gamma}^{B}\vec{F}\cdot d\vec{l}$

does depend on the chosen path (consider, for instance, the work done by a car to go from one city to another by taking different routes).

2.)

$$\oint_{\Gamma} \vec{\nabla} f \cdot d\vec{l} = 0 \,. \tag{1.144}$$

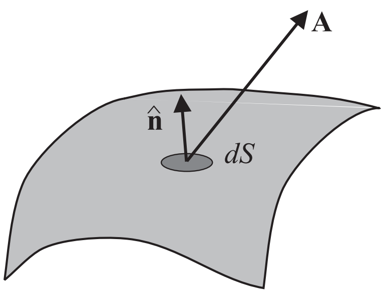
The above line integral is null along any closed path Γ.

Activity 1.14: The gravitational force (or weight), $\vec{P}$, can be expressed as $\vec{F} = -\vec{\nabla} E_p(y)$, where $E_p(y) = mg(y - y_0)$ with y_0 being a constant given reference and $g = 10\,\mathrm{m/s^2}$.

- For a particle of mass $m = 2\,\mathrm{kg}$, find the value of $\mathcal{C}_A^B = \int_{A,\Gamma}^{B} \vec{P} \cdot d\vec{l}$ when $A(2,3,0)\,\mathrm{m}$ and $B(1,0,0)\,\mathrm{m}$. Does the above integral (work done by the gravitational force) depend on the path followed by the particle to go from A to B? Find an explanation for that.
- Without doing any computation, deduce the value of the gravitational force required to translate the above particle between points $A(2,3,0)\,\mathrm{m}$ and $B(1,3,-5)\,\mathrm{m}$, and also between $A(2,3,0)\,\mathrm{m}$ and $B(2,3,0)\,\mathrm{m}$ after traversing an arbitrary closed path.

1.9.4 Flux

Another very useful integral appearing in Physics is the the surface integral of vector fields or **flux**. The flux of a vector field $\vec{A}$ through a given surface S is defined by the following surface integral:

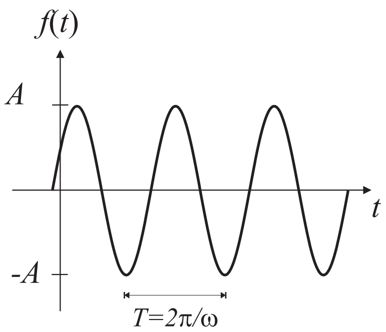


$$\Phi = \int_S \vec{A} \cdot d\vec{S} \tag{1.145}$$

where S is an arbitrary surface and $d\vec{S}$ is a vector called surface differential vector, which is defined by

$$d\vec{S} = dS\,\hat{\mathbf{n}} \tag{1.146}$$

and whose magnitude is the area of surface differential element and its direction is given by the unit vector normal to the surface differential element, $\hat{\mathbf{n}}$. For instance, for the case of a plane given by $z = \mathrm{Cte}$, the surface differential vector will be $d\vec{S} = dxdy\,\hat{\mathbf{z}}$.



1.10 General aspects of time-harmonic functions

A time-harmonic function $f(t)$ is a function of time with the following general mathematical form:

$$f(t) = A\cos(\omega t + \varphi) \tag{1.147}$$

where A is the **amplitude**, ω is the **angular frequency**, and φ the **initial phase**. The amplitude A determines the range in which the function can vary, that is,

$$-A \leq f(t) \leq A\,.$$

The angular frequency ω (its units are rad/s) is related to the frequency, f (units: Hz=s^{-1}), as follows:

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{1.148}$$

where T is the **period** of the function; namely, the value for which $f(t) = f(t+T)$. The initial phase φ determines the value of the function at the origin ($t = 0$):

$$f(0) = A\cos\varphi\,.$$

It is convenient to remind the following trigonometric identities:

$$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$$
$$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$$

from which it can be deduced, for instance, that

$$\cos(\omega t + \varphi - \pi/2) = \sin(\omega t + \varphi)\,. \tag{1.149}$$

1.10.1 Root-mean-square (rms) values

The root-mean-square (rms) value, $F_{\rm rms}$, of a time-harmonic function given by

$$F(t) = F_0\cos(\omega t + \varphi) \tag{1.150}$$

is defined as the square root of the average of the square of this function; namely,

$$F_{\rm rms} = \sqrt{\langle F^2(t)\rangle}\,. \tag{1.151}$$

The average value of any periodic function, $f(t)$, of period T is given by

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)\mathrm{d}t\,. \tag{1.152}$$

In the case of AC circuits, the rms value of the current $I(t) = I_0\cos(\omega t + \varphi)$ [or any other circuit variables with time-harmonic variation] is very important from a practical point of view, since this value is precisely the value given by the analog multmeters (ampmeter). Following the definition given in (1.151) and taking into account (1.152), it is obtained that

$$I^2_{\rm rms} = \langle I_0^2\cos^2(\omega t + \varphi)\rangle = \frac{1}{T}I_0^2\int_0^T \cos^2(\omega t + \varphi)\,\mathrm{d}t = \frac{I_0^2}{2}$$

and thus the rms value of the current is related to the current amplitude, I_0, as follows:

$$\boxed{I_{\rm rms} = \frac{I_0}{\sqrt{2}}} \ . \tag{1.153}$$

rms value of the alternating current

Similarly the rms value of any other time-harmonic quantity is defined as the amplitude of such quantity over $\sqrt{2}$.

It should be noted that the rms value, $I_{\rm rms}$, of the alternating current $I(t) = I_0 \cos(\omega t + \varphi)$ flowing through a resistance R coincides with the value of the *direct* current that dissipates the same energy during a period of time T. The energy $W_{\rm AC}$ dissipated in a resistance R by an alternating current during a time T can be computed as

$$W_{\rm AC} = \int_0^T P(t)\, {\rm d}t \tag{1.154}$$

where $P(t)$ is the *instantaneous power* dissipated in the resistor, which is given by the product of the current by the voltage,

$$P(t) = I(t)V(t) \ . \tag{1.155}$$

Since according to (7.6) the voltage drop in the resistor is $V(t) = RI(t)$, the energy dissipated by the AC in this resistor can be written as

$$W_{\rm AC} = I_0^2 R \int_0^T \cos^2(\omega t + \varphi){\rm d}t = I_0^2 R \frac{T}{2} = I_{\rm rms}^2 RT \ . \tag{1.156}$$

This value is precisely the amount of energy dissipated in a time T in that resistance R when a *direct* current of value $I_{\rm rms}$ flows through it; namely,

$$W_{\rm DC} = I_{\rm rms}^2 RT \ . \tag{1.157}$$

Activity 1.15:

- Find the amplitude, period, initial phase, and angular frequency of the following harmonic function $f(t) = 3\sin(100\pi t)$. First, write the above function in the general form given in (1.147).
- Given the following two harmonic functions: $f(t) = 5\cos(100\pi t)$ and $g(t) = -5\sin(100\pi t)$, find the resulting sum function. What is the frequency of this resulting sum function? Under what circumstances the resulting function of the sum of two harmonic functions will not have the same frequency?

1.11 Phasor Analysis

1.11.1 Complex numbers

In the resolution of algebraic equations of second degree we often find solutions that involves the square root of a negative number; for instance, the solution of $x^2 + 9 = 0$ is just $x = \sqrt{-9}$. However, there is not any *real* number whose square could be -9. The imaginary numbers are then introduced to solve this issue. These numbers are formed from the definition of the **imaginary unit**, j (this imaginary unit is also denoted as i):

$$\mathrm{j} = \sqrt{-1} \tag{1.158}$$

in such a way that

$$\sqrt{-9} = \sqrt{-1 \times 9} = \sqrt{-1} \times \sqrt{9} = \mathrm{j}3$$

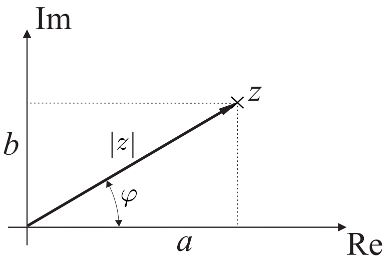
with j3 being an *imaginary number*. Numbers that have both a real part and an imaginary part are known as **complex numbers**. They can be written, in general, as

$$z = a + \mathrm{j}b \tag{1.159}$$

where $a = \mathrm{Re}(z)$ is the so-called *real part* of z and $b = \mathrm{Im}(z)$ the *imaginary part* of z. This form of writing the complex numbers is known as **rectangular form** or **binomic form**.

Usually, complex numbers are represented in a plane in such a way that the vertical axis stands for the imaginary values of the numbers and the horizontal axis for their real part. This way, the complex number z is characterized by a point in the plane (see the attached figure) which is located at a distance $|z|$ given by

$$|z| = \sqrt{a^2 + b^2} \tag{1.160}$$



known as **absolute value** (or magnitude, or modulus) of z, and with an angle φ (measured in a counter-clockwise way from the real axis) given by

$$\varphi = \arctan\left(\frac{b}{a}\right) \tag{1.161}$$

denoted as **argument** (or phase) of z.

In the figure it is easy to realize that the complex number z can also be written as

$$z = |z|(\cos\varphi + \mathrm{j}\sin\varphi)\ .$$

If we now take into consideration the so-called *Euler identity*, which states that

$$\boxed{\mathrm{e}^{\mathrm{j}\varphi} = \cos\varphi + \mathrm{j}\sin\varphi} \tag{1.162}$$

it is found that z can alternatively be written as

$$z = |z|\mathrm{e}^{\mathrm{j}\varphi}\ . \tag{1.163}$$

This way of writing complex number is known as **polar form**.

Activity 1.16:

- Write the following complex numbers given in binomic form in their polar form: $z_1 = 3$, $z_2 = -3$, $z_3 = \mathrm{j}$, $z_4 = -\mathrm{j}$, $z_5 = 1 + \mathrm{j}$, $z_6 = 1 - \mathrm{j}$, $z_7 = -1 + \mathrm{j}$, and $z_8 = -1 - \mathrm{j}$. [It is convenient to plot the complex numbers in the complex plane.]
- Write in their binomic form the following complex numbers $z_1 = \mathrm{e}^{-\mathrm{j}\pi}$, $z_2 = \mathrm{e}^{\mathrm{j}\pi/4}$, $z_3 = 7\mathrm{e}^{-\mathrm{j}\pi/2}$, $z_4 = 2\mathrm{e}^{\mathrm{j}\pi/3}$, $z_5 = 3\mathrm{e}^{-\mathrm{j}3\pi/4}$, $z_6 = 3\mathrm{e}^{\mathrm{j}3\pi/4}$, and $z_7 = 0\mathrm{e}^{\mathrm{j}\pi/2}$.
- Given $z_1 = 1 - \mathrm{j}$ and $z_2 = 2\mathrm{e}^{\mathrm{j}\pi/4}$, carry out the following operations: $z_1 - z_2$, $z_1 z_2$, z_1/z_2, z_1^4, $\sqrt{z_2}$, and $\ln z_2$.

1.11.2 Relation between harmonic functions and phasors

Making use of Euler's identity (1.162) we can recognize that the time-harmonic function

$$f(t) = A\cos(\omega t + \varphi)$$

can also be expressed as

$$f(t) = \mathrm{Re}\left(A\mathrm{e}^{\mathrm{j}(\omega t+\varphi)}\right) = \mathrm{Re}\left(A\mathrm{e}^{\mathrm{j}\varphi}\mathrm{e}^{\mathrm{j}\omega t}\right) \ . \tag{1.164}$$

If we now define the **phasor** $\tilde{f}$ associated with function $f(t)$ as

$$\tilde{f} = A\mathrm{e}^{\mathrm{j}\varphi} \tag{1.165}$$

it is obtained that

$$f(t) = \mathrm{Re}\left(\tilde{f}\mathrm{e}^{\mathrm{j}\omega t}\right) \ . \tag{1.166}$$

The identity (1.166) allows us to establish a univocal correspondence between the time-harmonic functions and its associated phasors. Thus, each time-harmonic function can be associated with a phasor:

$$f(t) \leftrightarrow \tilde{f} \qquad [\,A\cos(\omega t + \varphi) \leftrightarrow A\mathrm{e}^{\mathrm{j}\varphi}\,] \ . \tag{1.167}$$

Starting from the basic properties of the algebra of complex numbers, if

$$A_1\cos(\omega t + \varphi_1) \leftrightarrow A_1\mathrm{e}^{\mathrm{j}\varphi_1}$$
$$A_2\cos(\omega t + \varphi_2) \leftrightarrow A_2\mathrm{e}^{\mathrm{j}\varphi_2}$$

it is found that

$$\alpha A_1\cos(\omega t + \varphi_1) + \beta A_2\cos(\omega t + \varphi_2) \leftrightarrow \alpha A_1\mathrm{e}^{\mathrm{j}\varphi_1} + \beta A_2\mathrm{e}^{\mathrm{j}\varphi_2} \tag{1.168}$$

with α and β being real numbers.
An additional property of key practical importance is

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leftrightarrow \mathrm{j}\omega\tilde{f} \ . \tag{1.169}$$

This property can be deduced as follows:

$$\begin{aligned}\frac{df(t)}{dt} =& \frac{d}{dt}A\cos(\omega t+\varphi) = -\omega A\sin(\omega t+\varphi) = -\omega A\cos(\omega t+\varphi-\pi/2)\\ =&\mathrm{Re}\left(-\omega A e^{j(\omega t+\varphi-\pi/2)}\right) = \mathrm{Re}\left(-\omega A e^{j\varphi}e^{-j\pi/2}e^{j\omega t}\right)\\ =&\mathrm{Re}\left(j\omega A e^{j\varphi}e^{j\omega t}\right) = \mathrm{Re}\left(j\omega\tilde{f}e^{j\omega t}\right)\ ,\end{aligned} \tag{1.170}$$

from which it is deduced that the phasor associated with df/dt is $j\omega\tilde{f}$.

Activity 1.17:

- Find the phasors associated with the following harmonic functions (in binomic form): $f_1(t) = -5\cos(\pi t)$, $f_2(t) = -5\cos(2\pi t)$, $f_3(t) = -5\cos(2\pi t+\pi/6)$, $f_4(t) = -5\sin(\pi t)$, $f_5(t) = -5\sin(2\pi t)$, and $f_6(t) = -5\sin(2\pi t+\pi/6)$.
- Write the harmonic functions of period $T = 1/50$ s associated with the following phasors: $z_1 = e^{-j\pi}$, $z_2 = e^{j\pi/4}$, $z_3 = 7e^{-j\pi/2}$, $z_4 = 2e^{j\pi/3}$, $z_5 = 3e^{-j3\pi/4}$, $z_6 = 3e^{j3\pi/4}$, and $z_7 = 0e^{j\pi/2}$.
- Given the phasors $\tilde{f}_1 = 1 - j$ and $\tilde{f}_2 = 2e^{j\pi/4}$ associated with harmonic functions $f_1(t)$ and $f_2(t)$ of frequency $f = 50$ Hz, find the harmonic functions resulting of the following operations: $f_1(t) - f_2(t)$, $3f_1(t) + 2f_2(t)$ and $3f_1'(t) + 2f_2(t)$.
- For the case described in the above item, can you find the result of $f_1(t)f_2(t)$ making use of their associated phasors.

1.12 Problems

1. Write the expressions for
   (a) the length and area of a circumference of radius R,
   (b) the volume of a sphere of radius R,
   (c) the area and volume of a cylinder of radius R and height h,
   (d) the area and volume of a right square pyramide of side length a and height h.
2. Express $\cos(2\alpha)$, $\sin(2\alpha)$, $\sin^2(\alpha)$, and $\cos^2(\alpha)$ in terms of $\cos(\alpha)$ and $\sin(\alpha)$.
3. Two planes are given by the following equations:
   $$\pi \equiv 2x + 2y - z - 2 = 0 \qquad \pi' \equiv x - 2y + 2z + 4 = 0$$
   (a) Find the equation of the line that they define.
   (b) Find all the points that are equidistant to both planes π and π'.
4. Given the points $A(1, -1, 2)$, $B(2, 0, -1)$, $C(0, 1, 3)$,
   (a) Find all the points that are equidistant to A, B and C.
   (b) Which of them belong to the plane $2x + 2y + 2z + 1 = 0$?
   (c) Find the equation of the plane that contains A, B and C.

5. For the following system of linear equations:

$$\begin{aligned} x + y + 2z &= 2 \\ -3x + 2y + 3z &= -2 \\ 2x + my - 5z &= -4 \end{aligned}$$

(a) Discuss the properties of the system according to the values of m.

(b) Solve it for $m = 1$.

6. (a) Find the value of λ that makes the following function be continuous:

$$f(x) = \begin{cases} \dfrac{e^{\lambda x^2} - 1}{3x^2} & \text{si} \quad x > 0 \\[2ex] \dfrac{\sin(2x)}{x} & \text{si} \quad x \le 0\,. \end{cases}$$

(b) Plot the function in the range $-10 < x < 10$.

7. For the polynomial $P(x) = x^3 + ax^2 + bx + c$, find the values of a, b and c which verify the following conditions:

(a) $P(x)$ has relative extremes at points of abscissas $x = -1/3$, $x = -1$.

(b) The line that is tangent to $P(x)$ at point $(0, P(0))$ is given by $y = x + 3$.

8. Write the derivatives of the following functions:

(a) $(3t^2 - 1)\cos(4t)$

(b) $\dfrac{e^{z^2}}{z^6 - \cos^2 z}$

(c) $5x^{3x\ln(x)}$

9. Assuming that $f(x)$ is the derivative of function $F(x)$, and $f(x)$ is continuous in the closed interval $[2, 5]$ with $F(2) = 1$, $F(3) = 2$, $F(4) = 6$, $F(5) = 3$, $f(3) = 3$ and $f(4) = -1$; find:

(a) $\int_2^5 f(x)\,dx$

(b) $\int_2^3 [5f(x) - 7]dx$

(c) $\int_2^4 F(x)f(x)\,dx$

10. Obtain the primitives of the following integrands:

(a) $\int \cos^2(\omega t)\,dt$

(b) $\int \sin^3(2y)\,dy$

(c) $\int \dfrac{dz}{z^4 - 1}$

(d) $\int x^2 \ln^2(x)\,dx$

(e) $\int u e^{-3u}\,du$

11. Reminding that $e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$, with $j = \sqrt{-1}$ being the imaginary unit, write in binomic and polar forms the result of the following operations with complex numbers:

(a) $e^{j\pi/4} - 7j^3$

(b) $(4 + j)(2 - 5j)$

(c) $\dfrac{(3 + 2j)^2}{4 - 5j}$

(d) $[e^{(2+j\pi/3)}]^4$

12. Write vector $(9, 8)$ as a linear combination of vectors $(3, 1)$ and $(1, 2)$ and make a plot of the result.
Sol.: (9, 8)=2(3, 1)+ 3(1, 2).

13. Find the unit vector along the direction given by the points of coordinates $(3, 2, 0)$ and $(6, 8, 2)$.
Sol.: $\hat{\mathbf{n}}$=(3/7, 6/7, 2/7).

14. Calculate the unit vector orthogonal to the plane determined by the points $(0, 0, 0)$, $(1, 2, 3)$ and $(3, 3, 1)$.
Sol.: $(-7, 8, -3)/\sqrt{122}$

15. Find the angle between vectors $(3, 6, 2)$ and $(8, 6, 0)$ by means of two different procedures (dot product and cross product).
Sol.: $\alpha = 31.003^o$.

16. Using the cross product, find the area of the triangle whose vertices are the points of coordinates $(1, 0, 0)$, $(4, 5, 2)$ and $(3, 1, 2)$.
Sol.: Area=$\sqrt{117}/2$.

17. Find the radial and tangent unit vectors ($\hat{\mathbf{r}}$ and $\hat{\mathbf{t}}$) at points (x, y) of a circle of radius R that is on the XY plane and has its center at the origin. Repeat the above operation, assuming now that the circle has its center at point $(3, 2)$.
Sol.: center in $(0, 0)$: $\hat{\mathbf{r}} = (x/R, y/R)$, $\hat{\mathbf{t}} = (-y/R, x/R)$;
center in $(3, 2)$: $\hat{\mathbf{r}} = ((x-3)/R, (y-2)/R)$, $\hat{\mathbf{t}} = (-(y-2)/R, (x-3)/R)$.

18. Indicate which of the following statements are right or false:
a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$; b) $\vec{a} \cdot (\vec{b} \times \vec{c})$; c) $\vec{a}(\vec{b} \cdot \vec{c})$; d) $(\vec{a} \cdot \vec{b}) \times \vec{c}$.
Sol.: right: b), c); false: a) y d) .

19. Making use of $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$, prove that

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}}\,.$$

20. Find the component of vector $(7, 5, 2)$ along the direction given by the line that passes through the points $(5, 4, 3)$ and $(2, 1, 2)$.
Sol.: $(6, 6, 2)$.

21. Decompose the vector $\vec{A} = (1, 5, 5)$ into its parallel and perpendicular components to the direction given by the unit vector $\hat{\mathbf{n}}$=(0, 3/5, 4/5).
Sol.: $\vec{A} = \vec{A}_{||} + \vec{A}_{\perp}$, with $\vec{A}_{||}$=(0, 3/5, 4/5) and $\vec{A}_{\perp}$=(1, 21/5, 28/5).

22. The coordinates of a moving particle of mass $m = 2$ kg as a function of time are $\vec{r}(t) = (3t, t^2, t^3)$ m (t in seconds). Find a) the velocity and acceleration of the particle; b) the force that acts on the particle at time $t = 1$ s, and also the components of such force along the perpendicular and tangent directions to the trajectory.
Sol.: a) $\vec{v}(t) = (3, 2t, 3t^2)$ m/s, $\vec{a}(t) = (0, 2, 6t)$ m/s^2; b) $\vec{F} = (0, 4, 12)$ N; $\vec{F}_{\perp} = (-6, 0, 6)$ N, and $\vec{F}_{||} = (6, 4, 6)$ N.

23. Calculate the gradient of function $\phi(x, y, z) = 2xy/r$, with $r = \sqrt{x^2 + y^2 + z^2}$.
Sol. $\vec{\nabla}\phi = r^{-3}\left[2y(r^2 - x^2)\hat{\mathbf{x}} + 2x(r^2 - y^2)\hat{\mathbf{y}} - 2xyz\hat{\mathbf{z}}\right]$

Part II

ELECTROMAGNETISM

Lesson 2

Electrostatics

2.1 Introduction

One of the main objectives of this course is the basic study of the most fundamental electromagnetic phenomena, and since many of these phenomena are related to the interaction between electric charges, this first lesson will focus on the study of the interactions of **electric charges at rest** (stationary electric charges). The part of Electromagnetism dealing with this subject is known as *Electrostatics*.

The electric charge is a fundamental and intrinsic property of the matter (likewise *mass* is another of its intrinsic properties). The charge has the following relevant properties:

- It has **two polarities**: positive and negative. A piece of matter with the same amount of positive and negative charges will have zero total charge.
- The total quantity of charge in the universe (algebraic sum of all existing charges) is **constant**; that is, the charge cannot be created or destroyed. However, it does not preclude that positive charges counterbalance negative ones.
- In addition to the above global conservation property, the charge also has to be conserved *locally*. It means that if a certain charged matter disappears in one place and the same charged matter appears in another place, this is because this matter has "traveled" from one region of the space to another.
- Charge is **quantized**: any existing charge is actually an integer multiple of an elementary charge q_e. This elementary charge corresponds to the proton charge.

The charge unit in the International System of Units (SI) is the **coulomb** (C) and it corresponds to 6.2414959×10^{18} protons or, equivalently, the proton charge is $q_e = 1.60218 \times 10^{-19}$ C.

unit of electric charge: coulomb
1 C

It is interesting to note that out of the four fundamental interactions in Nature (strong nuclear, electromagnetic, weak nuclear, and gravitational), the electromagnetic interaction (more especifically, the electrostatic interaction when charges are at rest) is the second strongest. Indeed, the electric interaction between two electrons (charge *e* equal to $-q_e$) is about 10^{42} times stronger than the corresponding gravitational one. It can give us an idea of the crucial role played by electric forces in Nature. However, it should also be noted there are many situations where no electrical interaction occurs globally because of the precise *compensation* that happens in matter between its positive and negative constitutive charges. In fact, the matter aggregates generally appear with null total charge and, therefore, the interactions among large amounts of matter (planets, stars, etc) is mainly due to gravitational forces. However, this fact does not mean that interactions between electric charges are irrelevant. Rather, these interactions are behind many fundamental phenomena such as the formation and stability of the atoms, molecular forces, frictional forces, mechanical tensions and contact forces, etc...

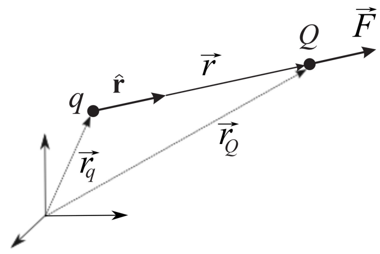
Activity 2.1:

- Is the geometric shape another intrinsic property of matter?
- Explain the difference between the "principle of conservation of the charge" and the "principle of local conservation of the charge."
- Describe some consequences of the "quantization" of the electric charge.
- Describe some simple and practical examples where the electrostatic interaction is the main force involved. Also give other practical examples where this interaction is absent.

2.2 Electric field of charge distribution

2.2.1 Coulomb's law

Our study of Electrostatic will start with Coulomb's law, an *experimental* law that describes the interaction between two **point charges at rest** in vacuum (the charges are assumed to be placed in free space). The concept of "point charge" is an idealization in which a certain amount of charge is assumed to be located at a single point in space. Although, this idealization might look like unrealistic, the experience shows that it is a very accurate assumption that works well in multiple practical situations. In fact, the charge evenly distributed on spherical bodies as well as charged bodies considered from far distances behave very approximately as point charges.



Coulomb's law ($\sim$1785) establishes that the force, $\vec{F}$, exerted by a so-called *source* charge q on a *test* charge Q, is given by

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{|\vec{r}|^2}\,\hat{\mathbf{r}} \equiv \frac{1}{4\pi\epsilon_0}\frac{qQ}{|\vec{r}|^3}\,\vec{r}} \qquad (2.1)$$

where ϵ_0 is a constant called **permittivity of vacuum** whose value in the International System of units (SI) is

$$\epsilon_0 = 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{Nm}^2} \qquad \left[\frac{1}{4\pi\epsilon_0} = 9 \times 10^9\,\frac{\mathrm{Nm}^2}{\mathrm{C}^2}\right]. \qquad (2.2)$$

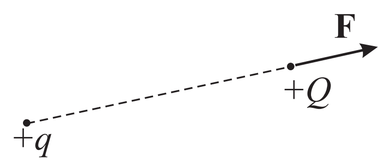
and

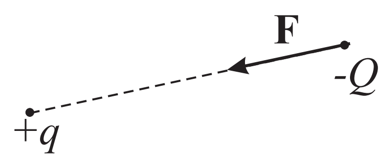
$$\vec{r} = |\vec{r}|\,\hat{\mathbf{r}} \qquad (r \equiv |\vec{r}|)$$

represents the vector from the position of the source charge, $\vec{r}_q$, to the position of the test charge, $\vec{r}_Q$; namely, $\vec{r} = \vec{r}_Q - \vec{r}_q$, with $r \equiv |\vec{r}|$ being the magnitude of this vector and $\hat{\mathbf{r}} = \vec{r}/r$ its associated unit vector. It is very important to realize that, in general, the position of the source charge will not coincide with the origin of coordinates.

[*It is highly recommendable to go over Sec. 1.8 for a brief survey of vectors. Also it should be noticed that in the figures of the present notes we will use an arrow to denote a vector ($\vec{u}$). Furthermore,* ***unit vectors*** *will be denoted in boldface type with a "hat" symbol above. This way,* $\hat{\mathbf{u}}$ *reads as "unit vector in the direction and sense of $\vec{u}$." Likewise, the* ***magnitude*** *of vector $\vec{u}$ will be denoted indistinctly either as $|\vec{u}|$ or u.*]

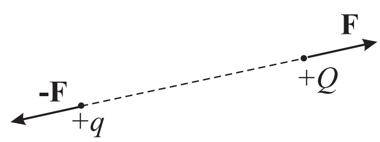
Some relevant features of Coulomb's law are [see Eq. (2.1)]:



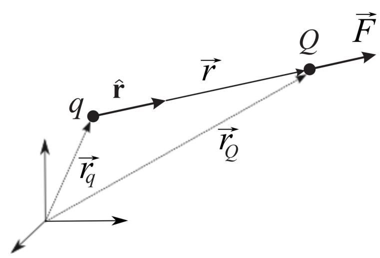


- The force between two charges is always directed along the line that joins these two charges (central force), and its direction comes determined by the sign of the product qQ. Therefore, the force between two charges will be *attractive* for charges of opposite sign and *repulsive* for charges with the same sign.

- The magnitude of the force decreases as the inverse of the square of the distance between charges. However, at short distances, this interaction grows intensely.



- The force exerted by the test charge on the source charge is just $-\vec{F}$ (satisfying the third Newton's law of action and reaction).



EXAMPLE 2.1 *Find the force that a charge of* $3\,\mu C$ *located at point* $(1, -2, 3)$ *exerts on a charge of* $-4\,\mu C$ *located at* $(2, 4, -1)$ *(distances in cm).*

According to the problem we have the following position vectors $\vec{r}_q = \hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$ and $\vec{r}_Q = 2\hat{\mathbf{x}} + 4\hat{\mathbf{y}} - \hat{\mathbf{z}}$, and then

$$\vec{r} = \vec{r}_Q - \vec{r}_q = \{2 - 1\}\hat{\mathbf{x}} + \{4 - (-2)\}\hat{\mathbf{y}} + \{(-1) - 3\}\hat{\mathbf{z}} = \hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 4\hat{\mathbf{z}}\ \mathrm{cm}$$

with its magnitude being

$$|\vec{r}| = \sqrt{1^2 + 6^2 + (-4)^2}\ \text{cm} = \sqrt{53}\times 10^{-2}\ \text{m}$$

and its associated unit vector

$$\hat{\mathbf{r}} = \frac{\hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 4\hat{\mathbf{z}}}{\sqrt{53}}\ .$$

The force, following (2.1), is given by

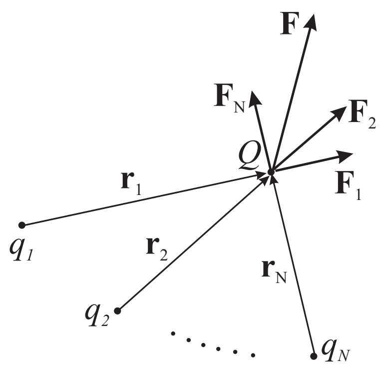
$$\begin{aligned}\vec{F} &= 9\times 10^9 \times \frac{(3\times 10^{-6})\times(-4\times 10^{-6})}{(\sqrt{53}\times 10^{-2})^2}\,\frac{\hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 4\hat{\mathbf{z}}}{\sqrt{53}} \\ &= \frac{-108\times 10^{-3}}{53\times 10^{-4}}\,\frac{\hat{\mathbf{x}} + 6\hat{\mathbf{y}} - 4\hat{\mathbf{z}}}{\sqrt{53}} = \frac{1080}{53}\,\frac{-\hat{\mathbf{x}} - 6\hat{\mathbf{y}} + 4\hat{\mathbf{z}}}{\sqrt{53}}\ \text{N}\ .\end{aligned}$$

We should note that the magnitude of the force, $|\vec{F}|$, is just $1080/53$ N.

Activity 2.2:

- Where does Coulomb's law come out from?
- Explain in words the meaning of $\vec{r}$ in Eq. (2.1). Why do we have two expressions for the force in (2.1)?
- What is the direction of the electric force between two point charges arbitrarily located?
- Plot the magnitude of the force between two point charges as a function of the distance between these charges. Draw some relevant conclusions from this plot.

2.2.2 Superposition Principle



Coulomb's law only describes the effect of a *single* source point charge, q, on a given test point charge, Q. In order to know the effect that a *set of charges* has on the test charge we have to resort to the so-called **superposition principle**. This principle states that

The interaction between any two point charges is completely independent of the presence of other charges.

It means that the computation of the effect of a set of point charges on certain test charge can be carried out by first calculating the partial effect of each single source charge on the test charge and then to obtain the total effect as the vector sum of the partial effects; namely, $\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$.

Thus, the force produced by the set of source charges, $\{q_1, q_2, \cdots, q_N\}$, on the test charge Q located at point P can be computed as

$$\vec{F}(P) = \sum_{i=1}^{N} \vec{F}_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i Q}{|\vec{r}_i|^2} \hat{\mathbf{r}}_i \tag{2.3}$$

$$= \frac{Q}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r}_i|^2} \hat{\mathbf{r}}_i \,. \tag{2.4}$$

2.2.3 Electric field of point charges

In the expression of the force given by (2.4) it can be seen that the summation only depends on *the configuration of the set of source charges*, so we can write

$$\vec{F}(P) = Q\vec{E}(P) \tag{2.5}$$

where vector $\vec{E}(P)$ is the so-called **electric field** produced by the set of source charges at point P. This electric field can be written as

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r}_i|^2} \hat{\mathbf{r}}_i \equiv \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\vec{r}_i|^3} \vec{r}_i \,. \tag{2.6}$$

Electric field of a distribution of point charges

The introduction of this vector $\vec{E}$ makes it possible to define a vector quantity varying from point to point and that only depends on the source charges. Thereby, each point P in space is endowed with a vector property such that the value of a test charge located at P times the value of the vector at that point gives the vector force exerted by the set of source charges on such test charge. The electric field $\vec{E}$ may then be defined as the force per unit charge, and its units are consequently N/C. It is worth noting that the electric field somehow "collects" the information of the source charges, "hiding" the specific geometric configuration and values of this distribution, while standing out only its global effect.

Unit of electric field: 1 N/C

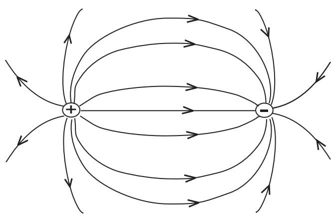
The way the electric field has been introduced might make us think that this field is only a mathematical entity only useful to calculate the force but without any particular physical meaning. However, as it will be made apparent later, $\vec{E}$ itself has a clear physical meaning and, therefore, from now on it is convenient to consider the electric field as an actual "physical" entity regardless of the presence or absence of any test charge (namely, the electric field has the same degree of physical reality as forces or momenta).

From (2.6) it is easily inferred that the electric field produced by a single point charge, $\vec{E}_q$, at point P ($\overline{OP} \equiv \vec{r}$) will be given by

Electric field produced by a single point charge

$$\boxed{\vec{E}_q(P) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r}|^2} \hat{\mathbf{r}}} \,. \tag{2.7}$$

A possible way to visualize this field is to plot the vector field $\vec{E}$ in certain points in space. However, it is more convenient to describe the field using the so-called **field lines**, which are those lines tangents in each point to the

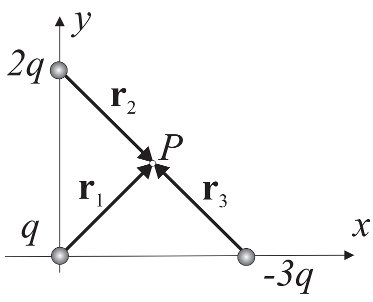


vector field at this point. For a system of two identical charges in magnitude, one positive and one negative, the field lines depart from the positive charge and end in the negative charge, according to the pattern shown in the figure. This particular feature is a general property of the electrostatic field; that is, the field lines always *depart* from positive charges and always *end* on negative charges (or go to infinity). Since electric charges are the only *sources* of the electrostatic field, we can say that there will be an electrostatic field whenever there are unbalanced electrical charges (that is, when they do not counterbalance each other at each point).

Activity 2.3:

- Where does the superposition principle come out from?
- Explain how the superposition principle can help us to compute the force that three charges exert on another fourth charge.
- What are the advantages of ascribing a vector quantity denoted as electric field to any point of space?
- The magnitude of the electric field always decreases as the inverse of the distance squared. Right of False? Justify your answer.
- Draw an approximate plot of the field lines of a configuration of three charges, two of them positives and the third negative.

EXAMPLE 2.2 *Find the electric field at point $P \equiv (1/2, 1/2)$ due to the global effect of a charge q located at $(0, 0)$, another charge $2q$ at $(0, 1)$ and a third charge $-3q$ located at point $(0, 1)$ (distances in m and $q = 12$ C). Also find the value of the magnitude of the electric field.*



We should apply the superposition principle to compute the electric field. To do this, first we have to obtain the field produced by each charge; although before this computation we should identify the vector going from each charge to the observation point P. According to the figure, we find that

$$\vec{r}_1 = \frac{1}{2}\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{y}} \quad , \quad \vec{r}_2 = \frac{1}{2}\hat{\mathbf{x}} - \frac{1}{2}\hat{\mathbf{y}} \quad , \quad \vec{r}_3 = -\frac{1}{2}\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{y}} \, ,$$

whose magnitude is the same and is given by

$$|\vec{r}_i| \equiv D = \sqrt{1/2} \, .$$

The electric field at P is then

$$\vec{E}(P) = \sum_{i=1}^{3} \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\vec{r}_i|^3} \vec{r}_i \, ,$$

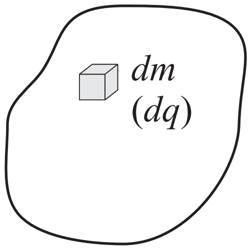
which, after substituting the value of $\vec{r}_i$ obtained above, can be written as

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\frac{q}{D^3}\left[(\frac{1}{2}\hat{\mathbf{x}}+\frac{1}{2}\hat{\mathbf{y}})+2(\frac{1}{2}\hat{\mathbf{x}}-\frac{1}{2}\hat{\mathbf{y}})-3(-\frac{1}{2}\hat{\mathbf{x}}+\frac{1}{2}\hat{\mathbf{y}})\right]\\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{D^3}(3\hat{\mathbf{x}}-2\hat{\mathbf{y}}) = \frac{2\sqrt{2}q}{4\pi\epsilon_0}(3\hat{\mathbf{x}}-2\hat{\mathbf{y}})\\ &= 9\times10^9\times2\sqrt{2}\times12(3\hat{\mathbf{x}}-2\hat{\mathbf{y}}) = 216\sqrt{2}\times10^9(3\hat{\mathbf{x}}-2\hat{\mathbf{y}})\\ &= 216\sqrt{26}\times10^9\,\frac{3\hat{\mathbf{x}}-2\hat{\mathbf{y}}}{\sqrt{13}}\,\frac{\mathrm{N}}{\mathrm{C}}\,.\end{aligned}$$

The magnitude of this field is finally $216\sqrt{26}\times10^9$ N/C.

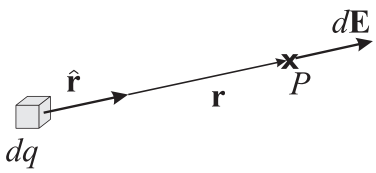
2.2.4 Electric field of a continuous distribution of charge

Despite the discrete nature of matter (matter is well known to be made of atoms), in many practical situations this discrete characteristic can be "overlooked" and instead consider that it can be described as a *continuum*. From a mathematical point of view, this implies that matter will be characterized as a *superposition of infinitesimal elements*; in such a way that, for instance, to calculate the mass of an object we should write $m = \int \mathrm{d}m$ instead of $m = \sum_i^N m_i$ (which would correspond to a description of matter as an aggregate of individual particles). This consideration of "continuum" for the mass of matter can likewise be extended to its charge, and so the charge can similarly be regarded as a **continuous distribution** in many situations. In that case, the total charge q of a given charge distribution will be obtained as the integral of differential charges,



$$q = \int \mathrm{d}q\,. \tag{2.8}$$

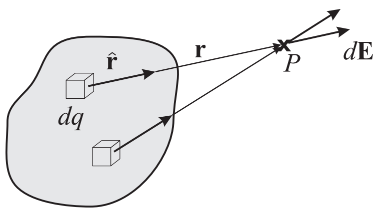
To compute the electric field caused by the previous charge distribution at the observation point P, we have to note that the differential contribution of each differential element of charge, $\mathrm{d}q$, to the electric field at P, $\mathrm{d}\vec{E}(P)$, corresponds to the electric field produced by a "point charge" of value $\mathrm{d}q$, whose expression is given by



$$\mathrm{d}\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}q}{|\vec{r}|^2}\,\hat{\mathbf{r}} \tag{2.9}$$

where vector $\vec{r}$ ($\equiv$ **r** in the figure) goes from the position of the differential charge $\mathrm{d}q$ until the observation point P.

The total field produced by the entire charge distribution can then be computed by using the superposition principle [as was already done for discrete charges in (2.6)] in order to sum all the infinitesimal contributions:



$$\begin{aligned}\vec{E}(P) &= \int \mathrm{d}\vec{E}(P)\\ &= \frac{1}{4\pi\epsilon_0}\int\frac{\mathrm{d}q}{|\vec{r}|^2}\,\hat{\mathbf{r}} \equiv \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{charge}\\ \text{region}}}\frac{\rho\,\mathrm{d}\mathcal{V}}{|\vec{r}|^2}\,\hat{\mathbf{r}}\,.\end{aligned} \tag{2.10}$$

In the above expression, $\rho(\cdot)$ is the volumetric charge density and $d\mathcal{V}$ the differential of volume, with the integral extended to all the volume occupied by the charge distribution.

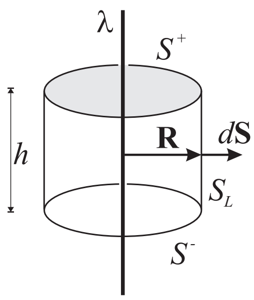
In practice, the computation of the field produced by a charge distribution requires the introduction of the concept of **charge density**, which relates the amount of charge in each differential element to the volume, surface, or length of that element. Only in very specific and simple cases it is possible to solve the integral (2.10) in closed form (i.e., to find its corresponding primitive). However, the integral can always be computed by means of numerical procedures. Fortunately, in many situations of practical interest, the geometry of the charge distribution is simple has a high degree of symmetry (for example, a cylinder, an sphere, or a plane) so that it will be possible to find a closed-form expression for the electric field by applying **Gauss's law**.

Gauss's law ($\sim$ 1867) states that the net electric flux (see Section 1.9.4) through any closed surface S is equal to $1/\epsilon_0$ times the net electric charge enclosed within that closed surface:

Gauss's law

$$\boxed{\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}}\,. \tag{2.11}$$

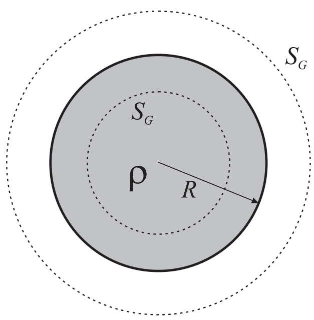
By applying this law we can find the expression of the electric field for the following cases of high symmetry:



- **Uniformly charged wire.**
If the charge of an infinite straight charged wire is uniformly distributed, then it will have a constant linear charge density (charge per unit length) λ. In that case, the electric field can be expressed as

$$\vec{E}(R) = \frac{\lambda}{2\pi\epsilon_0|\vec{R}|}\hat{\mathbf{R}} \tag{2.12}$$

where it should be noted that $\vec{R}$ is here the vector that goes perpendicularly from the straight wire to the observation point P (located at a distance $R = |\vec{R}|$ from the wire).



- **Uniformly charged sphere**
Consider a sphere of radius R with a uniform charge distribution (its volume charge density is ρ). The spherical symmetry of this charge distribution will also be reflected in the symmetries of its corresponding electric field, which allows us to write $\vec{E} = |\vec{E}(r)|\hat{\mathbf{r}}$ ($r \equiv |\vec{r}|$), and obtain finally that

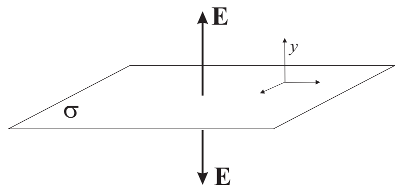
$$\vec{E} = \begin{cases} \dfrac{\rho}{3\epsilon_0}|\vec{r}|\hat{\mathbf{r}} & \text{if } |\vec{r}| < R \\[2ex] \dfrac{Q}{4\pi\epsilon_0|\vec{r}|^2}\hat{\mathbf{r}} & \text{if } |\vec{r}| \geq R\,. \end{cases} \tag{2.13}$$

It is worth noting here that the electric field of a uniformly charged sphere (total charge $Q = \rho \times 4/3\pi R^3$) at points outside the sphere has the same expression as the electric field of a point charge Q located at the center of that sphere.

- **Uniformly charged plane**
  An infinite charged plane with a constant charge density per unit surface, σ, will give rise to an electric field with the following dependence $\vec{E} = |\vec{E}(y)|\,\hat{\mathbf{y}}$, or more specifically

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\,\mathrm{sign}(y)\,\hat{\mathbf{y}}\;. \tag{2.14}$$

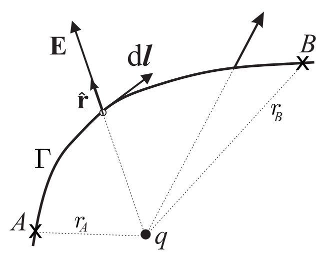
  It should be noted that the magnitude of the electric field does not depend on the position of the observation point; namely, it is constant in all the points of the space.



Activity 2.4:

- Explain the reasons that make the computation of the electric field created by a continuous distribution of charge more difficult than for the case of a set of discrete charges.
- Are there any situations in which the computation of the electric field produced by a continuous distribution of charge is simple? Justify your answer.
- For observation points far from a uniformly charged sphere, is there any difference between the field produced by such sphere and the field produced by a point charge with the same amount of charge located at the center of the sphere? Justify your answer.
- How would you compute the field created by a single point charge q placed at a distance d from a uniformly charged plane of density σ? Give a closed expression for the field at any point.

2.3 Electric potential

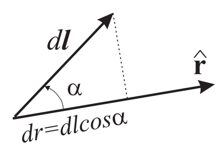


If we carry out the line integral (see Sec. 1.9.2) of the electric field created by a *point charge q* (here denoted as $\vec{E}_q$), along a curve Γ that connects two points A and B, it is found that

$$C_A^B = \int_{A,\Gamma}^{B} \vec{E}_q \cdot \mathrm{d}\vec{l} = \int_{A,\Gamma}^{B} \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}|^2}\,\hat{\mathbf{r}}\cdot\mathrm{d}\vec{l} = \frac{q}{4\pi\epsilon_0}\int_{A,\Gamma}^{B}\frac{\hat{\mathbf{r}}\cdot\mathrm{d}\vec{l}}{|\vec{r}|^2}\;. \tag{2.15}$$

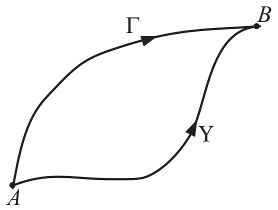
Taking into account the "projection" property of the dot product (see Sec. 1.8.3), the numerator of the last integrand above can be expressed as

$$\hat{\mathbf{r}}\cdot\mathrm{d}\vec{l} = \mathrm{d}l\cos\alpha = \mathrm{d}r$$



namely, dr is the projection of vector $\mathrm{d}\vec{l}$ onto the direction given by $\hat{\mathbf{r}}$. Therefore, (2.15) can be written as (for simplicity we write $r \equiv |\vec{r}|$)

$$C_A^B = \frac{q}{4\pi\epsilon_0}\int_{r_A}^{r_B}\frac{\mathrm{d}r}{r^2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right)\;. \tag{2.16}$$



It is interesting to note that the result in (2.16) tells us that the line integral is independent of the path (curve) chosen to go from point A to B; that is,

$$\int_{A,\Gamma}^{B} \vec{E}_q \cdot d\vec{l} = \int_{A,\Upsilon}^{B} \vec{E}_q \cdot d\vec{l} \,. \tag{2.17}$$

This fact implies that the line integral along any *closed* path (also denoted as *circulation*) of the electric field due to a point charge is null:

$$\oint_{\Gamma} \vec{E}_q \cdot d\vec{l} = 0 \,. \tag{2.18}$$

This result can easily be extended to the electric field created by any *discrete/continuous charge distribution* by applying the superposition principle. After doing that, we will find that the line integral of the electric field produced by an arbitrary charge distribution (this electric field is now denoted as $\vec{E}$, without the index q) will also have the above discussed properties (2.17) and (2.19). In particular, property (2.19) (the circulation of $\vec{E}$ is zero) allows us to state that

the electrostatic field is conservative.

This property, which is mathematically described as

$$\boxed{\oint_{\Gamma} \vec{E} \cdot d\vec{l} = 0} \tag{2.19}$$

implies that the line integral of an electrostatic field between points A and B along an arbitrary curve Γ can be written in general as

Definition of electric potential

$$\boxed{\int_{A}^{B} \vec{E} \cdot d\vec{l} = V(A) - V(B)} \tag{2.20}$$

where function $V(\cdot)$ is a scalar function called **electric potential**. The unit of electric potential are then the product of unit of electric field times the unit of length; that is, Nm/C in the SI. This unit of electric potential has its own name (and symbol): **volt** (V). Hence, the unit of the electric field is usually expressed as V/m.

Unit of electric potential: volt (V)
Unit of electric field: V/m

It is worth noting that the right-hand side term in (2.20) is a *difference* of electric potential. This electric potential difference is usally known as **voltage** or voltage drop, and it is this difference what actually has physical meaning rather than the electric potential at a given point.

Introducing the expression for the electric field of a single *point charge* into (2.20) and looking at (2.15), it can be concluded that the potential V_q produced by a point charge at point P, with $\vec{r}$ being the vector from the position of the charge to point P, is given by

Electric potential due to a point charge

$$\boxed{V_q(P) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}|}} \,. \tag{2.21}$$

If we consider a set of N point charges and apply the superposition principle, the above expression can be generalized to express the electric potential at point P as

$$V(P) = \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0} \frac{q_i}{|\vec{r}_i|} . \tag{2.22}$$

For a continuous charge distribution, following the same procedure as previously employed for the electric field, it will be found that

$$V(P) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathrm{d}q}{|\vec{r}|} = \frac{1}{4\pi\epsilon_0} \int_{\substack{\text{charge}\\ \text{region}}} \frac{\rho}{|\vec{r}|}\, \mathrm{d}\mathcal{V} . \tag{2.23}$$

Electric potential due to a charge distribution

Activity 2.5:

- Enumerate the relevant conclusions that can be drawn from the expression (2.16) concerning the line integral of the electrostatic field produced by a point charge.
- Try to understand identity (2.19) for the case of a uniform electric field and a round closed path. Find the mathematical reasons that makes the line integral be null in this situation.
- Is expression (2.21) the general definition of the electric potential? Justify your answer.
- Write the general expression for the electric potential when we know either the electrostatic field or the charge density.
- Sometimes we will say "electric potential at point P". However this statement is incorrect. Justify this fact.

EXAMPLE 2.3 *Electric potential due to a uniformly charged plane.*

After introducing expression (2.14) for the field produced by an infinite plane with charge density σ into (2.20), it is found that

$$V(y) - V(0) = -\int_0^y \frac{\sigma}{2\epsilon_0} \operatorname{sign}(y)\, \mathrm{d}y = \frac{\sigma}{2\epsilon_0} \operatorname{sign}(y)\, y = -\frac{\sigma}{2\epsilon_0}|y|$$

and then

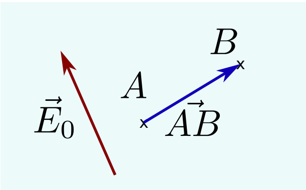
$$V(y) = V(0) - \frac{\sigma}{2\epsilon_0}|y|$$

where $V(0)$ is the value of the electric potential at the reference point $y = 0$.

At the light of the above expression it can be deduced that the locus of the points where the potential is constant (that is, the equipotential surfaces) is given by the following equation:

$$y = \mathrm{Cte}\ ,$$

namely, the equation for planes parallel to the charged plane. In general it is found that the equipotential surfaces are always *perpendicular* to the electric field vectors.



EXAMPLE 2.4 *Find the electric potential difference between two points A and B in a region where there is a uniform electric field $\vec{E}_0$.*

Making use of the definition of electric potential difference given in (2.20), it is found in the present case that

$$V(A) - V(B) = \int_A^B \vec{E}_0 \cdot d\vec{l} = \vec{E}_0 \cdot \int_A^B d\vec{l}$$

where the electric field has been move out from the integral because it does not vary along the integration path to go from A to B. Now it should be noted that the integral

$$\int_A^B d\vec{l} = \vec{AB}$$

with $\vec{AB}$ is a position vector going from A to B. It means that the potential difference can simply be simply as

$$V(A) - V(B) = \vec{E}_0 \cdot \vec{AB} \; .$$

2.3.1 Potential energy

Let $\vec{E}$ be the electrostatic field created by an arbitrary charge distribution. The work, W_E, done by this electrostatic field to move a test *point charge*, Q, from point A to point B is given by the following line integral (using the definition of work done by a force):

$$W_E = \int_{A,\Gamma}^B \vec{F} \cdot d\vec{l} = Q \int_{A,\Gamma}^B \vec{E} \cdot d\vec{l} \; . \tag{2.24}$$

From the results of the above section we already know that the last integral in (2.24) does not depend on the integration path and, therefore, the force is *conservative*. For *conservative* forces, it is well known that the work done by such forces can be expressed as minus the change of its associated potential energy, $\mathcal{E}_p$; namely,

$$W_E = -\Delta\mathcal{E}_p = -\left[\mathcal{E}_p(B) - \mathcal{E}_p(A)\right] = \mathcal{E}_p(A) - \mathcal{E}_p(B) \; . \tag{2.25}$$

Now if we use (2.20) to rewrite (2.24) in terms of the electric potential, it is obtained

$$W_E = Q \int_{A,\Gamma}^B \vec{E} \cdot d\vec{l} = Q[V(A) - V(B)] = QV(A) - QV(B) \tag{2.26}$$

which makes it apparent that the **potential energy** of a point charge Q at point P is given by

Potential energy of a point charge

$$\mathcal{E}_p(P) = QV(P) \; . \tag{2.27}$$

The above expression then allows us to identify

the electric potential as the potential energy per unit charge.

If we now remind the Work-Kinetic-energy theorem, which says that "the total work done on a particle is equal to the change in its kinetic energy" (namely, $W = \Delta\mathcal{E}_k$), and assume that the only force acting on the charged particle is the electrostatic force, we can write that

$$W_E = \Delta\mathcal{E}_k \,. \tag{2.28}$$

After equating the above equation with (2.25), it is found that

$$\Delta\mathcal{E}_k + \Delta\mathcal{E}_p = \Delta(\mathcal{E}_k + \mathcal{E}_p) = 0 \,. \tag{2.29}$$

If the mechanical energy of the particle, $\mathcal{E}_m$, is defined as

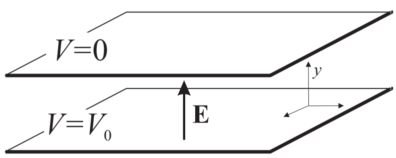
$$\mathcal{E}_m = \mathcal{E}_k + \mathcal{E}_p \,,$$

we can finally state that *the mechanical energy of a point charge is conserved under the action of the electrostatic field.* This statement is completely equivalent to have said that "the electrostatic field is conservative."

Activity 2.6:

- Can we say that the work done by *any* force to move a particle between two points is just minus the change in its potential energy? Justify your answer.
- Can we say that the work done by *any* force to move a particle between two points is just the change in its kinetic energy? Justify your answer.
- Is Eq. (2.27) a definition of the potential energy of a point charge in any electrostatic field? Can the above expression be applied to an arbitrary distribution of charges? Justify your answer.
- Is the mechanical energy always conserved for an arbitrary charge distribution in an electrostatic field? Justify your answer.
- Describe some reasons to explain why the conservation of the mechanical energy is so useful.

EXAMPLE 2.5 *Find the potential and kinetic energies of a point charge q within a parallel-plate system with a potential difference V_0 between its two conductor plates assuming that the electric field is uniform, $\vec{E} = E_0\hat{\mathbf{y}}$.*



Let V_0 be the potential difference that is established between the plates of a parallel-plate capacitor (located at $y = 0$ and $y = d$ respectively). This potential can be expressed as

$$\int_0^d \vec{E} \cdot d\vec{l} = V(0) - V(d) = V_0 - 0 = V_0 \,.$$

Taking into account that the electric field is uniform between the capacitor plates and that $d\vec{l} = dy\hat{\mathbf{y}}$, the above integral results in

$$V_0 \equiv \int_0^d \vec{E} \cdot d\vec{l} = \int_0^d E_0\hat{\mathbf{y}} \cdot dy\hat{\mathbf{y}} = \int_0^d E_0 dy = E_0 d \,,$$

which allows us to express the electric field within the capacitor as

$$\vec{E} = \frac{V_0}{d}\,\hat{\mathbf{y}} \,.$$

In order to compute the potential difference at an arbitrary point of coordinate y, we should consider that the potential at that point is given by

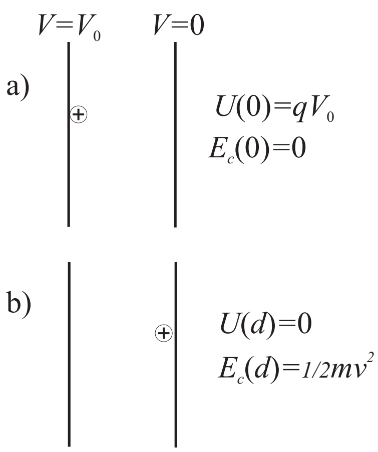
$$\int_0^y \vec{E} \cdot d\vec{l} = V(0) - V(y)$$

and taking into account that $V(0) = V_0$, it is finally obtained that

$$V(y) = V_0 - \int_0^y E(y)dy = V_0\left(1 - \frac{y}{d}\right) \,.$$

The potential energy, $\mathcal{E}_p(y)$, of a point charge q within the capacitor can then be written as

$$\mathcal{E}_p(y) = qV_0\left(1 - \frac{y}{d}\right) \,.$$



A point particle of positive charge that departs from rest ($\mathcal{E}_k = 0$) in the plate conductor at potential V_0 will necessarily move to regions of decreasing potential energy and increasing kinetic energy. According to the conservation of the mechanical energy, the kinetic energy when reaching the opposite capacitor plate [see (2.29)] will be

$$\mathcal{E}_k(d) = \frac{1}{2}mv^2 = qV_0 \,,$$

and therefore, the particle will have the following speed at that position:

$$v = \sqrt{\frac{2qV_0}{m}} \,. \tag{2.30}$$

The phenomenon of the increase of the kinetic energy of a charged particle between the two plates of a capacitor with certain potential difference is often employed to speed up charged particles (for instance, in cathodic ray tubes and particle accelerators). In practice, one of the plates can be substituted by a metallic grid that allows the particle to go through it.

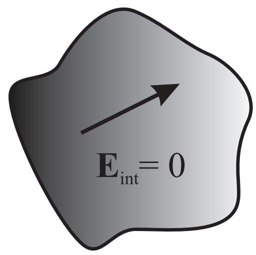
2.4 Conductors in electrostatic equilibrium

It is well known that matter is made of charged and neutral elementary particles. The positively charged particles (protons) are part of the nuclei of atoms and therefore are *fixed* (on average) in solids. In certain materials called **dielectrics**, negative elementary charges called electrons can also be considered *fixed*. However, in other materials called **conductors**, some of the electrons (the so-called *valence* electrons) are not bound to particular atoms but form a sort of "electron gas/cloud", so that the electrons of that "cloud" roam within the boundaries of the material object). In this section we will consider an *ideal model* for conductors in which the cloud electrons are charges that can move freely ("free charges") inside the conductors. This model is called **perfect conductor**.

2.4.1 Electric field of a charged conductor in electrostatic equilibrium

In general, conductors in Nature appears as neutral systems (equal number of positive and negative charges). However, the addition or removal of free charges to the conductor will make it be *charged*. **Electrostatic equilibrium** is defined as that situation in which all the free charges are at *rest*. Taking into account the aforementioned definition of perfect conductor, we can infer that the electric field produced by such charged conductors will have the following characteristics:

- **The electric field inside the conductor is null.**
  Was the electric field inside the conductor different from zero, its free charges would be moving bacause of that non-zero electric field. This situation would then be in contradiction with the assumed condition of electrostatic equilibrium, and therefore $\vec{E}_{\rm int} = 0$.
  Given that the electric field is null inside the conductor in equilibrium, the computation of the line integral of the electric field between two points A y B inside the conductor leads to

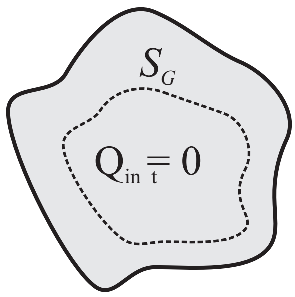


$$\int_A^B \vec{E}_{\rm int} \cdot d\vec{l} = V(A) - V(B) = 0 \;\Rightarrow\; V \equiv \text{Cte} \,. \tag{2.31}$$

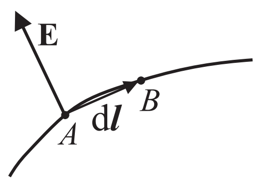
  It implies that conductors are **equipotential** regions and, specifically, the surface of that conductor is an *equipotential surface*.

Conductors are equipotential

- **The excess charge is placed on the surface of the conducting body.**
  If the field in *all* inner points of the *charged* conductor is zero, there cannot exist net charge inside the conductor.[1]



- **The electric field at the points of the surface is normal to the surface and its magnitude is σ/ϵ_0.**
  As the electric potential is the same in all the conductor region, given two arbitrary points A and B located on the conductor surface, it will be found



[1] This statement can be justified using Gauss's law. If there was net charge inside the conducting body, choosing a Gaussian surface that enclosed that interior region of the body, the flux of the electric field through this surface would be proportional to the enclosed net charge. But it would contradict the fact that the flux must be zero because the field inside the conductor is zero. Therefore, excess charge must be necessarily located at the surface.

that $V(A) - V(B) = 0$. If these points are infinitesimally close, we can write that

$$V(A) - V(B) = \mathrm{d}V = \vec{E} \cdot \mathrm{d}\vec{l}$$

where $\mathrm{d}\vec{l}$ is the infinitesimal vector that joins point A with B. Since $V(A) - V(B) = 0$, which implies that $\vec{E} \cdot \mathrm{d}\vec{l} = 0$, we find that the field at the conductor surface, $\vec{E}_S$, must be perpendicular to $\mathrm{d}\vec{l}$. As vector $\mathrm{d}\vec{l}$ is tangent to the conductor surface, it can be concluded that $\vec{E}_S = E\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ being the unit vector normal to the conductor surface at any point of this surface. If the surface charge density on the surface is σ, the application of Gauss's law would finally lead to the following expression for the electric field at the surface:

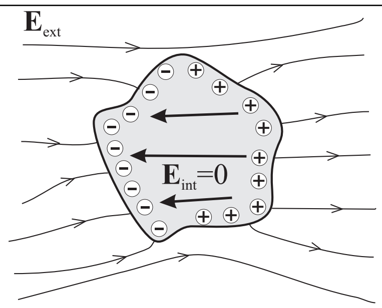
$$\vec{E}_S = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{n}}\,. \tag{2.32}$$

Activity 2.7:

- All the electrons in a perfect conductor are free to roam within the boundaries of the conductor. Right or False? Justify your answer.
- In dielectrics, the electron cloud is formed by the addition of a few electrons contributed by each atom. Right or False? Justify your answer.
- Under what conditions the electric field inside a conductor is zero?
- If a conductor is charged, the electric field inside can be different from zero. Right or False? Justify your answer.
- In a neutral conductor, the electric field at the points of its surface is normal to this surface. Right or False? Justify your answer.
- Find the value of the electric field and the electric potential in all points of the space created by a conductor sphere of radius R charged with a net charge Q.
- Find the values of the electric field and the electric potential in the points of the space created by a point charge q located at the origin of coordinates and a conductor sphere of radius R charged with a net charge Q and whose center is located at $(0, 2R, 0)$.

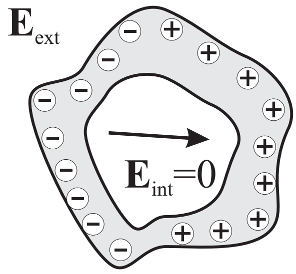
2.4.2 Neutral conductor under the influence of an external electric field

If a neutral conductor (equal amount of positive and negative charges) is placed in a region where there is an external electric field, the free-moving charges of the conductor will redistribute until the condition of electrostatic equilibrium is reached: $\vec{E}_{\rm int} = 0$. This trend to equilibrium is found in all natural processes. In particular, this process typically takes a time of the



order of 10^{-14} s in a copper conductor. The charge redistribution causes the appearance of an inhomogeneous surface charge density, which in turn gives place to an electric field inside the conductor that counteracts the external field. These two fields add up at each inner point of the conductor to make that the total electric inside the conductor fully canceled out.

Interestingly, the process resulting from the charge redistribution which leads to the electrostatic equilibrium can be regarded as if it only took place on the conductor surface, without any change in the interior of that conductor. Moreover, if part of the inner conductive material was removed to create an inner hollow, the same charge redistribution would be found at the *outer* conductor surface and, therefore, the field inside the conductor would keep on being null, also within the hollow region. This means that any interior hollow inside a conductor is electrically *isolated* from the outside and, consequently, external fields will not have any effect on any device sensitive to the electric field (for example, electronic circuits) if that device is placed inside the conductor. This phenomenon is used to design **Faraday cages** to isolate electrical systems. A simple metal casing (or a conductive plastic) isolates, for instance, electronic systems inside a computer from possible external electrical interferences.

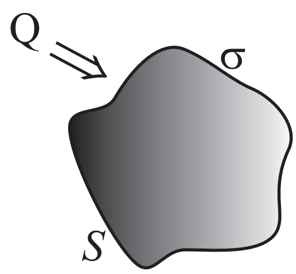


Activity 2.8:

- Dielectric material can also be employed to isolate electrically sensitive devices from external electric interferences. Right or False? Justify your answer.
- Give reasons to explain the *inhomogeneity* of the charge distribution on the surface of a conductor located in a region with an external electric field.
- If there are two inner hollows inside a conductor, will both hollows be electrically isolated from possible external electrical interferences? Justify your answer.

2.5 Capacitors

2.5.1 Capacitance of a conductor



If certain charge Q is added to a conductor initially uncharged, this charge is redistributed across the conductor surface and creates an electric field in the region outside the conductor; as well as a corresponding electric potential. In this situation we find that the ratio between the conductor net charge (Q) and the electric potential (V) at the surface of that conductor is constant and *only depends on the conductor geometry* ($V = 0$ is taken at infinity). This ratio is known as the **capacitance**, C, of the conductor and is given by

Capacitance of a conductor

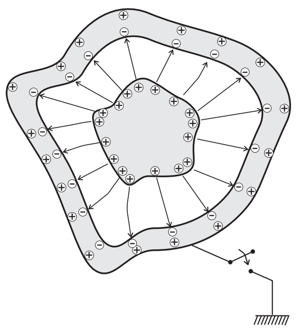
$$C = \frac{Q}{V} \, . \tag{2.33}$$

The capacitance of the conductor thus determines the charge that this conductor can "hold" for a given potential or, equivalently, the electric potential "acquired" by the conductor for a given charge. Note again that the capacitance C is a *purely geometrical parameter* and, therefore, only depends on the conductor size and shape.

Unit of capacitance: 1 farad (F)

The unity of capacitance is the **farad** (F), defined in SI as $1\,\mathrm{F} = 1\,\mathrm{C/V}$.

2.5.2 Influence between conductors



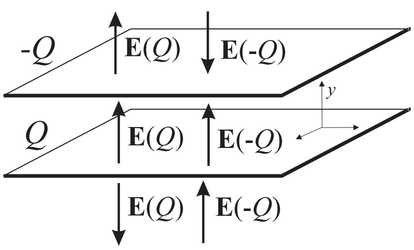
If a conductor charged with a net charge Q (assumed to be positive) is introduced inside another conductor initially uncharged, the charge of the initially neutral conductor is redistributed (see attached figure). This redistribution is a consequence of the electrostatic equilibrium conditions that appear in both conductors ($\vec{E}_{\text{int}} = 0$). If the outer surface of the neutral conductor is connected to ground (which can be regarded as a "charge store" with an infinite number of them), as many electrons from the ground as needed will move up to compensate the positive charges, with this global process leading to the appearance of a net charge $-Q$ in the conductor.

The two conductors of the above situation are in *total influence* since all the field lines emanating from the positive charges of one conductor end on the negative charges of the other conductor. In this situtation, the net amount of charge in one conductor is just the opposite to the one in the other. These two conductors under total influence form a system called **capacitor**, characterized by a capacitance given by

Capacitor: system of two conductors under total influence

$$C = \frac{Q}{\Delta V} \tag{2.34}$$

where Q is the absolute value of the charge in any of the conductors and ΔV is the absolute value of the electric potential difference (voltage drop) between the conductors. The capacitance can be regarded as a measure of the capacity to store charge for a given potential difference. Because the potential difference is proportional to the charge, this ratio does not depend on either Q or ΔV, but only on the size, shape, and relative position of the conductors.

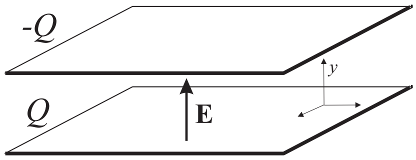


A typical example of capacitor is the **parallel-plate capacitor**. To calculate the potential difference between the parallel plates, it will be assumed that the size of the plates is much greater than the distance (d) between them, and therefore they can be modeled by two parallel charged infinite planes. Given the expression (2.14) for the field produced by a uniformly charged plane, in the case of two infinite planes with opposite charges, from the superposition principle, it is found that

$$\vec{E} = \begin{cases} \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{y}} \, , & \text{if } 0 < y < d \\ 0 \, , & \text{otherwise} \, . \end{cases} \tag{2.35}$$

Note that the electric field is uniform inside the capacitor and zero outside, which makes these parallel-plate capacitors be generally employed to produce an uniform electric field in a given region.

To find the potential difference between the capacitor plates, we can calculate the line integral of the electric field given by (2.35) from one plate to the other. Since the electric field is *uniform*, it is found that



$$\Delta V = \int_0^d \vec{E} \cdot d\vec{l} = |\vec{E}|d = \frac{\sigma}{\epsilon_0} d \,. \tag{2.36}$$

As the charge in each *finite* plate of area S is given by $Q = \sigma S$, the capacitance of the parallel-plate capacitor can finally be written as

Capacitance of a parallel-plate capacitor

$$C = \frac{\sigma S}{\dfrac{\sigma}{\epsilon_0} d} = \epsilon_0 \frac{S}{d} \,. \tag{2.37}$$

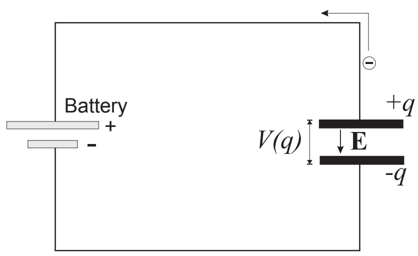
Activity 2.9:

- Two conductors, a cube and an sphere, have the same charge and voltges. Can they have the same capacitance? Justify your answer. If your answer is yes, how is that possible if the capacitance only depend on the shape of the conductor?
- Explain the difference between the capacitance of a conductor and the capacitance of a capacitor.
- What relevant characteristics have the electric field of a parallel-plate capacitor?
- How will you increase the capacitance of a given parallel-plate capacitor?

2.6 Electrostatic energy

2.6.1 Energy in a parallel-plate capacitor

Instead of obtaining the general expression for the electrostatic energy required to assemble an arbitrary charge distribution, our aim here will be to calculate the energy involved in the specific process of charging a parallel-plate capacitor. This energy will later be identified, without demonstration, as the electrostatic energy of an arbitrary charge distribution.



In the process of charging a parallel-plate capacitor (initially the two plates are neutral conductors), the task of the battery connected to the capacitor plates will be to extract negative charge from one of the plates and transfer this charge to the other plate. This way both plates are getting charged, giving rise to the *appearance of an electric field between the plates* as well as the corresponding potential difference $V(q) = q/C$ (this voltage will increase during this process). More specifically, in order to increment the charge in the capacitor an amount dq, the battery has to do a differential work, d$W_{\rm batt}$, that counteracts the effect exerted by the electric field on dq (the battery has to work against the force caused by the existing electric field).

Here it should be noted that the minimum work that an external agent has to do to move a point charge q in a region with an electric potential $V(P)$ is the one produced by an external force $\vec{F}_{\rm ext}$ which exactly counteracts the electrostatic force; that is, $\vec{F}_{\rm ext} = -q\vec{E}$ (a larger force would also increase the kinetic energy of the charge, which is not necessary to just move the charge from one point to another). Therefore, the minimum amount of work required to move the point charge from point A to point B is given by

$$W = \int_A^B \vec{F}_{\rm ext} \cdot {\rm d}\vec{l} = -q \int_A^B \vec{E} \cdot {\rm d}\vec{l} = q\left[V(B) - V(A)\right] = q\Delta V \left[= \Delta\mathcal{E}_p\right] \quad . \tag{2.38}$$

If the previous expression, valid for point charges, is now adapted to deal with a differential charge (dq), the resulting differential work can be written as

$$ {\rm d}W = {\rm d}q\,\Delta V\,. \tag{2.39}$$

Coming back to the charging of the capacitor, the differential work done by the battery to transfer a differential charge between the capacitor plates with a voltage drop between them ($\Delta V \equiv V(q) = q/C$ in the capacitor) can then be expressed as

$$ {\rm d}W_{\rm batt} = \frac{q{\rm d}q}{C}\,. \tag{2.40}$$

The total work required to charge the parallel-plate capacitor with a net charge Q can finally be obtained by integrating the above expression:

$$W_{\rm batt} = \int_0^Q {\rm d}W_{\rm batt} = \int_0^Q \frac{q}{C}\,{\rm d}q = \frac{1}{2}\frac{Q^2}{C}\,. \tag{2.41}$$

Now it should be considered that the work done by the battery, which is "delivered' to the capacitor to increase its potential energy, can be interpreted as energy *stored* in the capacitor. Thus, we finally identify this energy increase with the **electrostatic energy** of the capacitor, U_E, to write that

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV\,. \tag{2.42}$$

2.6.2 Density of electrostatic energy

In the particular case of a parallel-plate capacitor it was found that

$$V = |\vec{E}|d \qquad \text{and} \qquad C = \epsilon_0\frac{S}{d}\,,$$

which, after being introduced into (2.42), leads to

$$
\begin{aligned}
U_E &= \frac{1}{2}CV^2 = \frac{1}{2}\epsilon_0\frac{S}{d}|\vec{E}|^2 d^2 \\
&= \frac{1}{2}\epsilon_0|\vec{E}|^2 Sd = \frac{1}{2}\epsilon_0|\vec{E}|^2\mathcal{V} \,.
\end{aligned} \tag{2.43}
$$

Defining the **density of energy of the electric field**, u_E, as the electric energy per unit volume; namely,

$$
\mathrm{d}U_E = u_E \mathrm{d}\mathcal{V} \tag{2.44}
$$

it can be deduced from (2.43) that the density of electrostatic energy in the parallel-plate capacitor is given by

$$
u_E = \frac{1}{2}\epsilon_0|\vec{E}|^2 \,. \tag{2.45}
$$

Interestingly, the electrostatic energy of the parallel-plate capacitor can be expressed in terms of either the charge, expression (2.42), or the electric field, expression (2.43). These two expressions show the possible ambiguity found to determine whether the potential energy of the system is stored in the charges or in the electric field. Although the consideration that "the energy is stored in the field" could seem somewhat artificial, this concept turns out to be the most appropriate one.[2] Previous to the appearance of an electric field between the capacitor plates, the electrostatic energy in that region was null. Afterwards, when an electric field has already been set up in that region, the energy reaches a certain value. Therefore, it is reasonable to associate the existence of the electrostatic potential energy with the presence of an electric field.

It should be noted that the calculations that led us to (2.45) were carried out for a particular case: the parallel-plate capacitor. However, more elaborate calculations (beyond the scope of this lesson) would show that the same results are obtained for the *general* expression of the electrostatic energy density of any system of charges.

In consequence, the electrostatic energy of an arbitrary charge distribution is given by

Electrostatic energy

$$
\boxed{U_E = \int_{\substack{\text{all the}\\ \text{space}}} \frac{\epsilon_0|\vec{E}|^2}{2}\,\mathrm{d}\mathcal{V}} \,. \tag{2.46}
$$

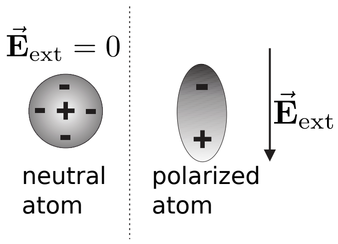
[2] This fact will be made more apparent in the further study of the energy associated with an electromagnetic wave, as we will see in a next lesson.

Activity 2.10:

- Explain the reasons why the increase of the potential energy in the charging process of a capacitor is given by $\Delta\mathcal{E}_p = \frac{1}{2}QV$ instead of just $\Delta\mathcal{E}_p = QV$.
- Why is it convenient to associate the electrostatic energy with the electric field rather than with charges and potentials?

2.7 Dielectrics

So far we have been studying different electrostatic phenomena involving charges and perfect conductors in *free space*. For instance, when considering the electric field created by a point charge in Section 2.2.3, it was assumed that there was no material in the space surrounding the point charge. In order to introduce the effect of a non-conductor material medium in our study, we have to consider that the media called *dielectrics* (see Section 2.4) are formed by atoms/molecules electrically neutral, with the center of their positive charges (protons) exactly matching the one of the negative charges (electrons). However, under the influence of an external electric field, the center of the negative charges can be slightly shifted with respect to the center of the positive charges; in which case, the atoms or molecules constituting the material medium are said to be *polarized*. This phenomenon of **polarization** gives rise to a new electric field (called electric polarization) that opposes to the original electric field. The overall effect of the superposition of the above two fields results in a partial decrease of the original electric field, as if the field was caused by a point charge with a lesser amount of charge.



The same previous effect can also be observed in a parallel-plate capacitor, where it is experimentally found that the introduction of a homogeneous and isotropic dielectric material between its plates increases the capacitance of such capacitor in a certain proportion that depends exclusively on the medium properties. To understand this effect, let us consider the *uncharged* capacitor in Fig. 2.1(a), which has a dielectric material (wood, paper, water, plastic,...) between its plates. If this capacitor is now charged with a charge Q_0 in one of the plates and $-Q_0$ in the other, these charges will create an electric field $\vec{E}_0$ between the capacitor plates. This field will in turn polarize the atoms of the dielectric material, resulting in a microscopic situation as the one plotted in Fig. 2.1(b). Note that, inside the dielectric material, positive and negative charges counterbalance themselves, however leaving certain uncompensated charge Q_p precisely at the edges of the material adjacent to the metallic plates. This charge will give rise to an electric field $\vec{E}_p$ that overlaps with the original field $\vec{E}_0$ to produce a new field $\vec{E}$, whose

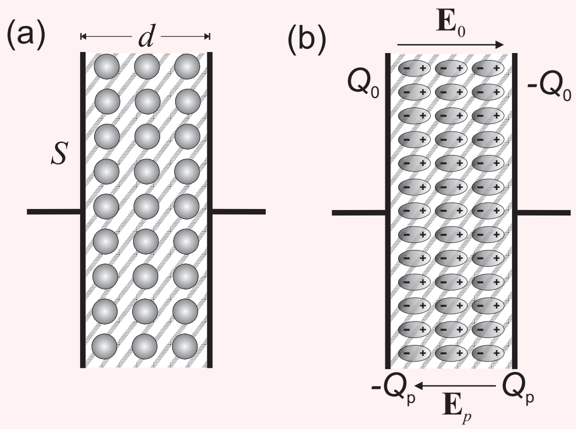


Figure 2.1: (a) Uncharged capacitor with a dielectric material between the plates. (Spheres stand for neutral atoms of the material.) (b) Charged isolated capacitor with a charge Q_0 at the plates which is counteracted by a charge Q_p coming from the polarization of the dielectric atoms

magnitude can be expressed as

$$|\vec{E}| = \frac{E_0}{\epsilon_r} \tag{2.47}$$

where ϵ_r is a dimensionless greater-than-one constant ($\epsilon_r \geq 1$). This constant is found to only depend on the material properties and is known as **relative permittivity** of the material.

As the capacitance of the parallel-plate capacitor in free space (namely, without any dielectric material between its plates) was given by

$$C_0 = \frac{Q_0}{V_0} = \epsilon_0 \frac{S}{d}$$

(with $V_0 = E_0 d$ being the potential difference between the plates), it can be observed that the introduction of a dielectric material reduces the value of the field between the capacitor plates and, consequently, will also reduce the potential difference between the plates by the same amount; namely,

$$V = |\vec{E}|d = \frac{V_0}{\epsilon_r} \ . \tag{2.48}$$

Since the presence of the dielectric does not affect the amount of initial charge placed on the metallic plates (note that the charge in the dielectric does appear on its own edges, not on the plates), we find that the capacitance is then given by

$$C = \frac{Q_0}{V} = \frac{Q_0}{V_0/\epsilon_r} = \epsilon_r C_0 = \epsilon_0 \epsilon_r \frac{S}{d} \tag{2.49}$$

thus explaining the increase of the capacitance experimentally observed.

It should be noted that the overall effect of introducing the homogeneous and isotropic dielectric material between the capacitor plates reduces to the apperance of a factor ϵ_r, that multiplies the vacuum permittivity. It means that we simply have to carry ou the following substitution:

$$\epsilon_0 \longleftrightarrow \epsilon_r \epsilon_0 \tag{2.50}$$

into the capacitance expression. Thus, the capacitance of a parallel-plate capacitor can finally be written as

$$C = \epsilon \frac{S}{d} \tag{2.51}$$

where

Electric permittivity

$$\boxed{\epsilon = \epsilon_0 \epsilon_r} \tag{2.52}$$

is the **electric permittivity** of the material and ϵ_r its relative permittivity.

Certainly we have that $\epsilon \geq \epsilon_0$, with the relative permittivities of some common materials given in the following table:

Material	Relative permittivity (ϵ_r)
Free space	1
Air	1.00059
Water (20°C)	80
Paper	3.7
Porcelain	7
Glass	5.6
Neoprene	6.9
Polystyrene	2.55

It is worth noting that, for any electrical practical purpose, the air behaves like the vacuum (free space) due to its relative permittivity very close to unity.

The above discussion about the effect of an homogeneous and isotropic dielectric on the expression of the electric field inside the dielectric could have been extended to the study of other quantities and situations. In that case, we will always find that the expressions previously obtained for free space only have to be modified by changing ϵ_0 by $\epsilon = \epsilon_r \epsilon_0$. Thus, for instance, the electrostatic energy of a region of space where there is a material with dielectric permittivity ϵ is given by

Electrostatic energy in a dielectric medium

$$\boxed{U_E = \int_{\substack{\text{region of}\\ \text{dielectric}}} \frac{\epsilon |\vec{E}|^2}{2} \, d\mathcal{V}} \,. \tag{2.53}$$

Activity 2.11:

- Write Coulomb's law for a source and test point charges submerged in water (an homogeneous dielectric medium).
- Describe the main consequences of having the two charges submerged in water.
- Give the physical reasons that explain the above consequences.
- For a given voltage drop between the plates of a plane capacitor, does the introduction of a dielectric increase or decrease the total charge that can be stored in the plates? Justify your answer.
- Deduce the expression of the total capacitance C of a capacitor formed by two capacitors of capacitances C_1 and C_2 respectively when they are connected in series and in parallel.

2.8 Problems

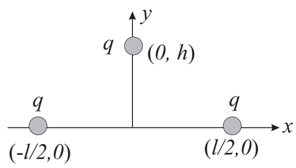
2.1: Find the repulsive force between two charged particles α separated a distance of 2 nm (each α particle is formed by two protons and two neutrons) and compare this force with the gravitational atractive force between them.
Sol. $F_{\text{elect}} = 2.30 \times 10^{-10}$ N; $F_{\text{grav}} = 7.51 \times 10^{-46}$ N.

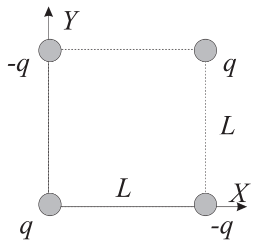
2.2: What is the magnitude of the electric field at a point located at 30 cm of a point charge of 10 μC.
Sol. $E = 10^6$ N/C.

2.3: Two positive point charges of equal value q lie on two opposite vertices of a square with side a. In another of the vertices, it is placed a charge q', the value of which makes that the electric field created by the three charges in the remaining empty vertex is zero. Find the value of q' in terms of q.
Sol.: $q' = -2\sqrt{2}q$.

2.4: Two point identical charges of charge q are located at points $(-a, 0, 0)$ and $(a, 0, 0)$. Find the electric potential and electric field produced by these charges at points of the Y axis.
Sol.: $V(0, y, 0) = [q/(2\pi\epsilon_0)](y^2 + a^2)^{-1/2}$, $\vec{E}(0, y, 0) = [q/(2\pi\epsilon_0)]y(y^2 + a^2)^{-3/2}\,\hat{\mathbf{y}}$.

2.5: Three identical charges of value q are located in the vertices of a triangle, as shown in the figure. Find: **a)** the electric field and potential produced by the charges at points along the segment joining the points $(0, 0)$ and $(0, h)$; **b)** the electric force exerted by the two charges on the X axis on the third charge at $(0, h)$.
Sol.: a) $V(0, y) = Kq[2((l/2)^2 + y^2)^{-1/2} + (h - y)^{-1}]$, $\vec{E}(0, y) = Kq[2y((l/2)^2 + y^2)^{-3/2} - (h - y)^{-2}]\,\hat{\mathbf{y}}$.
b) $\vec{F} = Kq^2 2h[(l/2)^2 + h^2]^{-3/2}\hat{\mathbf{y}}$.





2.6: The four charges of the figure are located at the vertices of a square of side L. **a)** Find the magnitude and direction of the electric force exerted on the charge located at the lower left vertex by the remaining three charges. **b)** Show that the total electric field at the middle point of any side of the square is parallel to that side, being directed to the negative charge of this side and that its value is given by $E = [2q/(\pi\epsilon_0 L^2)](1-\sqrt{5}/25)$ N/C.
Sol.: a) $\vec{F} = [q^2/(4\pi\epsilon_0 L^2)](1-1/\sqrt{8})(\hat{\mathbf{x}}+\hat{\mathbf{y}})$ N.

2.7: Two point charges q and $-q$ are located at points $(-a/2, 0)$ and $(a/2, 0)$, respectively, corresponding to two vertices of an equilateral triangle. (**a**) Find the electric field (vector) that these charges produce at the third vertex of the triangle. (**b**) If a third point charge Q is now placed at this third vertex and Q is moved to the x-axis at a distance $2a$ from q and $3a$ from $-q$, find the work, W_E, done by the electric force acting on Q along such displacement.

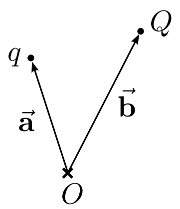
2.8: The electric potential in a given region of the space is given by $V = 2x^2 - y^2 + z^2$, where all the units are expressed in the S.I. Find the work done by the electric field on a point charge $q = 2$ C that travels from point $(1, 2, 3)$ to point $(3, 3, 3)$.
Sol.: -22 J.

2.9: An oil charged drop of mass 2.5×10^{-4} g is located inside a parallel-plate capacitor of area 175 cm^2. When the upper plate has a charge of 4.5×10^{-7} C, the oil drop remains floating. What is the charge of that drop?
Sol. $Q = 8.43\times10^{-13}$ C.

2.10: Two point charges $q_1 = -8\,\mu$C and $q_2 = 2\,\mu$C are located on the x axis at $x = 0$ and $x = 20$ cm, respectively. Find the point on the x axis where (**a**) the electric potential is null; (**b**) the electric field is null.

2.11: A solid conductor sphere of radius 10 cm at electrostatic equilibrium has a positive charge of 50 nC. Find the magnitude of the electric field (**a**) at 1 cm from the center of the sphere; (**b**) on the surface of the sphere.

2.12: At planes $x = 0$ and $x = 4$ there exist charge densities of value $\sigma_1 = 10^{-8}$ C/m^2 and $\sigma_2 = -10^{-8}$ C/m^2 respectively. Find: **a)** the electric force on a point charge $q = 1$ pC located at (1,0,0); **b)** the work done to move this charge to point (3,2,0); **c)** the potential difference between points (1,0,0) and (8,0,0).
Sol.: a) $36\pi\cdot10^{-11}\,\hat{\mathbf{x}}$ N; **b)** $72\pi\cdot10^{-11}$ J; **c)** 1080π V.



2.13: The figure shows two point charges of values q and Q, whose position vectors with respect to a given origin of coordinates O are $\vec{a}$ and $\vec{b}$, respectively. **(a)** Find the expression of the electrostatic force (vector) that q exerts on Q. **(b)** Find the expression of the potential difference that Q is creating between the origin and the point where q is located. **(c)** Find the work that we have to do to move charge q from its initial position to infinity (assume Q is fixed). Write all the results in terms of q, Q, ϵ_0, $\vec{a}$ and $\vec{b}$.

2.14: In a given region with a uniform electrostatic field $\vec{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$, the potential difference between point A (0,0) m and point B (3, 0) m is $V_A - V_B = 480$ V, and between B (3, 0) m and C (3, 4) m is $V_B - V_C = 200$ V. Find the value of $\vec{E}$.

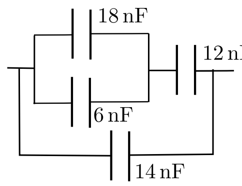
2.15: For a uniform electric field of value $\vec{E} = E_0\hat{\mathbf{y}}$ ($E_0 > 0$), **(a)** describe how are the field lines (plot these lines in the plane xy) and the equipotential surfaces. Write the potential difference between two points A and B (namely, $V_A - V_B$) located at (x_A, y_A, z_A) and (x_B, y_B, z_B). **(b)** If a positively charged particle (mass m and charge q) is released from point (x_0, y_0) at $t = 0$ with initial velocity $\vec{v}_0 = v_{0x}\hat{\mathbf{x}}$, find the position at a time $t > 0$. Also find the variation of potential and kinetic energy associated with this particle. What about the sum of both energies?
Sol. (a): $V(A) - V(B) = E_0(y_A - y_B)$. **(b)**: $x(t) = x_0 + v_{0x}t$, $y(t) = y_0 + 1/2(qE_0/m)t^2$.

2.16: An initially uncharged capacitor is being charged by means of a process in which the charge is carried from one plate to the other. At a given time, the capacitor plates have charges $-q$ and $+q$. **(a)** What amount of work should we do at this time to increase the charge of the capacitor in a differential dq? At the light of the above result, discuss whether the work to move a dq from one plate to the other is always the same or if it depends on the already accumulated charge. **(b)** Using an integration process, find the total energy supplied to the capacitor to charge it with a total charge Q.

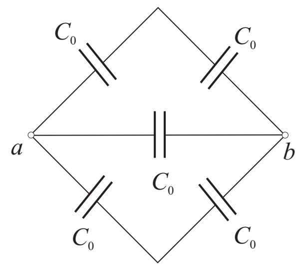
2.17: **a)** What is the capacitance of a parallel-plate capacitor of area 1 dm^2 separated 1 mm? **b)** Find the work to charge this capacitor with a total charge 10^{-3} C. **c)** What is the force exerted by one plate on the other?
Sol.: a) $C = 88.5$ pF; **b)** $W = 5649,7$ J; **c)** $F = 11.29 \times 10^6$ N.

2.18: **a)** What amount of charge should be added to an isolated conductor sphere of radius $R_1 = 10$ cm to increase the electric potential in 500 V?. **b)** If the above charge is shared with another isolated conductor sphere of radius $R_2 = 5$ cm (both spheres are connected through a very thin conductor wire), find the charge and electric potential of each sphere.
Sol.: a) $Q = 5.6 \times 10^{-9}$ C; **b)** $Q_1 = 3.74$ nC, $Q_2 = 1.86$ nC, $V_1 = V_2 \approx 336.6$V.

2.19: Two capacitors of capitance C_1 and C_2 can be connected either in series (both capacitors will have the same charge) or in parallel (both capacitors will have the same potential difference). Find the resulting capacitance of the connection.
Sol.: Series) $1/C = 1/C_1 + 1/C_2$. Parallel) $C = C_1 + C_2$.



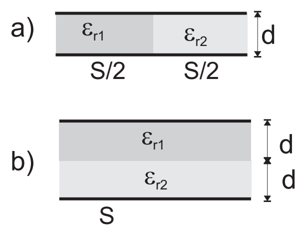
2.20: In the capacitor association shown in the figure, the potential difference between the plates of the 12 nF capacitor is 3 V. Find the charge Q_{14} of the 14 nF capacitor.
Sol.: $Q_{14} = 63$ nC .



2.21: Five identical capacitors of capacitance C_0 are connected in a bridge circuit as shown in the figure. **a)** What is the equivalent capacitance between points a and b?. **b)** Find the equivalent capacitance if the central capacitor between a and b now changes to $10C_0$.
Sol.: a) $C_{\rm equiv} = 2C_0$; **b)** $C_{\rm equiv} = 11C_0$;

2.22: A charged capacitor of 1 μF has reached a voltage of 10 V. Find: **a)** the stored charge and the required work to charge the capacitor.; **b)** the volumetric energy density inside the capacitor assuming that it can be modeled as an ideal parallel-plate capacitor whose plates are separated 10 cm; **c)** the work required to double the charge of the capacitor. Compare this work with the one obtained in (a) (**Data**: $\epsilon_0 = 8.854 \times 10^{-12}$ F/m).
Sol.: a) $Q = 10\ \mu$C, $W = 5 \times 10^{-5}$ J; **b)** $\rho_E = 4.427 \times 10^{-8}$ J/m^3; **c)** $W = 15 \times 10^{-5}$ J.

2.23: A parallel-plate capacitor of area $S = 0,5$ m^2 (no material inside the plates) has a breakdown electric field of 3 MV/m. Find (**a**) the electric field when the capacitor is charged with 20 nC. (**b**) the maximum charge before reaching the dielectric breakdown. (**c**) If now a material of $\varepsilon_r = 9$ is inserted between the plates, find the charge of this new capacitor so that it has the same energy as the previous capacitor with 20 nC.



2.24: (*) Find the capacitance of parallel-plate capacitors shown in the figure.
Sol.: a) $C = C_0(\epsilon_{r,1} + \epsilon_{r,2})/2$; **b)** $C = C_0\epsilon_{r,1}\epsilon_{r,2}/(\epsilon_{r,1} + \epsilon_{r,2})$, with $C_0 = \epsilon_0 S/d$ in both cases.

Lesson 3

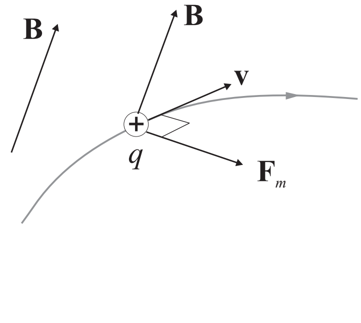
Magnetostatics

3.1 Magnetic Interaction

In the previous lesson we have studied interactions between charge distributions that are time invariant, and whose related phenomena were described in terms of the electric field and/or electric potential. It has long been well known that there is another interaction in Nature whose origin is not related to static electric charges but does affect *moving* electric charges. This interaction is known as *magnetic* interaction and is found, for example, in forces of attraction and repulsion between magnets and/or wires carrying electric currents, as well as in the attraction of pieces of iron (or other metals) by magnets or in the permanent orientation of a compass needle along the magnetic North of the Earth. The study of this new interaction, similarly to the case of Electrostatics, is carried out by means of a vector field now called **magnetic field**, $\vec{B}$. The introduction of this field will allow us to study the magnetic interaction, regardless of its sources. In the present lesson we will study *magnetic fields that do not vary in time*; i.e., **magnetostatic** fields.

3.2 Lorentz force law

Let us start the study of the magnetic interaction by assuming that there is a magnetic field $\vec{B}$ in certain region of the space. *Experimentally* it is found that there appears a force $\vec{F}_m$ on a test moving point charge, q, whose instantaneous velocity $\vec{v}$ is measured in the same reference frame as the one for $\vec{B}$, with the following characteristics:



- The force is proportional to the product $q|\vec{v}|$. It implies that this force does not act on either neutral particles or charged particles at rest.
- The force is also proportional to the magnitude of the magnetic field $|\vec{B}|$.

- The direction of the force is normal to the plane formed by vectors $\vec{v}$ and $\vec{B}$, with its magnitude being null if these vectors are parallel ($\vec{v} \parallel \vec{B}$) and maximum if the vectors are perpendicular ($\vec{v} \perp \vec{B}$).

The above experimental results can all be accounted for by the following mathematical expression:

$$\vec{F}_m = q\vec{v} \times \vec{B}\,. \tag{3.1}$$

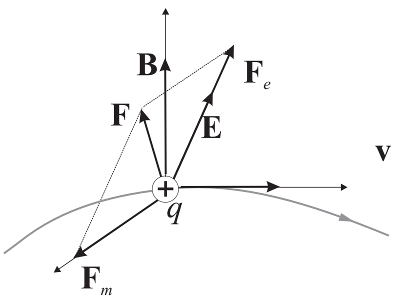
The vector product (see Sec. 1.8.4) of $q\vec{v}$ and $\vec{B}$ fully determines the magnetic force on a moving charge. From the above expression it can be deduced that the S.I. units of the magnetic field, known as **teslas** (T), are given by

Unit of magnetic field
1 tesla (T)

$$1\,\mathrm{T} = 1\,\frac{\mathrm{N/C}}{\mathrm{m/s}}\,. \tag{3.2}$$

The S.I. unit of magnetic field is relatively large; that is, it is not very usual to have magnetic fields of the order of teslas. For instance, the Earth's magnetic field at its surface is of the order of some tens of microteslas. Thus, in many practical cases, another unit of magnetic field is used, called **gauss** (G) and defined as

$$1\,\mathrm{T} = 10^4\,\mathrm{G}\,. \tag{3.3}$$



H. A. Lorentz (1853-1928) proposed that the net force acting on a point charge, q, in the presence of both an electric, $\vec{E}$, and a magnetic field, $\vec{B}$, is given by the so-called **Lorentz force law**:

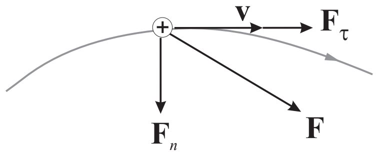
$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)} \tag{3.4}$$

which results from the superposition of the electric force $\vec{F}_e = q\vec{E}$ plus the magnetic force $\vec{F}_m = q\vec{v} \times \vec{B}$.

Activity 3.1:

- Describe the possible advantages of defining a magnetic field in order to study the magnetic interaction.
- Hypothetically, could the magnetic force have been defined in (3.1) with a dot product instead of a vector product? Justify your answer.
- Enumerate the main differences of the magnetic force with respect to the electric force when a magnetic/electric field acts on a point charge.
- Can you anticipate some physical consequences of the fact that the magnetic force is defined as a vector product of its velocity and the magnetic field?

3.2.1 Motion of a charged particle in a magnetic field



Before dealing with the magnetic force, it is convenient to remind here that the net resultant of the external forces, $\vec{F} = \sum \vec{F}_{\rm ext}$, applied on a particle with a curvilinear trajectory can be split into two components, one tangent to the particle trajectory, $\vec{F}_\tau$, and another normal to the trajectory, $\vec{F}_n$:

$$\vec{F} = \vec{F}_\tau + \vec{F}_n = F_\tau \hat{\tau} + F_n \hat{\mathbf{n}} \ .$$

In consequence, the equation of motion of a particle of mass m (namely, second Newton's law),

$$m\frac{{\rm d}\vec{v}}{{\rm d}t} = \sum \vec{F}_{\rm ext} \ ,$$

can be rewritten (taking into account that $\vec{v} = |\vec{v}|\hat{\tau}$) as

$$\begin{aligned} m\frac{\rm d}{{\rm d}t}(|\vec{v}|\hat{\tau}) &= m\frac{{\rm d}|\vec{v}|}{{\rm d}t}\hat{\tau} + m|\vec{v}|\frac{{\rm d}\hat{\tau}}{{\rm d}t} \\ &= m\frac{{\rm d}|\vec{v}|}{{\rm d}t}\hat{\tau} + m\frac{|\vec{v}|^2}{r}\hat{\mathbf{n}} \\ &= F_\tau\hat{\tau} + F_n\hat{\mathbf{n}} \end{aligned}$$

or equivalently,

$$F_\tau = m\frac{{\rm d}|\vec{v}|}{{\rm d}t} \qquad (3.5)$$

$$F_n = m\frac{|\vec{v}|^2}{r} \qquad (3.6)$$

where r is the radius of curvature of the trajectory.

Magnetic force is a normal force

Now let us focus on the case of a particle of mass m and charge q in a magnetic field $\vec{B}$. In this case, the equation of motion is given by

$$m\frac{{\rm d}\vec{v}}{{\rm d}t} = \vec{F}_m = q\vec{v}\times\vec{B} \ . \qquad (3.7)$$

In the above equation we can observe that the magnetic force is always a normal force, since $\vec{F}_m$ is perpendicular to $\vec{v}$ due to the vector product $\vec{v}\times\vec{B}$. Hence, we can make the following considerations:

- As the tangent component of the force is null ($F_\tau = 0$), from (3.5) it is obtained that ${\rm d}|\vec{v}|/{\rm d}t = 0$; namely, *the magnetic force does not change the magnitude of the velocity (speed) but only its direction* ($|\vec{v}|$ = cte).
- *The magnetic force does not do work* on a charged particle with velocity $\vec{v}$. Taking into account that the differential work along the particle trajectory is given by the dot product $\vec{F}_m\cdot{\rm d}\vec{l} = \vec{F}_m\cdot\vec{v}{\rm d}t$ (${\rm d}\vec{l} = \vec{v}{\rm d}t$), we can observe that this dot product is always zero because $\vec{F}_m \perp \vec{v}$.
- Since $F_n = |\vec{F}_m|$, (3.6) and (3.7) lead to

  $$m\frac{|\vec{v}|^2}{r} = q|\vec{v}||\vec{B}|\,{\rm sen}\,\theta \qquad (3.8)$$

  (with θ being the angle between $\vec{v}$ and $\vec{B}$), and then the particle speed is given by

  $$|\vec{v}| = \frac{q|\vec{B}|r}{m}\,{\rm sen}\,\theta \ . \qquad (3.9)$$

Trajectory when $\vec{B}$ has a fixed direction

In the frequent case that the magnetostatic field has a fixed direction in the space, say $\vec{B} = B\hat{\mathbf{z}}$, we can carry out a decomposition of the the velocity in terms of its components parallel and perpendicular to the magnetic field $\vec{B}$; namely,

$$\vec{v} = v_z\hat{\mathbf{z}} + \vec{v}_\perp ,$$

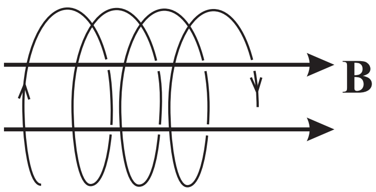
which allows us to express the magnetic force as

$$\vec{F}_m = q\vec{v} \times \vec{B} = q\vec{v}_\perp \times \vec{B} .$$

As $\vec{F}_m$ has no projection along $\vec{B}$, we can write the following equations for the velocities $\vec{v}_z = v_z\hat{\mathbf{z}}$ and $\vec{v}_\perp$:

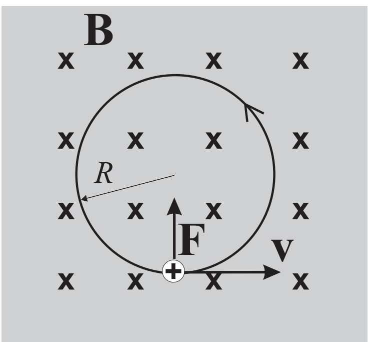
$$m\frac{\mathrm{d}v_z}{\mathrm{d}t}\hat{\mathbf{z}} = 0 \tag{3.10}$$

$$m\frac{\mathrm{d}\vec{v}_\perp}{\mathrm{d}t} = \vec{F}_m = q\vec{v}_\perp \times \vec{B} . \tag{3.11}$$



These equations imply, on the one hand, that the component of the velocity parallel to $\vec{B}$ does not change under the influence of the magnetic field, given that the acceleration in this direction is null, $\vec{a}_z = 0 \Rightarrow \vec{v}_z =$ Cte. On the other hand, the perpendicular component, $\vec{v}_\perp$, is affected by a normal force that only changes its direction.

If we now assume that the magnetic field is uniform and its magnitude fixed, the resulting motion of the particle is the superposition of an uniform motion along the direction of $\vec{B}$ (provided that $\vec{v}_z \neq 0$) plus a uniform circular motion in a perpendicular plane; that is, the trajectory of the particle is a helical motion with the line of sight along $\vec{B}$.



If the initial velocity of the particle has no component parallel to the magnetic field, $\vec{v}_z = 0$, the particle will have a purely uniform *circular* motion. The radius R of the circular trajectory can be deduced from (3.8) ($\theta = \pi/2$):

$$m\frac{|\vec{v}|^2}{R} = q|\vec{v}||\vec{B}| ,$$

namely,

$$R = \frac{m|\vec{v}|}{q|\vec{B}|} . \tag{3.12}$$

Recalling now the relation between the angular velocity $\vec{\omega}$ and the linear velocity $\vec{v}$, it is obtained that $|\vec{\omega}| = |\vec{v}|/R = 2\pi/T$ and, therefore, the period of the uniform circular motion is given by

$$T = 2\pi\frac{m}{q|\vec{B}|} . \tag{3.13}$$

Activity 3.2:

- What are the most relevant consequences of the fact that the magnetic force is always normal to the particle trajectory?
- Explain the reasons why the magnetic field never does work on moving charged particles.
- Describe the conditions under which the trajectory of a charged particle in a region with $\vec{B}$ is helical.
- When does the above trajectory become circular? Deduce the period of this motion and explain how it can be increased.

EXAMPLE 3.1 *Find the mass of a charged particle with* $q = 1.6 \times 10^{-19}$ *C that, after passing through a region with a magnetic field of* $|\vec{B}| = 4000$ *G, describes a circular motion with* $R = 21$ *cm, with its speed being selected by a setup as the one shown in the figure,* $|\vec{E}| = 3.2 \times 10^5$ *V/m.*

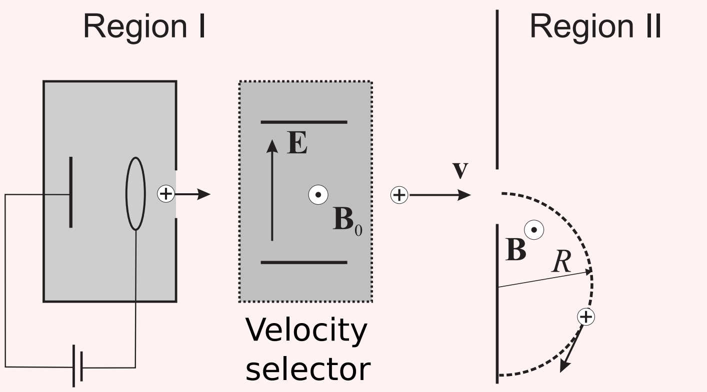


FIGURE: Circular trajectory of a charge particle after passing through a velocity selector.

The velocity selector imposes that only those particles satisfying

$$|\vec{F}_e| = |\vec{F}_m| \quad \Rightarrow \quad |\vec{E}| = |\vec{v}||\vec{B}_0|$$

will pass through the aperture and get to region II. Thus, the particles in this region will have a speed given by

$$|\vec{v}| = \frac{|\vec{E}|}{|\vec{B}_0|} = \frac{3.2 \times 10^5}{0.4} \text{ m/s} = 8.05 \times 10^6 \text{ m/s} .$$

Once in region II, the magnetic force makes the particles describe a circular trajectory of radius

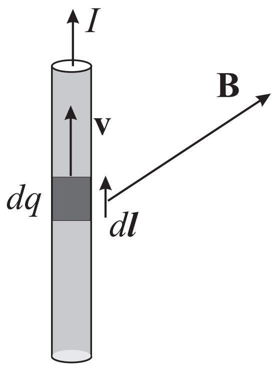
$$R = \frac{m|\vec{v}|}{q|\vec{B}|}$$

and then their mass will be

$$m = \frac{qR|\vec{B}|}{|\vec{v}|} = \frac{1.6\times10^{-19}\times0.21\times0.4}{8.05\times10^{6}} = 1.67\times10^{-27}\,\text{kg}\,.$$

Given the mass and charge of the particles, it can be deduced that they are protons.

3.3 Magnetic forces on conductors



3.3.1 Magnetic force on a current-carrying wire

Expression (3.1), $\vec{F}_m = q\vec{v}\times\vec{B}$, described the magnetic force exerted by a magnetic field $\vec{B}$ on a point charge q with velocity $\vec{v}$. From this expression we can easily obtain the value of the magnetic force on a wire segment that carries a current I.[1] First, we can deduce from Eq. (3.1) that the differential force on a differential segment of a current-carrying wire with charge dq is given by

$$\mathrm{d}\vec{F}_m = \mathrm{d}q\vec{v}_d\times\vec{B} \tag{3.14}$$

where $\vec{v}_d$ is the drift velocity of the differential charge flowing through the wire. This differential charge can be written in terms of the current as $\mathrm{d}q = I\mathrm{d}t$, and therefore

$$\mathrm{d}q\vec{v}_d = I\vec{v}_d\mathrm{d}t = I\mathrm{d}\vec{l}$$

where $\mathrm{d}\vec{l} = \vec{v}_d\mathrm{d}t$ is a vector whose magnitude is a differential length along the wire and its direction is given by that of the current in the wire. The quantity $I\mathrm{d}\vec{l}$ is called a **current element**. Introducing $I\mathrm{d}\vec{l}$ into (3.14) we can write as well that

$$\mathrm{d}\vec{F}_m = I\mathrm{d}\vec{l}\times\vec{B} \tag{3.15}$$

and, therefore, the force on a finite segment of a current-carrying wire can finally be expressed as

Magnetic force on a segment of a current-carrying wire

$$\vec{F}_m = \int_{\text{wire}} \mathrm{d}\vec{F}_m = \int_{\text{wire}} I\mathrm{d}\vec{l}\times\vec{B}\,. \tag{3.16}$$

When both I and $\vec{B}$ are constant along the length of the wire, the above expression can be simplified to

$$\vec{F}_m = I\left(\int_{\text{wire}}\mathrm{d}\vec{l}\right)\times\vec{B} = I\vec{l}\times\vec{B} \tag{3.17}$$

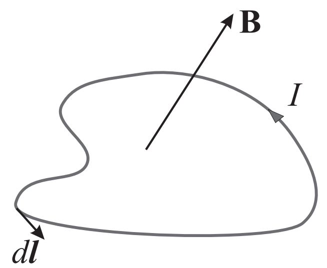
where $\vec{l}$ is a vector going from the initial to the final points of the wire segment (if the wire is straight, its magnitude is just the total length of the wire and its direction is given by the direction of the electric current).

[1] At this point it should be reminded that the current $I = \mathrm{d}q/\mathrm{d}t$ in a wire is the charge per unit time passing through the wire. For more details on this quantity, please see Sec. 5.2.

Activity 3.3:

- Which are the charges that take part in the electric current flowing through a wire?
- Explain how to deduce (3.14) from (3.1).
- Describe the conditions under which Eq. (3.17) is valid.
- If a portion of a current-carrying wire is curved, draw the vector $\vec{l}$. What happens if the segment turns into a loop?

3.3.2 Forces and torques on current loops



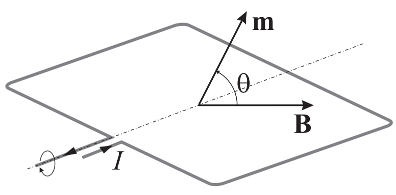
If we have a current loop (a current-carrying wire loop) with I, the magnetic force on it according to (3.16) is given by

$$\vec{F}_m = I \oint_{\text{loop}} \mathrm{d}\vec{l} \times \vec{B} \,. \tag{3.18}$$

In the particular and usual case that $\vec{B}$ is uniform in the region where the loop is placed, (3.17) tells us that

$$\vec{F}_m = I \left[\oint_{\text{loop}} \mathrm{d}\vec{l} \right] \times \vec{B} = 0 \qquad \text{since} \qquad \oint_{\text{loop}} \mathrm{d}\vec{l} = 0 \,,$$

namely, the net force on the loop is null. However, it should be noticed that this fact does not necessarily imply that the loop has no motion. We should realize that it simply implies that the loop will not have *translational* motion. Indeed, the loop would have a *rotational* motion provided that the torque exerted on this loop was not zero.



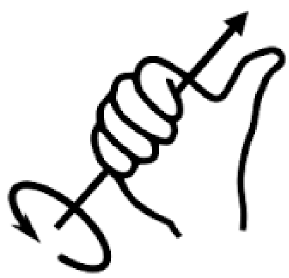
If we proceed to calculate the torque, $\vec{M}$, due to the magnetic force we will find the following expression:

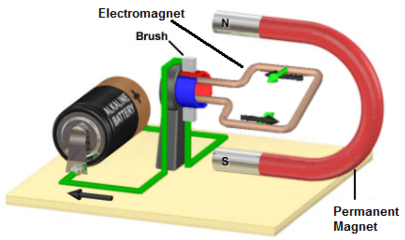
$$\boxed{\vec{M} = \vec{m} \times \vec{B}} \tag{3.19}$$

where

$$\vec{m} = NI\vec{S} \tag{3.20}$$

is a vector quantity known as **magnetic dipole moment** (or, simply, magnetic moment). N stands for the number of turns in the loop and $\vec{S}$ is a vector whose magnitude is the area of the loop and its direction is given by the normal to the loop surface (as shown in the adjacent figure, the normal direction [thumb] is determined by the traveling direction of the current [remaining fingers] according to the right-hand rule). It is worth noting that expression (3.19) is valid for any geometry of the loop provided that $\vec{B}$ is uniform.

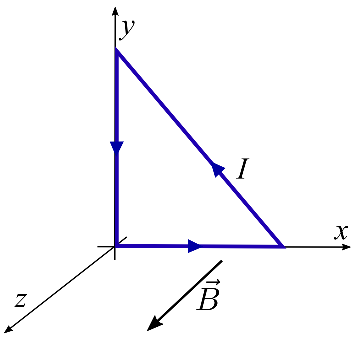




The torque exerted on the current-carrying loop causes a twist of the loop so that $\vec{m}$ tends to align with $\vec{B}$. The appearance of this "magnetic" torque is the physical basis of operation of the **electric motor**. An elementary schematic of an electric motor is precisely a current-carrying coil (a loop with many turns) which, under the influence of a magnetic field, undergoes a torque that causes a rotational motion. However, this simple scheme would not give place to a uniform rotation but rather to an oscillatory motion. In order to have the desired uniform rotational motion we should devised some way of making that the torque changes its direction at the right time, which is achieved by the brush in the attached figure. Thus, attaching the coil to some rotor will make it possible to transform electric/magnetic energy into rotational kinetic energy; energy which can further be transformed into any other type of translational motion.

Activity 3.4:

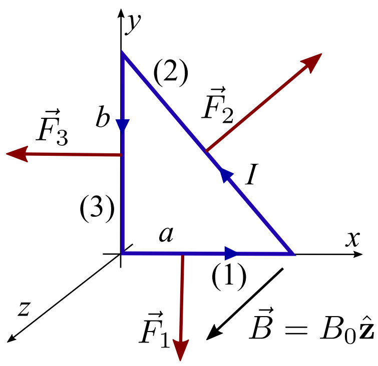
- Find examples of rotational motions where the net force acting on the system is null.
- Explain the operating mechanism of the DC motor shown in the figure above.



EXAMPLE 3.2 *A wire loop with a current of 1 A has a right-triangle shape and is located in the xy plane. The side on the x axis measures a = 60 cm and the one on the y axis b = 80 cm. The current flows through the loop in the counter-clockwise direction. In the region there is a uniform magnetic field of magnitude B_0 = 2 T in the positive direction of the z axis ($\vec{B} = B_0\hat{\mathbf{z}}$).* ***(a)*** *Find the magnetic force on each side and draw a picture of them. Check that the resultant of the three forces is zero.* ***(b)*** *Find the torque on the current-carrying loop.*

(a) As we have already studied, the magnetic force exerted by a *uniform* magnetic field $\vec{B}$ on a current-carrying segment $I\vec{l}$ is given by

$$\vec{F}_m = I\vec{l} \times \vec{B} \,.$$



In the present case, following the notation given in the attached figure, we have that the length vectors associated with each side are given by

$$\vec{l}_1 = a\hat{\mathbf{x}}$$
$$\vec{l}_3 = -b\hat{\mathbf{y}}$$
$$\vec{l}_2 = -a\hat{\mathbf{x}} + b\hat{\mathbf{y}} \,.$$

Taking into account the expression for the magnetic force on each side, it is found

$$\vec{F}_1 = I\vec{l}_1 \times \vec{B} = Ia\hat{\mathbf{x}} \times B_0\hat{\mathbf{z}} = -IaB_0\hat{\mathbf{y}}$$
$$\vec{F}_3 = I\vec{l}_3 \times \vec{B} = -Ib\hat{\mathbf{y}} \times B_0\hat{\mathbf{z}} = -IbB_0\hat{\mathbf{x}}$$
$$\vec{F}_2 = I\vec{l}_2 \times \vec{B} = I(-a\hat{\mathbf{x}} + b\hat{\mathbf{y}}) \times B_0\hat{\mathbf{z}} = IaB_0\hat{\mathbf{y}} + IbB_0\hat{\mathbf{x}}$$

and therefore

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \,.$$

(b) The torque is given by $\vec{M} = \vec{m} \times \vec{B}$, with the magnetic moment given by

$$\vec{m} = I\frac{ab}{2}\,\hat{\mathbf{z}}\,.$$

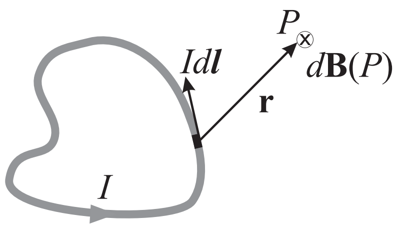
Since $\vec{B}$ is in the same direction as $\vec{m}$, the torque is null.

3.4 Biot-Savart law

Up to now, we have discussed some effects caused by the magnetic field $\vec{B}$ without referring to the possible *sources* of this field. **Magnets** are well know for long to be sources of magnetostatic fields. These magnets are pieces of certain materials (for instance, magnetite) that have the fancy property of attracting pieces of iron. Also, a charge moving close to a magnet is affected by a magnetic force according to (3.1). Although magnets have been widely known and employed for many centuries, the study on the origin of their magnetic properties was only possible in the frame of Quantum Physics and, therefore, it is beyond our scope.

The experiments carried out by H. C. Oersted (∼1820) showed that the effects on moving charges and current-carrying wires due to the magnetic fields produced by magnets were also reproduced when these magnets were substituted by current-carrying wires. Since an electric current is formed by a flux of moving charges, it can then be concluded that, in general, *moving charges are sources of magnetic fields*. As this lesson only deals with *magnetostatic* fields, we will focus on the sources responsible for these time-invariant fields. Experimentally it was found that

the sources of magnetostatic fields are the steady electric currents.



How these steady currents produce magnetostatic fields is accounted for by the **Biot-Savart law** (∼ 1830). This law establishes that the differential of the magnetostatic field at the observation point P due to a current element $I d\vec{l}$, which is part of a steady-current loop, is given by

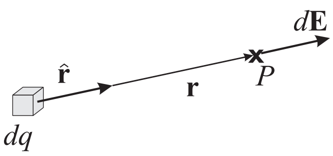
$$d\vec{B}(P) = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \hat{\mathbf{r}}}{|\vec{r}|^2} \equiv \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \vec{r}}{|\vec{r}|^3} \tag{3.21}$$

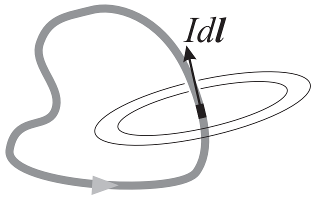
where $\vec{r}$ is the vector from the current element, $I d\vec{l}$, to the observation point P. The constant μ_0 is known as **vacuum permeability** and its value is

$$\boxed{\mu_0 = 4\pi \times 10^{-7}\frac{\mathrm{T\cdot m}}{\mathrm{A}}}\,. \tag{3.22}$$

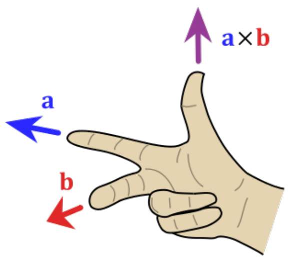
It should be noted that expression (3.21) is similar to (2.9), which gave us the electrostatic field produced by a differential charge element:

$$d\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\frac{dq}{|\vec{r}|^2}\,\hat{\mathbf{r}}\,.$$





Líneas de campo **B** debidas a *Idl*



Both expressions have the same dependence with respect to the distance r between the differential element and the observarion point; namely, $1/r^2$. However, a relevant difference is that the direction of the field is distinct for $\vec{E}$ and $\vec{B}$. For the electrostatic field, the field direction is determined directly by the position vector $\vec{r}$ going from the charge to the observation point. On the contrary, for the magnetostatic field, the direction of $\mathrm{d}\vec{B}$ comes determined by the cross product

$$I\mathrm{d}\vec{l} \times \hat{\mathbf{r}} \, ,$$

which makes that the direction of $\mathrm{d}\vec{B}$ at the observation point is always perpendicular to $\vec{r}$ (that is, $\mathrm{d}\vec{B} \perp \hat{\mathbf{r}}$). This direction can be obtained following the "right-hand rule" with the thumb pointing at the direction of the current element, the index finger along $\vec{r}$ and the middle finger then giving the field direction. This way, the magnetic field lines due to a current-carrying element will be *circumferences* concentric to an axis directed along the current element. This also implies that the *the magnetic field lines are endless lines*; in this case being *closed* lines.

The total field produced by an steady-current loop will then be given by the integral of (3.21) along the complete closed current path:

Magnetostatic field due to a steady-current loop

$$\boxed{\vec{B}(P) = \frac{\mu_0}{4\pi} \oint_{\text{loop}} \frac{I\mathrm{d}\vec{l} \times \vec{r}}{|\vec{r}|^3}} \, . \tag{3.23}$$

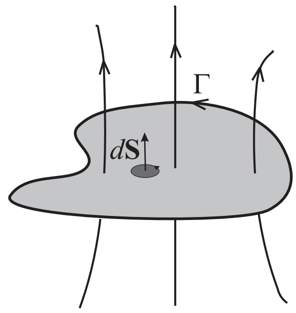
Activity 3.5:

- Starting from the differential sources that produce electrostatic and magnetostatic fields respectively, plot the differential fields in each case. Explain the main differences.
- Why is the direction of $\mathrm{d}\vec{B}$ always perpendicular to the current element that produces it?
- Has any sense to talk about the magnetostatic field produced by a segment of a current-carrying wire?
- Can you find any case where it is easy to perform the integration in Eq.(3.23)? Justify your answer.

3.5 Ampere's law

Similar to what happened with the integral expression for the electrostatic field given in (2.10), the use of Eq. (3.23) to compute the magnetostatic field is not adequate to obtain closed-form expressions in most cases. However, this expression is easily implementable as a numerical algorithm to be used in a computer, and therefore it is always useful for computing numerically the magnetostatic field produced by a given current-carrying loop.

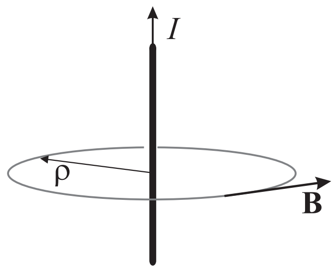
Fortunately, in highly symmetric situations we can make use of **Ampere's law** (∼1830) to obtain a closed-form expression for the magnetostatic field.[2] This law states that



$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = \mu_0 I_{\Gamma} \tag{3.24}$$

namely, the circulation integral of the magnetostatic field, $\vec{B}$, along an arbitrary *closed* curve Γ is μ_0 times the value of the net current, I_Γ, that passes through any surface $S(\Gamma)$ whose boundary is defined by the curve Γ. The application of this law allows us to find the magnetic field in the following situations with a high degree of symmetry:

- **Magnetic field due to a straight current-carrying infinite wire**

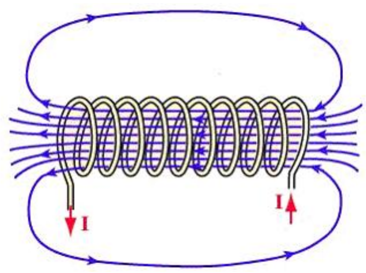
  

  Due to the symmetry of the problem, the magnetostatic field will have the following dependence: $\vec{B} = |\vec{B}(\rho)|\hat{\tau}$ (ρ is the minimum distance from the observation point to the straight wire and $\hat{\tau}$ is the unit vector tangent to the circumference whose axis is the wire). The application of (3.24) leads us to find that

  $$\boxed{\vec{B}(P) = \frac{\mu_0 I}{2\pi\rho}\,\hat{\tau}}\ . \tag{3.25}$$

- **Magnetic field due to a solenoid with current I**

  A solenoid is a coil of wire wound as a tightly packed helix. A long solenoid (longer than wide) is usually employed to create strong uniform magnetic fields, since the field inside this type of solenoid has this particular characteristic, and it is null outside the solenoid. In this way, the solenoid plays the same role as the parallel-plate capacitor does for the electrostatic field.

  As the theoretical deduction of the shape of the magnetic-field lines produced by a solenoid is somewaht complex, we use *experimental* evidences to determine the shape of these lines. The experiments show that the field lines inside the solenoid are very approximately straight lines parallel to the axis of the solenoid, and that these lines get apart each other outside the solenoid with a decreasing amplitude as the solenoid increases its length. For an infinitely long solenoid, which can be taken as a model of a long solenoid, *the magnetic field can be considered null outside*.

  When we apply Ampere's law to this case, it is finally found that

  $$\boxed{\vec{B}(P) = \begin{cases} \mu_0 n I \hat{\mathbf{u}} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases}} \tag{3.26}$$

  with $n = N/l$ being the number of turns per unit length in the solenoid and $\hat{\mathbf{u}}$ the unit vector in the direction of the solenoid axis.

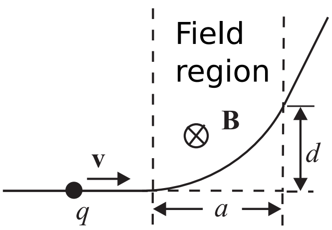
[2] Likewise Gauss's law in (2.11) was useful to obtain closed-form expressions for the electrostatic field of highly symmetric charge distributions.

Activity 3.6:

- Explain when Ampere's law is valid and also when it is useful. Justify your answer.
- What is the exact meaning of the term I_Γ in Eq. (3.24)?
- If the circulation integral of $\vec{B}$ along a given closed path Γ is zero, then it means that the magnetostatic field is zero. Right or False? Justify your answer.
- Explain why Eq. (3.25) is not valid for a finite length of a straight current-carrying wire.
- When cannot Eq. (3.26) be considered the right expression for the magnetic field of a current-carrying solenoid?

3.6 Problems

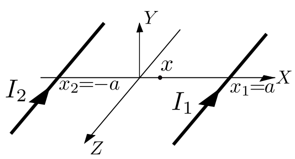
3.1: What is the radius of the orbit of a proton of energy 1 MeV in a magnetic field of 10^4 G.
Sol. R = 14.4 cm.



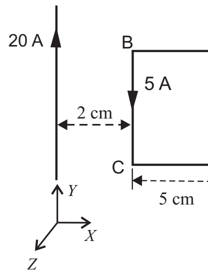
3.2: A particle of charge q and initial velocity $\vec{v}$ enters in a region with a uniform magnetic field directed inward to the page. As shown in the figure, the magnetic field deviates the particle a distance d from its original trajectory. Discuss if the charge of the particle is positive or negative and find the value of its linear momentum p in terms of a, d, B and q.
Sol.: es positiva; $p = qB(a^2 + d^2)/(2d)$.

3.3: A conductor wire parallel to the y axis is moving with a velocity $\vec{v} = 20\,\hat{\mathbf{x}}$ m/s in a magnetic field $\vec{B} = 0.5\,\hat{\mathbf{z}}$ T. **a)** Find the magnitude and direction of the magnetic force exerted on a electron in the wire. **b)** Due to this magnetic force, the electrons will move to one extreme of the wire (leaving the other extreme positively charged) until the electric field caused by the charge separation exerts a force on the electron that cancels out the effect of the magnetic force. Find the magnitude and direction of this electric field in the stationary regime. **c)** If the wire is 2 m long, what is the potential difference between its extremes due to the appearance of the electric field?
Sol.: **a)** $\vec{F} = 1.6 \times 10^{-18}$ N $\hat{\mathbf{y}}$; **b)** $\vec{E} = 10$ V/m $\hat{\mathbf{y}}$; **c)** $V = 20$ V.;

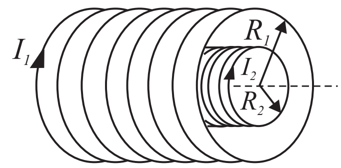
3.4: Two sections of straight and parallel wires of length 90 cm are separated a distance of 1 m. If the currents in both conductors is 5 A flowing in opposite directions, what is the magnitude and direction of the forces between these sections of both wires?
Sol.: 4.5 μN, being a repulsive force.



3.5: Two straight conducting wires of infinite length, parallel to the Z axis, carry currents I_1 and I_2 in the same direction, as shown in the figure. The wires intersect the X axis at $x_1 = a$ and $x_2 = -a$, respectively. Find the magnitude of the magnetic field, B, that they create at any point x (see figure) of the segment in the X axis between the two wires.
Sol.: $B = \dfrac{\mu_0}{2\pi}\left|\dfrac{I_1}{a-x} - \dfrac{I_2}{a+x}\right|$.



3.6: A straight infinite wire carries a current of 20 A, as shown in the figure. Besides this wire there is a rectangular loop of sides 5 cm and 10 cm. A current of 5 A is flown through this loop in the direction shown in the figure. **a**) Find the force of each side of the loop and also the net force on the complete loop; **b**) Also find the flux through this loop due to the magnetic field $\vec{B}$ produced by the wire.
Sol. a) side AB: -2.5×10^{-5} N $\hat{\mathbf{y}}$, side BC: 10^{-4} N $\hat{\mathbf{x}}$, side CD: 2.5×10^{-5} N $\hat{\mathbf{y}}$, side DA: -2.85×10^{-5} N $\hat{\mathbf{x}}$, $\vec{F}_{\text{net}} = 7.15 \times 10^{-5}$ N $\hat{\mathbf{x}}$; **b**) $\Phi = 5.01 \times 10^{-7}$ Tm^2.



3.7: A long solenoid with n_1 turns per unit length carries a current I_1 and has a cross section of radius R_1. Inside this solenoid and coaxial with it, there is a second solenoid with n_2 turns per unit length and cross section of radius R_2 ($R_2 < R_1$). If a current I_2 is flowing through this second solenoid, find: **a**) the magnetic field created in all the points; **b**) the magnitude and direction of I_2 so that, fixed I_1, the magnetic field inside the second solenoid is null.

Sol.: **a**) $B(r) = \begin{cases} \mu_0 n_1 I_1 \pm \mu n_2 I_2 & \text{if } r < R_2 \\ \mu_0 n_1 I_1 & \text{if } R_2 < r < R_1 \\ 0 & \text{if } r > R_1 \end{cases}$

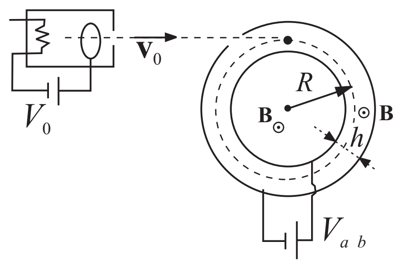
where r is the distance from the solenoid axes and plus/minus sign applies if the solenoid current have the same/opposite direction; **b**) $I_2 = -n_1 I_1 / n_2$.

3.8: (*) Two straight infinite wires with currents I_1 and I_2 are perpendicular to plane XY and cut this plane at points $(0, a, 0)$ and $(0, -a, 0)$. Find the magnetic field produced by these current at all points of the space (the given solution should be valid independently of the direction of the carrying currents).
Sol.:

$$\vec{B}(P) = \frac{\mu_0}{2\pi}\left\{\left(\frac{(a-y)I_1}{x^2 + (a-y)^2} - \frac{(a+y)I_2}{x^2 + (a+y)^2}\right)\hat{\mathbf{x}} + \left(\frac{xI_1}{x^2 + (a-y)^2} + \frac{xI_2}{x^2 + (a+y)^2]}\right)\hat{\mathbf{y}}\right\},$$

where the currents are taken positive if directed along the z axis and negative otherwise.

3.9: A wire of length l is wound to form a circular coil with N turns. Prove that if a current I is flowing through this coil, its magnetic dipole moment is given by $Il^2/(4\pi N)$.



3.10: The figure shows a device to speed up electrons and a cylindrical capacitor of radius R and distance between conductors $h \ll R$, such that the electric field between the conductors is radially directed and uniform. In the direction of the capacitor axis there is a uniform magnetic field $\vec{B}$. The accelerating device injects electrons in the capacitors tangential to the circumference of radius R, as shown in the figure. Assuming that the accelerating potential difference is V_0, find: **a**) the speed, v_0, of the injected electrons (mass m_e and charge -e); **b**) the voltage V_{ab} applied between the conductors of the capacitor so that the electrons have a circular motion of radius R when the magnetic field is null. **c**) Repeat the above problem assuming now that the magnetic field is not null $\vec{B} \neq 0$.
Sol.: **a**) $v_0 = \sqrt{\dfrac{2qV_0}{m_e}}$; **b**) $V_{ab} = \dfrac{2V_0 h}{R}$; **c**) $V_{ab} = \dfrac{2V_0 h}{R} + hB\sqrt{2qV_0/m_e}$.

Lesson 4

Electromagnetic Induction

4.1 Introduction

In Chapter 3 we have seen that electric currents are sources of magnetic fields; in particular, around the year 1820 H.C. Oersted found out that a steady-current-carrying wire loop created a static magnetic field (detected, for example, by the effect it has on a magnetic needle). Since electric currents produce magnetic fields, on the basis of reciprocity it is reasonable to question if magnetic fields can produce electric currents. In this regard it was carried out an intensive experimental research that seemed to deny that possibility. However, experiments carried out by M. Faraday (1791-1867) around 1830 revealed that the generation of an electric current in a wire loop was related to the time variation of the magnetic flux through the surface of the wire-loop circuit. Consequently, magnetostatic magnetic fields will never produce electrical currents in static circuits.

Can magnetostatic fields produce electric currents?

We should consider here that there will always be dissipation of energy in any *real* circuit. This energy loss is caused by the inelastic collisions of the mobile charge carriers forming the current with the atoms of the conductor material in the circuit. This phenomenon gives rise to the appearance of a certain amount of resistance, R, in any circuit. Thus, in order to have an electric current flowing in the circuit, it is always required a continuous supply of electric energy. In other words, the charge carriers have to be constantly "pushed" by an external driving force produced by the generator or battery.[1] To account for this fact we use a quantity called **electromotive force** (**emf**), ξ, which is defined as the tangential force per unit charge on the wire, $\vec{f}$ (namely, the force on the wire that "pushes" a mobile unit charge), integrated along

Current always gives rise to dissipation of energy in any real circuit

[1] These questions will be studied with more detail in Secs. 5.3.3 and 5.4.

the length of the circuit; namely,

Definition of electromotive force

$$\xi = \oint \vec{f} \cdot \mathrm{d}\vec{l}\,. \tag{4.1}$$

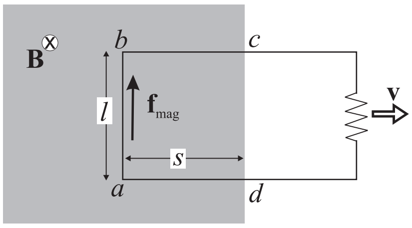
Consequently, the existence of an electric current in a real circuit has to be necessarily linked to the action of a source of emf that keeps it flowing. The origin of the emf can be diverse: chemical potentials in batteries, mechanical forces in the Van der Graff generator, optical effects in photo-voltaic cells, etc. In general we can say that the effect of an emf generator is to transform some kind of energy into the electrical energy required to create the electric current. In the experiments carried out by Faraday, the generation of that emf was directly related to the existence of a time-varying magnetic flux. This so-called *induced* emf (in contrast with *direct* emf produced, for instance, by batteries) will have very relevant consequences, both conceptual and technological, being the basis for the generation of electricity in power plants, the operation of AC circuits, and the existence of electromagnetic waves.

Activity 4.1:

- Why is it necessary to have an external agent that supplies energy to the circuit in order to keep a flowing electric current?
- Can we have a resistance R in a wire made up of a perfect conductor? Justify your answer.
- Deduce the units in the S.I. of the electromotive force from its definition in (4.1).
- Can you guess why the emf related to the time-varying magnetic flux is called "induced"?

4.2 Faraday's law

4.2.1 Motional emf



A possible way of generating an emf in a circuit consists in making use of the appearance of a magnetic force on the mobile charge carriers of a conducting wire in a region where there is a magnetic field. Thus, the motion of a conductor bar in a magnetic field, $\vec{B}$, would give rise to the so-called *motional emf*. Let us consider the situation shown in the figure, where the shaded region indicates the presence of a uniform magnetic field ($\vec{B}$) pointing into the page and the circuit is pulled to the right with velocity $\vec{v} = |\vec{v}|\hat{\mathbf{x}}$ [the magnetic field $\vec{B}$ can be produced, for instance, by a magnet]. In this situation, the charge carriers of the segment $\overline{ab}$ will experience the following Lorentz force per unit charge:

$$\vec{f}_{\mathrm{mag}} = \frac{\vec{F}_{\mathrm{mag}}}{q} = \vec{v} \times \vec{B} \tag{4.2}$$

whose net effect is to drive the charge carriers from a to b. This driving force will give place to a current in the circuit (the direction of which is determined by $\vec{f}_{\rm mag}$) caused by the appearance of an emf. According to to (4.1) this emf has the following value:

$$\xi = \oint \vec{v} \times \vec{B} \cdot {\rm d}\vec{l} \,. \tag{4.3}$$

In the present case $\vec{v} \times \vec{B}$ is parallel to ${\rm d}\vec{l}$ in the segment $\overline{ab}$ and perpendicular in the horizontal conductor segments inside the shaded region ($\vec{f}_{\rm mag} \cdot {\rm d}\vec{l} = 0$). As both $\vec{v}$ and $\vec{B}$ are invariant, the above integral then reduces to

$$\xi = \int_a^b \vec{v} \times \vec{B} \cdot {\rm d}\vec{l} = \int_a^b |\vec{v}||\vec{B}|{\rm d}l = |\vec{v}||\vec{B}| \int_a^b {\rm d}l = |\vec{v}||\vec{B}|l \tag{4.4}$$

where l is the length of the segment $\overline{ab}$. According to Ohm's law (see Sec. 5.3.2), the induced current, I, flowing in the circuit with resistance R is given by

$$I = \frac{\xi}{R} = \frac{|\vec{v}||\vec{B}|l}{R} \,. \tag{4.5}$$

Although the motional emf has been deduced from the Lorentz force on the charge carriers, it is worth noting that this emf could also have been deduced as minus the time rate of change of the magnetic flux, Φ_m, through the loop:

$$\xi = -\frac{{\rm d}\Phi_m}{{\rm d}t} \,. \tag{4.6}$$

To check this fact, we should consider that the magnetic flux is defined as

$$\Phi_m = \int_S \vec{B} \cdot {\rm d}\vec{S} \tag{4.7}$$

where ${\rm d}\vec{S} = {\rm d}S\,\hat{\mathbf{n}}$ stands for the differential surface vector, being ${\rm d}S$ its magnitude (that is, the area of a differential of surface) and $\hat{\mathbf{n}}$ the unit vector normal to the surface. Since $\vec{B}$ is parallel to ${\rm d}\vec{S}$ in the present case (and therefore $\vec{B} \cdot {\rm d}\vec{S} = |\vec{B}|{\rm d}S$), it can be written that

$$\Phi_m = \int_S |\vec{B}|{\rm d}S = |\vec{B}| \int_S {\rm d}S = |\vec{B}|S = |\vec{B}|ls \tag{4.8}$$

where ls is the area of the circuit inside the shaded region where $\vec{B} \neq 0$. The time rate of change of the magnetic flux is then given by

$$\frac{{\rm d}\Phi_m}{{\rm d}t} = \frac{\rm d}{{\rm d}t}|\vec{B}|ls = -|\vec{B}|l|\vec{v}|$$

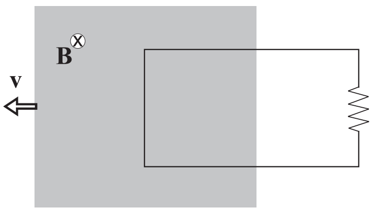
(the minus sign accounts for the fact that ${\rm d}s/{\rm d}t = -|\vec{v}|$ in the present case). From (4.6) it is apparent that we have obtained the same emf as the one computed in (4.3) after integrating the Lorentz force per unit charge.

Activity 4.2:

- What kind of transformation of energy has taken place in the situation discussed above?
- Will there still be electromotive force in the above case if the circuit is not closed? Justify your answer.
- Try to obtain by yourself the results shown in (4.4) and (4.8). You are strongly encouraged to understand all the steps that have been carried out.
- If the resistor in the analyzed case was a light bulb, would its performance be different if we had simply added a battery without moving the circuit?
- Can you find any connection between the two given procedures to obtain the same emf?

4.2.2 Induced emf

The discussion of the situation analyzed in Sec. 4.2.1 has shown that the appearance of a electromotive force in the *moving* circuit could be attributed to the effect of the Lorentz force. However, if we now consider that the circuit remains stationary and the agent that creates the magnetic field (for example, a magnet) is what is moving to the left, it is reasonable to assume that there will also appear a emf equal in magnitude and direction as in the previous case. According to the principle of relativity, what actually matters is the *relative* motion between the field $\vec{B}$ and the circuit rather than which of them is moving away.



Experiments show that the above assumption is effectively true. However, if we analyze the case of a stationary circuit and a moving magnet, since the charge carriers are now at *rest*, there will not be any Lorentz force driving these charges. Therefore, if there is not Lorentz force acting on the charges, where does the emf induced in the circuit come from? We can speculate that the cause that now produces the emf is the appearance of an electric field. This electric field cannot be of electrostatic nature if it is responsible for the emf (see discussion on Sec. 5.4) but *a new kind of electric field that should be associated with a time-varying magnetic field*.

The common point of the above two equivalent phenomena is that, in both cases, there is a time change of the magnetic flux through the circuit. This fact was found experimentally by M. Faraday ($\sim$ 1830) in what it is known as Faraday's law:

An ξ_{ind} is induced in the circuit that is equal in magnitude to the rate of change of the magnetic flux through the surface bounded by the circuit.

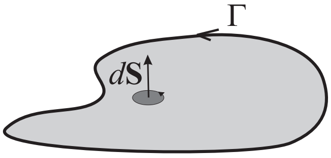
Mathematically, this law can be expressed as

Faraday's law

$$\mathcal{E}_{\mathrm{ind}} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \tag{4.9}$$

where the minus sign is related to the direction of the induced emf. Taking into account that the origin of the induced emf can in general be attributed to the appearance of a non-electrostatic electric field $\vec{E}$, Faraday's law can alternatively be expressed as

$$\oint_{\Gamma} \vec{E}\cdot\mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(\Gamma)} \vec{B}\cdot\mathrm{d}\vec{S} \tag{4.10}$$



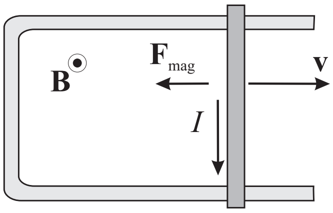
where curve Γ coincides with the wire loop. The minus sign in Faraday's law is now completely determined since the direction of the line integral around Γ is related to $\mathrm{d}\vec{S}$ according to the right-hand rule. Expression (4.10) makes it apparent that the induced emf is, in general, *distributed* along all the wire loop of the circuit.[2]

A useful and easy way to determine the direction of the induced emf and current is provided by **Lenz's law**:

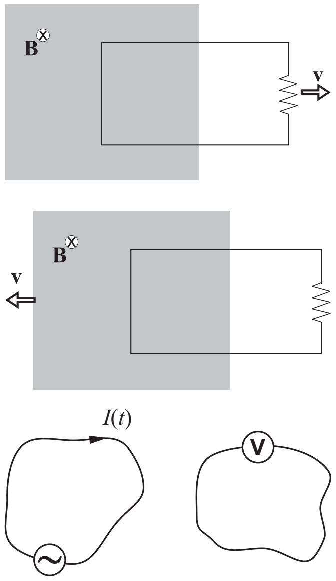
*The induced **emf** and current are in such a direction as to oppose, or tend to oppose, the magnetic-flux change that produces it.*

Lenz's law does not address the specific cause (or causes) that produces the magnetic flux change, and which results in the appearance of the induced emf. It rather suggests that the reaction of the system does generate an induced emf and current that will always counteract the flux change causing them. Otherwise, the circuit would *favor* the underlying cause of the induced emf/current, intensifying its effect indefinitely.

For practical purposes, the deduction of the direction of the induced emf and current may be given by the increasing or decreasing character of the time rate of change of the magnetic flux (this character is given by the sign of the time derivative). If, for instance, the flux increases at a certain instant of time, then the induced emf/current will have a direction that gives rise to a magnetic field able to counteract the increasing rate of the magnetic flux. Let us consider the effect of Lenz's law in the moving circuit shown in the figure. In this example, the sliding bar moves to the right with a velocity $\vec{v}$ due to the action of an external agent. As it has been discussed in Sec. 4.2.1, and according to Lenz's law, the direction of the induced current that appears in the circuit is such that the appearing magnetic force acting on the moving bar, $\vec{F}_{\mathrm{mag}} = I\vec{l}\times\vec{B}$, opposes to the externally imposed movement. Had the induced current an opposite direction to that shown in the figure, the magnetic force on the moving bar would favor the rightward motion of the bar so that it will continuously accelerate, causing a constant increase in its kinetic energy, which obviously makes no sense.



[2] Contrary to what happens, for instance, in a battery where the *emf* (and therefore the non-conservative electric field) are located inside the battery.



A current can be induced without physical contact between the circuits

We have found that a change in the magnetic flux through the circuit would always give rise to an induced emf in that circuit. Thus, some of the possible causes that can produce the appearance of an induced emf are:

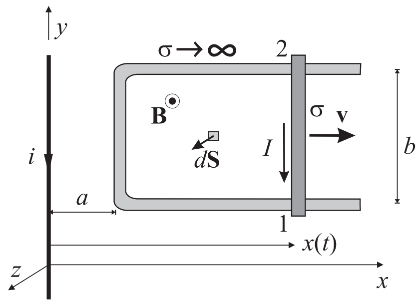
- Motion of the circuit, or deformation of its shape, in a region in which there exists a time-constant magnetic field.
- Motion of the agent that produces the magnetic field in such a way that a fixed loop intercepts a time-varying magnetic field. For instance, the motion of a magnet would produce an induce emf in the loop.
- Change of the current that flows in a primary circuit so that the magnetic flux intercepted by a neighbor secondary circuit is changing in time.
- Simultaneous combination of some of the above causes.

In the case of a time-varying current in a primary circuit which induces a current in a secondary circuit, it is important to note that this induced current is generated *without any physical contact between the circuits*. From an energy standpoint, the energy associated with the induced current in the secondary circuit should obviously be supplied by the source of emf in the primary circuit. As there is not physical contact between the two circuits, the only explanation for the appearance of energy in the secondary circuit is that this energy was transmitted from the primary circuit to the secondary one by means of the electromagnetic field through the empty space. This indicates that the field is capable of transmitting energy and, therefore, it must be considered an entity with physical reality itself.

Activity 4.3:

- Can Faraday's law be deduced starting from the Lorentz magnetic force? Justify your answer.
- A magnetostatic field cannot generate an induced current. Right or false? Justify your answer.
- At the light of (4.10), why the electric field caused by a varying magnetic field cannot be of electrostatic nature?
- Give an example where Lenz's law can explain the direction of the induced current.
- Find additional examples where Faraday's law explain the appearance of induced current in a circuit.
- Has Faraday's law something to do with the performance of wireless chargers for smartphones?
- How is it possible that energy (or data) can be transfered using wireless/contactless systems if there is nothing material connecting the different devices?

EXAMPLE 4.1 *Obtain the direction and value of the induced current in the device shown in the figure. Data. Moving bar: $\sigma = 10^8\,(\Omega m)^{-1}$, $b = 10\,cm$, $r = 2\,mm$, $v = 5\,m/s$; $i = 200\,mA$, $a = 20cm$.*



In the situation shown in the figure, as the vertical bar is moving to the right, the magnetic flux through the circuit loop (due to the magnetic field of the straight wire) is increasing and, therefore, an induced emf ξ_{ind} will appear in the loop. The resistance of the circuit, R, causes that the flowing current is given by

$$I_{ind} = \frac{\xi_{ind}}{R} \,. \tag{4.11}$$

Introducing the data above in expression (5.19), the resistance of the moving bar is

$$R = \frac{b}{\sigma S} = \frac{0.1}{10^8 \cdot \pi(2 \times 10^{-3})^2} = \frac{10^{-3}}{4\pi}\,\Omega \,.$$

Before computing the induced ξ_{ind} we should note that the magnetic field produced the straight wire (in the the plane $z = 0$ where the moving loop is placed) is given by

$$\vec{B}(x) = \frac{\mu_0 i}{2\pi x}\,\hat{\mathbf{z}} \,. \tag{4.12}$$

Due to the increasing nature of the magnetic flux through the loop, Lenz's law tells us that the reaction of the circuit should be to create a current that counteracts the "growing" flux. Thus, the magnetic field $\vec{B}_{ind}$ created by the induced current, I_{ind}, should oppose the external magnetic field, which is achieved by an induced current flowing in the direction shown in the figure. The magnitude of the induced emf and current are obtained next.

The induced emf can be calculated in the present case by two procedures:

- **Lorentz force**.

  As the charge carriers of the vertical bar are in a region with a magnetic field, there will appear a magnetic force per unit charge, $\vec{f}_m = \vec{v} \times \vec{B}$, on these charges. After applying (4.3), this magnetic force gives rise to an induced emf in the loop given by

$$\xi_{ind} = \int_1^2 \vec{v} \times \vec{B} \cdot d\vec{l} = \int_1^2 |\vec{v}||\vec{B}| dy = |\vec{v}||\vec{B}|b \,.$$

  Taking into account expression (4.12) for the magnetic field and considering that the motion of the moving bar is described by

$$x(t) = a + vt \tag{4.13}$$

  it is found the following expression for ξ_{ind}:

$$\xi_{ind}(t) = \frac{\mu_0 i v b}{2\pi(a + vt)} \,. \tag{4.14}$$

- **Faraday's law.**

  In order to apply Faraday's law (4.9), we need to obtain the magnetic flux Φ by integrating the differential of this flux. First we should note that the differential of surface can be written as $\mathrm{d}\vec{S} = \mathrm{d}x\mathrm{d}y\,\hat{\mathbf{z}}$, and therefore the differential of magnetic flux, $\mathrm{d}\Phi$, through that differential of surface is

$$\mathrm{d}\Phi = \vec{B}\cdot\mathrm{d}\vec{S} = |\vec{B}|\mathrm{d}S = \frac{\mu_0 i}{2\pi x}\,\mathrm{d}x\mathrm{d}y\,.$$

  The total flux can then be computed by integrating the above expression over the total surface of the loop:

$$\begin{aligned}\Phi &= \int_0^b\left\{\int_a^x \frac{\mu_0 i}{2\pi x}\,\mathrm{d}x\right\}\mathrm{d}y = \int_0^b\left\{\frac{\mu_0 i}{2\pi}\ln\frac{x}{a}\right\}\mathrm{d}y\\ &= \frac{\mu_0 i}{2\pi}\ln\frac{x}{a}\int_0^b \mathrm{d}y = \frac{\mu_0 ib}{2\pi}\ln\frac{x}{a}\,.\end{aligned} \tag{4.15}$$

  As ξ_{ind} is the time derivative of the magnetic flux, this derivative applied to (4.15) gives

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mu_0 ib}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln\frac{x}{a}\right) = \frac{\mu_0 ib}{2\pi}\frac{\mathrm{d}x/\mathrm{d}t}{x(t)} = \frac{\mu_0 ib}{2\pi}\frac{v}{x(t)}\,.$$

  In order to apply Lenz's law, we should observe that the sign of $\mathrm{d}\Phi/\mathrm{d}t$ in the above expression is always positive, and therefore the induced current has to create a magnetic field that opposes to this increasing flux. The direction of this induced field has to be $-\hat{\mathbf{z}}$ and, consequently, it should be generated by a counterclockwise directed current (as it was previously deduced).

  Given the equation for $x(t)$, the magnitude of ξ_{ind} can finally be written as

$$\xi_{\text{ind}}(t) = \frac{\mu_0 ib}{2\pi}\frac{v}{a+vt}\,, \tag{4.16}$$

  whose expression coincides with the one obtained previously in (4.14).

Finally, the value of the induced current is $I_{\text{ind}} = \xi_{\text{ind}}/R$; namely,

$$I_{\text{ind}}(t) = \xi_{\text{ind}}(t)\frac{\sigma S}{b} = \frac{\mu_0 i\sigma S}{2\pi}\frac{v}{a+vt}\,. \tag{4.17}$$

After one minute of motion, the induced current takes the following value:

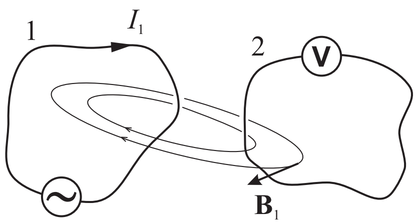
$$I_{\text{ind}}(60) = \frac{4\pi\times10^{-7}\cdot0.2\cdot10^{8}\cdot4\pi^2\times10^{-6}}{2\pi}\,\frac{5}{0.2+5\cdot60} \approx 2.6\mu\mathrm{A}\,.$$

4.3 Inductance

4.3.1 Mutual inductance

If we compute the magnetic flux, Φ_{21}, through the surface of the loop circuit 2 (see attached figure), due to the magnetic field $\vec{B}_1$ generated by the current I_1 flowing through the loop circuit 1, we would obtain that

$$\Phi_{21} \propto I_1 ,$$



namely, the magnetic flux through circuit 2 is found proportional to the current flowing in circuit 1. The proportionality factor between the magnetic flux through a loop due to the current flowing in another loop is called **mutual inductance**, and denoted as M. In our case we will have that

$$\Phi_{21} = MI_1 . \tag{4.18}$$

The SI unit of inductance is the **henry** (symbol H):

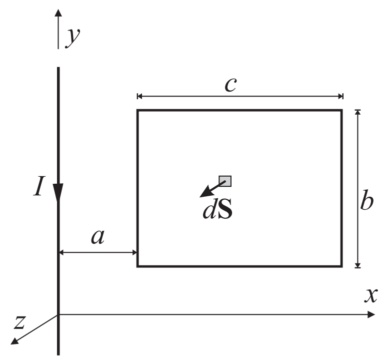
Unit of inductance
1 henry (H)

$$1\ \mathrm{H} = 1\ \frac{\mathrm{T\cdot m^2}}{\mathrm{A}} . \tag{4.19}$$

Using a rationale that will not be discussed here, we would find that the ratio between the magnetic flux Φ_{12} through loop 1 due to a magnetic field $\vec{B}_2$ generated by a current I_2 flowing in loop 2 will be the same ratio as before; namely,

$$\Phi_{12} = MI_2 . \tag{4.20}$$

EXAMPLE 4.2 *Find the mutual inductance of a rectangular loop (shown in the attached figure) and an infinite straight wire.*



What we can do is to obtain the magnetic flux through the rectangular loop due to the magnetic field created by the infinite straight wire with a flowing current I. In the plane $z = 0$ where the rectangular loop is placed, the magnetic field created by the current flowing in the infinite straight wire is given by

$$\vec{B}(x) = \frac{\mu_0 I}{2\pi x}\,\hat{\mathbf{z}} .$$

In the present case, the differential of surface can be written as $d\vec{S} = dxdy\,\hat{\mathbf{z}}$, and the differential of magnetic flux , $d\Phi$, through this surface is then

$$d\Phi = \vec{B}\cdot d\vec{S} = |\vec{B}|dS = \frac{\mu_0 I}{2\pi x}\,dxdy .$$

The total magnetic flux is the integral of the above expression over the surface of the rectangular loop:

$$\begin{aligned}\Phi &= \int_0^b \left\{\int_a^{a+c} \frac{\mu_0 I}{2\pi x}\,dx\right\} dy = \int_0^b \frac{\mu_0 I}{2\pi} \ln\frac{a+c}{a}\,dy \\ &= \frac{\mu_0 I}{2\pi} \ln\frac{a+c}{a} \int_0^b dy = \frac{\mu_0 I b}{2\pi} \ln\frac{a+c}{a} .\end{aligned}$$

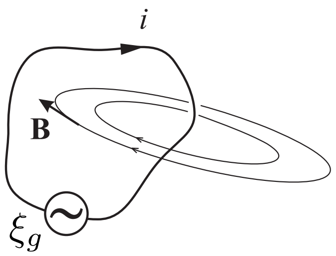
The above expression tells us that the mutual inductance in the present case is

$$M = \frac{\Phi}{I} = \frac{\mu_0 b}{2\pi} \ln \frac{a+c}{a} .$$

Activity 4.4:

- Can you think of any reason to justify the proportionality that exists between the magnetic flux through a loop due to the current flowing in another loop and this current? [Hint: The magnetic field produced by a loop current is proportional to the current flowing in the loop]
- Does the mutual inductance depend on the amount of flowing current? Justify your answer.
- Starting from the definition of henry (H) given in (4.19), obtain the units of μ_0 in terms of H.

4.3.2 Self inductance



Considering now the case of a single loop circuit connected to a generator of emf ξ_g that produces a current i flowing through it, a similar calculation as in previous section will show that the magnetic flux, Φ, through this same circuit is equally proportional to the current:

$$\Phi \propto i .$$

When the magnetic flux through a circuit is caused only by the current flowing through the circuit itself, this flux is known as **self-flux** and the proportionality constant between the self-flux and the current is called **self-inductance** (or simply inductance) and denoted by L. In consequence we can write

$$\Phi = Li . \tag{4.21}$$

The self-inductance SI unit is obviously the same as that of the mutual inductance (1 H).

For the very usual case in which ξ_g is time-varying and the circuit remains at rest, the magnetic flux given by (4.21) will also be time-varying (remind that L is constant in the present case). In this situation there will appear an induced emf in the circuit given by

Induced emf caused by self-inductance

$$\xi_{\text{ind}} = -L\frac{\text{d}i}{\text{d}t} . \tag{4.22}$$

Since any real circuit will have certain resistance, R, and consequently an associated voltage drop $V = Ri$ due to this fact, it has to be satisfied that the combined effect of the electromotive forces acting in the circuit has to equal the voltage drop through the resistance; namely,

$$\xi_g + \xi_{\text{ind}} = Ri \tag{4.23}$$

or, equivalently,

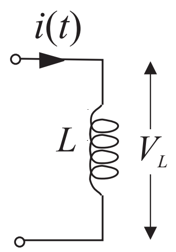
$$\xi_g - L\frac{\mathrm{d}i}{\mathrm{d}t} = Ri\,. \tag{4.24}$$

If (4.24) is now rewritten as

$$\xi_g = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t} \tag{4.25}$$

$$= V_R + V_L \tag{4.26}$$

the above equations can be interpreted as if the emf supplied by the generator equals the sum of the voltage drop in the resistance, $V_R = Ri$, and an additional voltage drop, V_L, due to the self inductance L. The *distributed effect* of the induced emf in the circuit loop can thus be regarded as a voltage drop in a circuit element that will be denoted in general as **inductor**, and that is characterized by its inductance L (see attached figure):



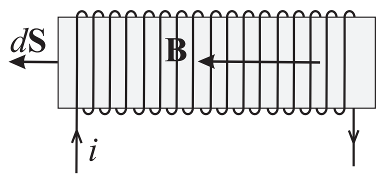
$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t}\,. \tag{4.27}$$

In general, the effects of the electromagnetic induction are assumed to be localized in the inductors. Usually, these inductors are elements purposely added to the circuit in order to increase these inductive effects: **solenoids** or **coils**. Given the high value of the magnetic field inside the solenoids and the possibility of miniaturization, these elements are an essential part of the electrical and electronic circuits.

EXAMPLE 4.3 *Find the self inductance of a long solenoid with* $N = 100$ *turns, length* $l = 1$ *cm and radius* $r = 1$ *mm.*

For a long solenoid with N turns and length l, the magnetic field inside can be written according to (3.26) as

$$\vec{B} = \mu_0 n\, i\, \hat{\mathbf{u}}$$



where $n = N/l$ is the number of turns per unit length and $\hat{\mathbf{u}}$ is the unit vector along the solenoid axis. As the differential of surface of one of the loops is given by $\mathrm{d}\vec{S} = \mathrm{d}S\hat{\mathbf{u}}$, the flux through N turns of the solenoid will be

$$\Phi = N\int_S \vec{B}\cdot\mathrm{d}\vec{S} = N\int_S |\vec{B}|\mathrm{d}S = N|\vec{B}|\int_S \mathrm{d}S = \mu_0\frac{N^2}{l}iS\,,$$

from which it can be deduced that the self inductance L is

$$L = \mu_0\frac{N^2}{l}S = \mu_0 n^2 l S\,.$$

Taking into account the data of this example we finally obtain that

$$L = 4\pi\times 10^{-7}\frac{10^4}{10^{-2}}\pi\times 10^{-6} \approx 3.95\mu\,\mathrm{H}\,.$$

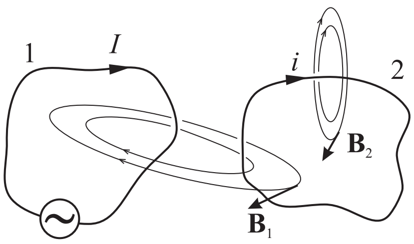
Activity 4.5:

- Does the inductance of a wire loop depend on the current flowing through it? Justify your answer. Can we talk about the inductance of a loop if there is no current flowing through it?
- When a solenoid is introduced in a circuit, explain why we usually say that the inductance of the circuit is the one provided by the solenoid.
- Is expression (4.27) valid for inductors with moving parts? Justify your answer.

4.3.3 General case

In the general case where we have two loop circuits with flowing currents through both of them, the total flux, Φ_{tot}, through the surface of, say, the loop circuit 2 can be expressed as

$$\begin{aligned}\Phi_{\text{tot}} &= \Phi_{21} + \Phi_{22} \\ &= \Phi_{\text{ext}} + \Phi_{\text{self}}\end{aligned} \tag{4.28}$$



where Φ_{ext} is the flux through circuit 2 due to the *external* agents (in this case, the field generated by the current, I, flowing in circuit 1) and Φ_{self} is the self-flux through circuit 2. At the light of the proportionality relations between the fluxes and the currents given in (4.18) and (4.21), the total flux can be written as

$$\Phi_{\text{tot}} = MI + Li\ . \tag{4.29}$$

According to Faraday's law and taking into account (4.28), the induced emf in circuit 2 is given by

$$\xi = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{\text{ext}} + \Phi_{\text{self}}\right)\ . \tag{4.30}$$

In the very common case where both the self inductance and the mutual inductance do not change in time (namely, if the shape of the motionless circuits does not vary), the induced emf can be written as

$$\xi = -M\frac{\mathrm{d}I}{\mathrm{d}t} - L\frac{\mathrm{d}i}{\mathrm{d}t}\ . \tag{4.31}$$

The computation of the induced emf in circuit 2 from (4.31) is not trivial because this emf depends on the time variations of i, and this current itself depends on the value of the induced emf. Fortunately, there are many practical situations where the time rate of change of the magnetic self-flux is negligible w.r.t. that of the external flux, and therefore the induced emf in the circuit can approximately be obtained as

Induced emf if self-flux linkage is negligible

$$\xi = -\frac{\mathrm{d}\Phi_{\text{ext}}}{\mathrm{d}t}\ . \tag{4.32}$$

However, there are other situations where the self-flux linkage cannot be neglected. A particular and very interesting case occurs when the variations of the external flux are null (for instance when $I = 0$). In that situation, the induced emf has to be calculated as

$$\xi = -\frac{d\Phi_{self}}{dt} \ . \qquad (4.33)$$

Induced emf when the external flux is null

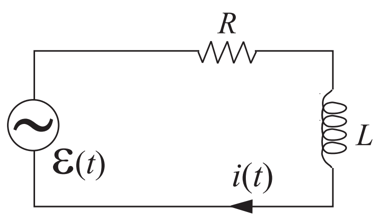
Activity 4.6:

- What are the reasons to separate the magnetic flux through a given circuit between external and self fluxes?
- Explain when expression (4.32) can be used and why. Do the same for expression (4.33).

4.4 Magnetic energy

We have already seen in Sec. 4.3.2 that the time evolution of *RL* circuit as the one shown in the figure is determined by the following differential equation:

$$\xi = Ri + L\frac{di}{dt} \ . \qquad (4.34)$$



Multiplying both sides of the equation by the current, i, we obtain

$$\xi i = Ri^2 + Li\frac{di}{dt} \qquad (4.35)$$

where the left member of (4.35) can be identified as the power supplied by the emf generator[3] and the right member has then to be the power "consumed" in the circuit. As the first term of this second member, Ri^2, is the power dissipated due to Joule effect in the resistance [see Eq. (5.21)], we can conclude that the second term, $Li di/dt$, has to be related to the inductive effect caused by the time-varying magnetic flux in the inductor. Hence, this term can be identified as the time rate of change of the *magnetic* energy stored in the inductor (remind that the induction effects are assumed to be localized exclusively in this element and also that the rate of change of energy has units of "power"). If the magnetic energy stored in the inductor is called U_B, the rate of change of this energy can be written as

$$\frac{dU_B}{dt} = Li\frac{di}{dt} = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) \ . \qquad (4.36)$$

Consequently, the magnetic energy stored in the inductor is given by

$$\boxed{U_B = \frac{1}{2}Li^2} \ . \qquad (4.37)$$

Magnetic energy stored in the inductor

[3] This fact is the result shown in Eq. (5.30), which will be studied in next chapter.

The energy stored in the inductor can be attributed to the existence of a magnetic field inside this element. To make this fact more apparent, let us consider that the inductor is a long solenoid (whose inductance $L = \mu_0 n^2 Sl$ was obtained in Example 4.3), which allows us to write

$$U_B = \frac{1}{2}\mu_0 n^2 Sl i^2 = \frac{1}{2\mu_0}\mu_0^2 n^2 i^2 Sl \,. \tag{4.38}$$

Since the magnitude of the magnetic field inside the solenoid was found to be $|\vec{B}| = \mu_0 ni$ [see Eq. (3.26)], the above expression can be rewritten as

$$U_B = \frac{|\vec{B}|^2}{2\mu_0}\mathcal{V} \tag{4.39}$$

with $\mathcal{V} = Sl$ being the volume of the solenoid. Thus we find that the volumetric density of magnetic energy of this inductor, u_B, is given by

Density of magnetic energy

$$\boxed{u_B = \frac{|\vec{B}|^2}{2\mu_0}} \,. \tag{4.40}$$

Although the above result has been obtained for the previous specific case (a long solenoid), more rigorous calculations show that the expression (4.40) is valid in general.

Activity 4.7:

- Why the term $Li di/dt$ can be identified with the time rate of change of the magnetic energy?
- Where is the magnetic energy "stored"?
- Does an inductor store electric energy? Does a capacitor store magnetic energy? Justify your answers.

4.5 (*) Maxwell's equations

In previous sections we have studied a number of laws (taken directly from experiments) that determine the behavior of the electric and magnetic fields. Among the many laws and expressions already considered, four out of them can be chosen so that they constitute the *basis* of electromagnetism, from which *all* electromagnetic phenomena can further be derived. These laws were formulated by James C. Maxwell ($\sim$1860) in a work that is recognized as one of the most successful synthesis in all history of physics. Besides this compilation work, Maxwell also noted an inconsistency in Ampere's law, which he solved by adding to this law an additional term related to a new type of current called *displacement current*. Maxwell's equations are four differential or integro-differential equations (here the equations will be presented in their integro-differential form) that summarize all the information we have acquired on the behavior of the electric and magnetic fields and the relations with their sources.

Maxwell made a revision of the laws of the electrostatic and magnetostatic field, extending them to general situation of **time-varying** electric and magnetic fields. His contributions can be summarized as follows.

- **Gauss's law for the electric field**
  Maxwell extended the validity of Gauss's law (which in its original form (2.11) was only applicable to static electric fields) to electric field that vary in time, $\vec{E} = \vec{E}(\vec{r}, t)$. Thus, Gauss's law can be written in general as

  Gauss's law for $\vec{E}(\vec{r}, t)$

  $$\oint_S \vec{E}(\vec{r}, t) \cdot d\vec{S} = \frac{Q_S(t)}{\epsilon_0} \qquad (4.41)$$

  where $Q_S(t)$ is the *total* charge (that now can be non-stationary) within the closed surface S and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m.

- **Gauss's law for the magnetic field**
  As the magnetic field lines are found to be endless (namely, there are no magnetic charges where the lines can start/finish), Maxwell proposed the following law for time-varying magnetic fields, $\vec{B} = \vec{B}(\vec{r}, t)$:

  $$\oint_S \vec{B}(\vec{r}, t) \cdot d\vec{S} = 0 \ . \qquad (4.42)$$

  Gauss's law for $\vec{B}(\vec{r}, t)$

  The magnetic flux through any closed surface is always null.

- **Faraday-Maxwell's law**
  The law of electromagnetic induction as established by Faraday was fundamentally linked to the presence of conductors, so the curve Γ in expression (4.10) coincided strictly with the loop wire of a circuit. Maxwell noted that the mathematical identity expressed by (4.10) did not have to be linked to the existence of conductors; that is, there is nothing in (4.10) requiring that the geometrical path Γ should correspond to the circuit wire. With that in mind, Faraday-Maxwell's law:

  $$\oint_\Gamma \vec{E}(\vec{r}, t) \cdot d\vec{l} = -\int_{S(\Gamma)} \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \cdot d\vec{S} \qquad (4.43)$$

  Faraday-Maxwell's law

  can be read in the following way: the circulation of the electric field along an *arbitrary curve*, Γ, is minus the time rate change of the magnetic flux through a surface $S(\Gamma)$ bounded by curve Γ. This conception of Faraday's law has a lot more physical insight than the original one, since it allows us to say that

  there will appear an electric field at any point of the space where there is a time-varying magnetic field.

- **Ampère-Maxwell's law**
  The extension of Ampère's to deal with non-steady electric currents and magnetic fields has to be done in the following way:

Ampère-Maxwell's law

$$\oint_{\Gamma} \vec{B}(\vec{r},t)\cdot d\vec{l} = \mu_0 \int_{S(\Gamma)} \left[\vec{J}(\vec{r},t) + \epsilon_0 \frac{\partial \vec{E}(\vec{r},t)}{\partial t}\right]\cdot d\vec{S}\ . \qquad (4.44)$$

One of the most important consequences to be drawn from the above equations comes after combining Faraday-Maxwell's equation with Ampère-Maxwell's equation for the case of free space [in the absence of any electric charges and currents, $\vec{J}(\vec{r},t) = 0$ in (4.44)]:

$$\oint_{\Gamma} \vec{E}(\vec{r},t)\cdot d\vec{l} = -\int_{S(\Gamma)} \frac{\partial \vec{B}(\vec{r},t)}{\partial t}\cdot d\vec{S} \qquad (4.43)$$

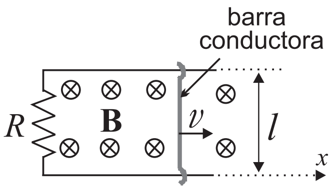
and

$$\oint_{\Gamma} \vec{B}(\vec{r},t)\cdot d\vec{l} = \mu_0\epsilon_0 \int_{S(\Gamma)} \frac{\partial \vec{E}(\vec{r},t)}{\partial t}\cdot d\vec{S}\ . \qquad (4.44)$$

At the light of the above two equations it can be inferred the existence of a electromagnetic disturbance in free space that can be *self-sustained*. Equation (4.43) states that the existence of a time-varying magnetic field causes the appearance of an electric field and, in turn, equation (4.44) states that the existence of a time-varying electric field results in the appearance of a magnetic field. Consequently, the existence of a time-varying magnetic field generates another magnetic field, which in turn generates other magnetic field, and so on (it would also happen starting from time-varying electric fields). Therefore, we have a situation where the electromagnetic fields would be sustaining themselves and they are their own cause and effect. In a further lesson we will see that this wave phenomenon is precisely the origin of electromagnetic waves.

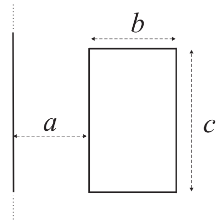
4.6 Problems

4.1: Inside a long solenoid of 600 turns, it is observed that the magnetic flux drops from 8.0×10^{-5} Wb to 3.0×10^{-5} Wb in 15 ms. Find the induced emf (1 Wb = 1 Tm^2).
Sol.: $\xi = -2$ V.



4.2: In a region with a uniform magnetic field $\vec{B}$, a metallic bar is moving at a constant speed v over two conductor bars joined at their left extremes by a resistor R, as shown in the figure. **a**) Find the induced emf in the loop and also the induced current and direction; **b**) What force is being applied to the moving bar for it to have such a constant speed?; **c**) Discuss the power balance between the applied power and the power dissipated in the resistor. **Note**: Neglect the self-flux.
Sol.: **a**) $\xi = -|\vec{B}|l|\vec{v}|$; **b**) $\vec{F}_a = Il|\vec{B}|\hat{\mathbf{x}}$; **c**) Power applied=$F_a v$ = Power dissipated = I^2R.

4.3: Find the mutual inductance between the straight and loop conductors in the figure.
Sol: $M = \dfrac{\mu_0 c}{2\pi} \ln\left(\dfrac{a+b}{a}\right)$.

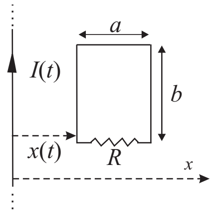


4.4: An straight infinite wire carries a current $I(t)$. A rectangular loop of sides a, b and resistance R is coplanar to this conductor and is moving according to a given motion equation $x(t)$. Find: **a**) the magnetic flux, $\Phi(t)$, through the loop; **b**) the induced emf in the loop, pointing out the terms related to the loop motion and to the time-changing magnetic field; **c**) the value of the induced current at time t if $I(t) = I_0$ and $x(t) = vt$ ($v > 0$). **Note**: Neglect the self flux.

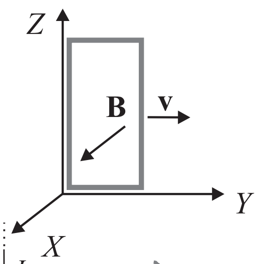
Sol.: a) $\Phi(t) = \dfrac{\mu_0 b I(t)}{2\pi} \ln\left(\dfrac{a + x(t)}{x(t)}\right)$;

b) $\xi(t) = -\dfrac{\mu_0 b}{2\pi}\left[\dfrac{\mathrm{d}I(t)}{\mathrm{d}t} \ln\left(\dfrac{a+x(t)}{x(t)}\right) - \dfrac{aI(t)v}{x(t)(a+x(t))}\right]$;

c) $I_{\text{ind}} = \dfrac{ab\mu_0 I_0}{2\pi R(at+vt^2)}$ counter-clockwise direction.



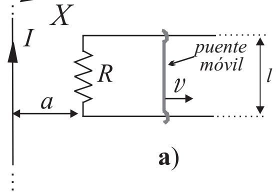
4.5: A rectangular loop of 2 Ω resistance is moving in the YZ plane in a region with a magnetic field $\vec{B} = (6 - y)\hat{\mathbf{x}}$ T. The dimensions of the loop are 0.5 m of height and 0.2 m of width. Assuming that at $t = 0$ the left side of the loop coincides with axis Z (as shown in the figure), find the induced current in the loop for the following cases: **a**) the loop is moving uniformly to the right with a speed of 2 m/s; **b**) after 100 s, if the loop is moving to the right with a uniform acceleration $a = 2$ m/s^2 (at $t = 0$ the loop was at rest). **c**) Repeat the above questions assuming that the motion is now parallel to axis Z. **Note**: Neglect the self flux.
Sol.: a) 0.1 A; **b**) 10 A; **c**) 0 A in both cases since the magnetic flux is not changing in time.

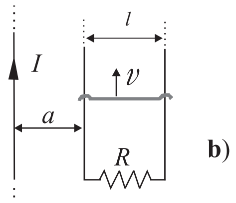


4.6: An straight infinite wire carries a current I. Another U-shaped conductor is coplanar to the wire with its short side having a resistance R and, through a mobile bridge moving at speed v, makes a rectangular loop of variable area. As shown in the figure, two cases will be considered: (**a**) R parallel and (**b**) R perpendicular to the straight wire. For each case, find the induced current and the required external force to be exerted on the mobile bridge to make it have a uniform motion at speed v. **Note**: the self flux is assumed to be negligible and the power delivered by a force $\vec{F}$ moving an object at velocity $\vec{v}$ is given by $P = \vec{F} \cdot \vec{v}$.

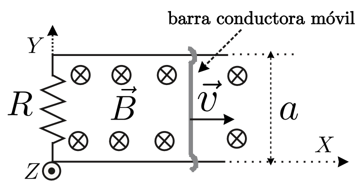
Sol.: a) $I_{\text{ind}} = \dfrac{\mu_0 I l v}{2\pi R(a+vt)}$, $F = \dfrac{v}{R}\left[\dfrac{I\mu_0 l}{2\pi(a+vt)}\right]^2$;

b) $I_{\text{ind}} = \dfrac{\mu_0 I v}{2\pi R} \ln\left(\dfrac{a+l}{a}\right)$, $F = \dfrac{v}{R}\left[\dfrac{\mu_0 I}{2\pi} \ln\left(\dfrac{a+l}{a}\right)\right]^2$.

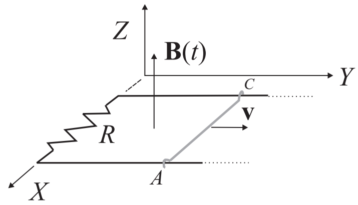


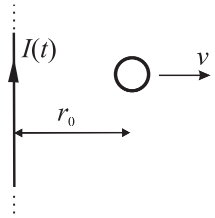


4.7: A circuit of varying area is made up of a bar conductor of mass m that can move without friction on two conductor rails separated a distance $a = 0.8$ m. The two rails are connected through a resistor $R = 10\,\Omega$, as shown in the figure. The circuit is in a region with a uniform magnetic field $\vec{B} = -0.5\,\hat{\mathbf{z}}$ T directed towards the paper. If we apply a force $\vec{F}_a = 0.032\hat{\mathbf{x}}$ N, the bar has a uniform motion with velocity $\vec{v}$ (constant). In this situation, find: (**a**) the magnetic force, $\vec{F}_m$, acting on the bar; (**b**) the current induced in the circuit by reasoningly indicating its direction according to Lenz's law; (**c**) the induced electromotive force; (**d**) the velocity $\vec{v}$ of the bar.
Sol.: (**a**) $\vec{F}_m = -0.032\hat{\mathbf{x}}$ N; (**b**) 80 mA, counterclockwise ; (**c**) $\xi = 0$ V; (**d**) $\vec{v} = 2\hat{\mathbf{x}}$ m/s .



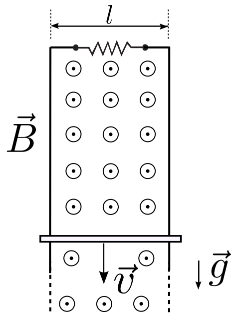
4.8: In a region with a magnetic field $\vec{B}(t) = (2 + 0.5t^2)\,\hat{\mathbf{z}}$ T (t en seconds) there is an U-shaped conductor with a resistor $R = 10\ \Omega$. This conductor is form a rectangular loop with a moving conductor bar AC, of length $l = 1$ m and mass m kg. If the motion equation of the bar AC is $y(t) = 3t^2$ m, find: **a**) the magnetic flux through the loop; **b**) the induced emf in the loop; **c**) the induced current with its direction; **d**) the y-component of the force exerted by the magnetic field on the moving bar; **e**) the external force that has to be applied to the bar so that it has the given motion equation. **Note**: Neglect the self flux.
Sol.: a) $\Phi(t) = 6t^2 + 1.5t^4$ weber; **b**) $\xi(t) = -(12t + 6t^3)$ V; **c**) $I_{\text{ind}}(t) = 1.2t + 0.6t^3$ clockwise; **d**) $\vec{F}_{\text{mag}}(t) = -(2.4t + 1.8t^3 + 0.3t^5)\,\hat{\mathbf{y}}$; **e**) $\vec{F}_{\text{appl}}(t) = (6m + 2.4t + 1.8t^3 + 0.3t^5)\,\hat{\mathbf{y}}$.





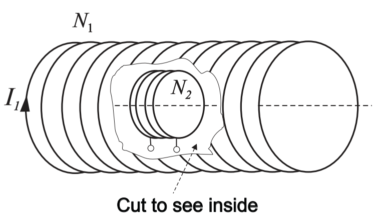
4.9: A current $I(t) = at$ (with $a = 0.7$ A/s) is flowing through a long straight wire. Next to this wire there is a circular loop of radius $b = 5$ mm and resistance $R = 0.2$ mΩ. This loop is moving away from the wire with uniform speed v, being at $t = 0$ at a distance $r_0 = 12.5$ mm from the wire. Find: **a)** the magnetic flux through the loop; **b)** the induced emf; **c)** the induced current at $t = 0$ with its direction. **Note**: Due to the small size of the circular loop it can be assumed that the magnetic field is *uniform* inside the loop with (the value is then determined by the magnetic field at the center of the loop).

Sol: **a)** $\Phi(t) = \dfrac{\mu_0 b^2 at}{2(r_0 + vt)}$; **b)** $\xi(t) = \dfrac{\mu_0 b^2 a r_0}{2(r_0 + vt)^2}$; **c)** $I(0) = 4.39\,\mu$A, counter clockwise.



4.10: (*) A conductor bar has a mass m and length l. This bar slides with no friction under the gravitational action on the two parallel arms of an U-shaped conductor. As shown in the figure, there is a uniform magnetic field in that region. **a**) Find the motion equation of the bar (it should be considered that the forces acting on the bar are the gravitational and magnetic forces). **b**) Find the falling limit speed of the bar (this limit speed is reached when the bar is no longer being accelerated)); **c**) In the previous situation of limit speed, carry out a power balance involving the potential gravitational energy and the energy dissipated in the bar due to Joule's effect). **Note**: neglect the self flux.

Sol.: **a**) $-mg - \dfrac{B^2 l^2}{R}\dfrac{dy}{dt} = m\dfrac{d^2y}{dt^2}$; **b**) $\mathbf{v}_{\rm lim} = -\dfrac{mgR}{B^2 l^2}\,\hat{\mathbf{y}}$; **c**) $-\dfrac{d}{dt}(mgy) = I^2_{\rm ind}R$; namely, the potential energy of the bar is decreasing and turning into heat in the resistor.

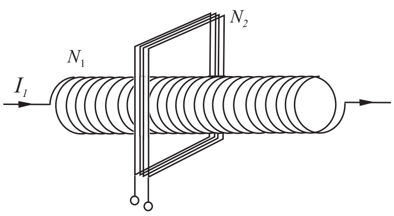


4.11: The attached figure shows a long solenoid of length l_1 with N_1 turns. Inside this solenoid and coaxial with it there is another coil with radius R_2 and N_2 turns. Find: **a**) the mutual inductance of this system; **b**) the induced emf in the smaller solenoid when the bigger one carries a current $I_1(t) = I_0\cos(\omega t)$. **c**) Repeat the above questions if the axis of smaller coil is forming and angle θ with the axis of the long solenoid.

Sol.: **a**) $M = \mu_0 n_1 n_2 \mathcal{V}_2$, with $\mathcal{V}_2$ being the volume of the inner solenoid and $n_i = N_i/l_i$; **b**) $\xi = M\omega I_0 \mathrm{sen}(\omega t)$; **c**) in this case, the above results should be multiplied by a factor $\cos(\theta)$.

4.12: Find the induced emf in a current-carrying coil of inductance $L = 23$ mH when: **a**) the current is 25 mA at $t = 0$ and increases with a rate of 37 mA/s; **b**) the current is null at $t = 0$ and increases with a rate of 37 mA/s; **c**) the current is 125 mA at $t = 0$ and decreases with a rate of 37 mA/s; **d**) the current is 125 mA at $t = 0$ and does not vary.

Sol.: in cases **a**), **b**) and **c**), $\xi = 851 \times 10^{-6}$ V, with the polarity of the emf being different in case (c); **d**) $\xi = 0$.



4.13: The figure shows a long solenoid of length l, cross section S, N_1 turns which carries a current $i(t) = I_0\mathrm{sen}(\omega t)$. Around this solenoid there is rectangular loop with N_2 turns. Find: **a**) the magnetic field, $B(t)$, inside the solenoid; **b**) the self inductance, L, of the solenoid; **c**) the voltage, $V_1(t)$, at the terminals of the solenoid; **d**) the magnetic flux through the rectangular loop, $\Phi_2(t)$, and **e**) the induced emf, $\xi_2(t)$ in this loop. **f**) Verify that the ratio of voltage drops in the above two elements is given by $V_2/V_1 = N_2/N_1$ (namely, this system acts as a *transformer*).

Sol.: **a**) $B(t) = \dfrac{\mu_0 N_1 I_0}{l}\mathrm{sen}(\omega t)$; **b**) $L = \dfrac{\mu_0 N_1^2 S}{l}$; **c**) $V_1(t) = L\omega I_0\cos(\omega t)$;
d) $\Phi_2(t) = \mu_0 I_0 N_1 N_2 S\mathrm{sen}(\omega t)/l$; **e**) $\xi_2(t) = -\mu_0 I_0 N_1 N_2 S\omega\cos(\omega t)/l$.

Part III

CIRCUITS

Lesson 5

Circuits of Direct Current

5.1 Introduction

In the previous lessons we have been studying the laws and behavior of the electric and magnetic fields. Now in the present lesson we will start to study the flow of charged particles in circuits; namely, the electric current. Depending on the type of motion of the charge carriers, the electric current is classified as steady current or time-varying current. The steady current or **direct current** (DC) is characterized by a flow of charges that remains stationary; for example, when the free electrons in a wire have a uniform motion.[1] When the flow of charges varies in time in a harmonic way (i.e., a mathematical variation described by functions cosine or sine), then this current is called **alternating current** (**AC**).

The goal of the following three lessons is then the analysis of DC, transient regime, and AC **linear** circuits, both for their own importance in current technology as well as a first step to the further study of more complex electronic circuits. In general we can define a linear circuit as the circuit that obeys the superposition principle. It means that the output of the circuit $F(x)$ when a linear combination of signals $ax_1(t) + bx_2(t)$ is applied to it is equal to the linear combination of the outputs due to the signals $x_1(t)$ and $x_2(t)$ applied separately: $F(ax_1+bx_2) = aF(x_1)+bF(x_2)$. In practice, it means that a linear circuit is composed of any combintation of the three following basic linear passive components: resistors (R), capacitors (C), and inductors (L). These elements can be driven by DC and/or AC voltage sources and may be combined in any arbitrary way: circuits with only resistors, RC circuits, RL circuits, LC circuits, and RLC circuits (with the acronyms indicating the components involved).

[1] It is interesting to note that if the charge flow remains time-invariant (DC), then the charge per unit time through any surface does not increase/decrease and, therefore, the charge distribution remains constant in time. This means that even though the charges are *moving*, we can still apply Electrostatics. However, the *charges* inside the conductors do not generate any electric field because of the perfect cancellation among positive and negative charges.

Circuits with only resistors and driven by DC voltage sources will be analyzed in the present lesson by using Kirchhoff's rules, which will be deduced starting from the basic laws of Electrostatic and charge conservation. After deducing these rules, we will discuss the power supply in DC circuits and, in particular, the concept of electromotive force (emf). In the following lesson, it will be studied the transient responses of RC and RL circuits, and in Chapter 7 we will deal with alternating-current RLC circuits.

5.2 Current density vector

A measure of the electric current is provided by the intensity of the current (or simply "current") I, defined as

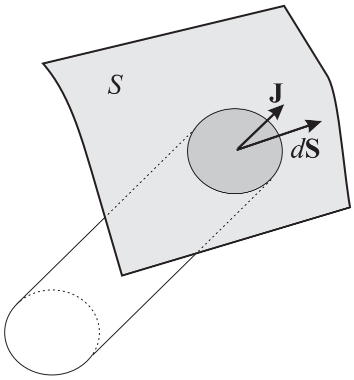
Current

$$I = \frac{dQ}{dt} \tag{5.1}$$

namely, *the time rate at which charge Q flows through certain surface S*. The SI unit of current is the **ampere** (A), defined as

Unit of current: 1 ampere (A)

$$1\,\text{ampere} = \frac{1\,\text{coulomb}}{1\,\text{second}} \quad ; \quad 1\,\text{A} = 1\,\text{C/s}\,.$$



The previous definition of current clearly shows a dependence of this quantity with respect to the surface S the current is passing through. This fact makes it apparent the convenience of expressing the current in terms of the flux of certain vector (see Appendix 1.9.4), which is called **current density** vector $\vec{J}$, through the surface S:

$$\boxed{I = \int_S \vec{J} \cdot d\vec{S}}\,. \tag{5.2}$$

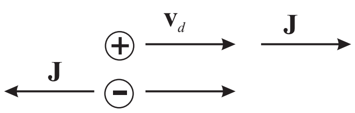
We can observe that the units of $\vec{J}$ will be of current per unit area; namely, A/m^2. The magnitude of this vector represents the amount of charge per unit time that passes through a unit surface element perpendicular to the direction of the charge flow. Thus, a flow of charges with n particles per unit volume, each one of charge q and drift velocity $\vec{v}_d$, will make a corresponding current density vector given by

Current density vector

$$\boxed{\vec{J} = nq\vec{v}_d}\,. \tag{5.3}$$

If there are more than one type of charge carriers, expression (5.3) can be generalized as

$$\vec{J} = \sum_i n_i q_i \vec{v}_{d,i}\,. \tag{5.4}$$



It is interesting to note (see attached figure) that if we have positive and negative charges flowing in the same direction, the current associated with the negative charges will be directed opposite to the one with positive charges. Thus we can see that a current formed by electrons (the usual current in circuits) has a direction opposed to the motion of the electrons.

Activity 5.1:

- Can we have current in a finite segment of a cut wire? Justify your answer.
- Explain the reasons why the definition of current given in Eq. (5.1) is incomplete? Why is it more convenient to define the current as the flux of the current density vector?
- Try to guess from the definition of $\vec{J}$ given in Eq. (5.3) the conditions that a DC has to satisfy.
- Starting from Eq. (5.3), deduce the expression for the DC that flows through the transverse section of a wire.

EXAMPLE 5.1 *Find the drift speed of the mobile electrons in a wire of cooper with radius r = 0.8 mm and a carrying-current I = 20 mA (mass density of Cu ρ = 8.93 g/cm^3, atomic mass A = 63.55 g).*

First we should note that, in the case of direct current in a wire (assuming constant cross section), the current is found to be

$$I = \int_S \vec{J} \cdot d\vec{S} = \int_S |\vec{J}| dS = |\vec{J}| \int_S dS = |\vec{J}| S \tag{5.5}$$

where it has been assumed that $\vec{J} \parallel d\vec{S}$ and that $|\vec{J}|$ is uniform over the cross section of the wire (i.e., both n and $\vec{v}$ do not change over the cross section of the wire).

As $|\vec{J}| = nq|\vec{v}_d|$, from (5.5) it can be deduced that the magnitude of the drift velocity of the mobile charge carriers can be written as

$$|\vec{v}_d| = \frac{I}{nqS} \ .$$

Since the current, the elementary charge q, and the cross section can be calculated from the data of the problem, $|\vec{v}_d|$ can be determined after obtaining the value of n. To calculate the number of free electrons per m^3 in copper, we assume that each atom of Cu provides one free electron to the metal, so the number of free electrons will coincide with the number of Cu atoms per m^3, n_a. To obtain this last quantity, we can calculate the number of moles per m^3, χ, and then multiply this number by the number of atoms in a mol, $N_A = 6.02 \times 10^{23}$; i.e., $n_a = \chi N_A$. In turn, the number of mols per m^3 can be obtained as

$$\chi = \frac{\text{mass of 1 m}^3}{\text{mass of 1 mol}} = \frac{\rho}{A}$$

and thus n can be obtained from the following expression:

$$n = N_A \frac{\rho}{A} \ .$$

Using now the data of the problem,

$$n = 6.02 \times 10^{23} \, \frac{8.93 \times 10^6}{63.55} = 8.46 \times 10^{28} \text{ electrons/m}^3 \ .$$

The drift speed is then given by

$$|\vec{v}_d| = \frac{20 \times 10^{-3}}{8.46 \times 10^{28} \cdot 1.6 \times 10^{-19} \cdot \pi(0.8 \times 10^{-3})^2} = 7.43 \times 10^{-7}\ \mathrm{m/s}\ .$$

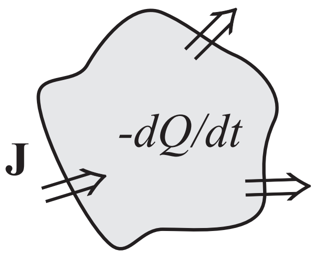
Note the extremely small value of the drift speed of electrons inside the wire. However, this small speed does not mean we have to wait a long time for the electric current to flow when you turn on a light switch. Something similar occurs in a column of soldiers responding to the voice of "go"; although the speed of each soldier may be slow, the column itself starts to move almost instantaneously.

Charge continuity equation

The principle of local conservation of charge (see Sec. 2.1) stated that if certain charge disappears from one place, this same charge must have traveled to pop up in another place. Since the traveling charge constitutes an electric current, this principle can be expressed in terms of the electric current as

the current flowing through the closed surface that bounds a certain region is minus the time rate of change of the charge inside that region.

This law simply states that if there are 5 charges coming into a given region every second and 2 charges coming out from that region per second, then the charge inside the region is increasing at a rate of 3 charges per second. Mathematically, the above principle is known as the **continuity equation** and can be expressed as



$$\oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ}{dt} \tag{5.6}$$

where the minus sign indicates that an *outward* flux (with positive sign) corresponds to a *decreasing* inner charge.

If the current is steady (DC), the total amount of charge in the considered region cannot change (otherwise it would not be a "direct current"), which implies that

$$\frac{dQ}{dt} = 0$$

and thus the continuity equation now becomes

Continuity equation in stationary regime

$$\oint_S \vec{J} \cdot d\vec{S} = 0\ . \tag{5.7}$$

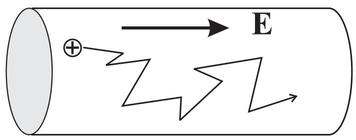
It means that the direct current flowing through any closed surface that bounds a given region is null; or equivalently, the same amount of charges that comes into a region is the one that comes out from such a region.

Activity 5.2:

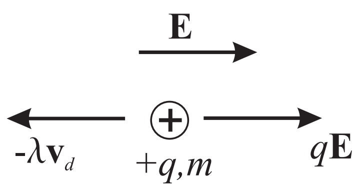
- Express with words the meaning of Eq. (5.6).
- Apply Eq. (5.6) to the DC carried by a wire and comment the resulting conclusions.
- Can there be accumulation of charges in any point of a DC circuit? Justify your answer.

5.3 Conductivity. Ohm's law

5.3.1 Electric conductivity



In the most basic model of a **real conductor**,[2] the mobile charges of the conductor are accelerated under the influence of an applied electric field, but this continuous gain of kinetic energy is compensated by an equivalent energy loss coming from the continuous bounces suffered by the mobile charges (electrons) when they strike the *fixed* ions (positively charged) of the conductor material. This simultaneous process of acceleration caused by the external electric field and deceleration from the continuous bounces turns out to be equivalent to an average motion of the charge carriers at constant speed.[3] This complicated internal process can globally be regarded as the combined result of a driving force, $q\vec{E}$, plus a *dissipative* force opposing the motion given by $\vec{F}_{\rm dis} = -\lambda\vec{v}_d$ (friction). According to this simple model, the law of motion (2nd Newton's law) of a charge particle inside a real conductor can be written as



$$m\frac{\mathrm{d}\vec{v}_d}{\mathrm{d}t} = q\vec{E} - \lambda\vec{v}_d\ . \tag{5.8}$$

In stationary regime, when the drift velocity of the charge carriers remains constant (namely, $\mathrm{d}\vec{v}_d/\mathrm{d}t = 0$), the velocity will become

$$\vec{v}_d = \frac{q}{\lambda}\vec{E}$$

or, equivalently written as

$$\vec{v}_d = \mu\vec{E} \tag{5.9}$$

which shows the linear relation existing between the drift velocity and the applied electric field through the parameter $\mu = q/\lambda$ known as **mobility** [its

[2] It is very important to distinguish the present case of a *real* conductor from the case of a *perfect* conductor, which was already studied in Sec. 2.4. It should be reminded that there cannot be electric field inside perfect conductors. On the contrary, there can be electric fields inside a real conductor.

[3] A similar situation is found for the falling drops of rainwater. Every drop of water is accelerated by the gravitational field and, in turn, slowed down because of the collisions with the air molecules found in its fall. The overall result is that water droplets fall at an approximately constant rate.

SI units are m²/(Vs)]. It should be noted that this parameter depends exclusively on the characteristics of the conductor material.

Now, taking into account that $\vec{J} = nq\vec{v}_d$ as well as expression (5.9), the current density vector can be written as

$$\vec{J} = nq\mu\vec{E} \tag{5.10}$$

or, equivalently,

$$\vec{J} = \frac{nq^2}{\lambda}\vec{E}\,. \tag{5.11}$$

The above expression makes it apparent the existence of a linear relation between the current density vector and the applied electric field, which can be expressed as

Ohm's law for $\vec{J}$ and $\vec{E}$

$$\boxed{\vec{J} = \sigma\vec{E}} \tag{5.12}$$

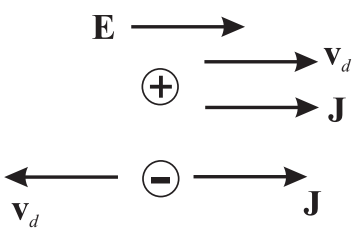
with σ being a parameter associated with the material called **electrical conductivity**, given by

Electrical conductivity

$$\boxed{\sigma = \frac{nq^2}{\lambda}} \tag{5.13}$$

or

$$\sigma = qn\mu\,. \tag{5.14}$$



The electric conductivity measures the "capability" of electrical conduction of the material, being higher for those materials where the electric current flows more easily (note that σ is inversely proportional to the parameter λ).

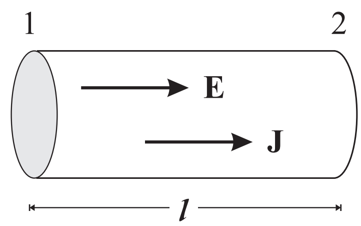
Interestingly, regardless of the sign of the charge [since this charge appears squared in (5.13)], the direction of the flowing current is always the same as the one of the applied electric field.

Activity 5.3:

- Describe and explain the main differences between perfect and real conductors.
- Find common physical situations where a motion at constant speed is caused by the counterbalance of two opposed forces.
- Give some reasons why it is somewhat expected that $\sigma \propto n$ but $\sigma \propto \lambda^{-1}$.
- A perfect conductor is the limiting case of a real conductor when the electrical conductivity tends to infinity ($\sigma \to \infty$). Explain why there can exist current in perfect conductors although $\vec{E}_{\rm cond} = 0$.
- Is the direction of $\vec{J}$ always the same as that of $\vec{E}$? Is the direction of $\vec{v}_d$ always the same as that of $\vec{E}$? Justify your answers.

5.3.2 Circuital Ohm's law

As commented above, there can exist an applied electric field, $\vec{E}$, *inside* a wire conductor with conductivity σ (unlike to what happens inside a perfect conductor, where $\vec{E}_{\text{int}} = 0$). This electric field will "push" the free charges of the real conductor, thus giving rise to an electric current, I. According to (2.20), the line integral of the electric field between two points of the conductor will be the potential difference between these two points; namely,



$$\int_1^2 \vec{E} \cdot d\vec{l} = V(1) - V(2) \equiv V_{12} \,.$$

This potential difference is usually called **voltage drop** or electric tension (or simply **tension**). As the electric field can be related to the current through Ohm's law [see (5.12)], we can write

$$V_{12} = \int_1^2 \frac{\vec{J}}{\sigma} \cdot d\vec{l} \,. \tag{5.15}$$

If the area of the transverse section of the wire conductor is S, the current density vector can be written as

$$\vec{J} = \frac{I}{S}\,\hat{\mathbf{u}} \tag{5.16}$$

(with $\hat{\mathbf{u}}$ being the unit vector in the direction of the conductor), and the voltage drop (5.15) results in

$$V_{12} = \int_1^2 \frac{\vec{J}}{\sigma} \cdot d\vec{l} = \frac{I}{\sigma S} \int_1^2 \hat{\mathbf{u}} \cdot d\vec{l} = \frac{I}{\sigma S} \int_1^2 dl = \frac{I}{\sigma S}\, l \tag{5.17}$$

where l is the distance between points 1 and 2 along the wire.

It should be noted that the above expression shows a *linear* relation between the voltage drop, V_{12}, and the flowing current, I. This relation can be written in general as

Circuital Ohm's law

$$\boxed{V = RI} \tag{5.18}$$

which is known as *circuital Ohm's law* (stated by G. S. Ohm at 1827), where the parameter R, called **resistance**, of the wire conductor is given by

Resistance of a wire conductor

$$R = \frac{l}{\sigma S} \,. \tag{5.19}$$

The resistance is a characteristic of each conductor and depends on its material constitution (through σ) and its geometry (section and length). The SI unit of resistance is the **ohm** (Ω), being

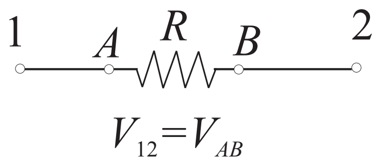
Unit of Resistance:
1 ohm (Ω)

$$1\,\text{ohm} = \frac{1\,\text{volt}}{1\,\text{amp}} \quad , \quad 1\,\Omega = 1\,\text{V/A} \,.$$

Unlike what happens in a *perfect* conductor, which is equipotential, the presence of a resistance (that is, the existence of power loss of the mobile charge carriers because of the bounces with fixed ions) gives rise to a potential difference, or tension, along the current-carrying *real* conductor.

Unit of electrical conductivity: $1\,(\Omega\text{m})^{-1}$

From (5.19) we find that the SI units of conductivity σ are inversely proportional to those of resistance and length, and thus the units of conductivity are usually given as $(\Omega\text{m})^{-1}$. The electrical conductivity is one of the most varying quantities from one material to another: from $10^{-15}(\Omega\text{m})^{-1}$ for very poor conductors (dielectrics) up to $10^{8}(\Omega\text{m})^{-1}$ for metals as cooper or silver. Since the conductivity of metals is typically very high (and thereby their resistance is very low), in most practical situations involving circuits we will consider that there is no voltage drop in the metallic wires and that all the voltage drops occur in specific elements with lower conductivity called **resistors**.

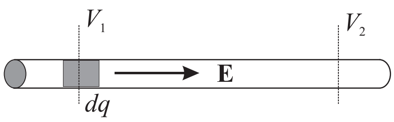


Activity 5.4:

- Can we have voltage drop along perfect conductors? Justify your answer.
- In which practical circumstances we should not neglect the voltage drop along wire conductors?
- Which are the physical assumptions taken to derive Ohm's law? Try to find a practical example where this law is not applicable.

5.3.3 Joule effect

In the previous sections it has been discussed that the presence of a flowing current in a real conductor gives rise to an energy dissipative process caused by the continuous bounces of the mobile carriers with the fixed ions. This dissipative process implies that the charge carriers are steadily loosing part of their kinetic energy in the form of heat supplied to the conductor material as well as to the environment. The presence of a voltage drop in a current-carrying real conductor implies then that the applied electric field has to do certain work to move a differential charge, dq, from a point of potential V_1 to a point of potential V_2.



If the voltage drop between these two points is expressed as $V = V_1 - V_2$, from (2.39) the required differential work done by the electric field to translate a differential charge dq between these two points is given by

$$dW = dq(V_1 - V_2) = dqV\,.$$

Taking now into account that the moving charge, dq, is part of a current I flowing through the conductor, we can write that dq = Idt. Thus, the differential work done by the electric field can be expressed as

$$dW = IVdt\,. \tag{5.20}$$

Consequently, the time rate at which this work is done, which will coincide with the power $P = \mathrm{d}W/\mathrm{d}t$ dissipated as heat in the resistor, is given by

$$\boxed{P = IV = I^2R = V^2/R} \ . \tag{5.21}$$

Joule's law

This important law was experimentally deduced by J. P. Joule at 1841.

Activity 5.5:

- Explain the reasons that make it necessary to do an external work to move a charge between two points of a real conductor.
- Is the power given in Eq. (5.21) linear with the current and voltage drop? Justify your answer.
- Describe when it is convenient to use each one of the three possible expressions of the power.

EXAMPLE 5.2 *Two conductors of the same length and the same radius are connected through the same potential difference. If one of the conductors has twice the resistance as the other, which of the two conductors will dissipate more power?*

If the resistance of conductor #1 is $R_1 = R$ and the one of conductor #2 is $R_2 = 2R$, then from (5.21) the powers dissipated in each conductor are

$$\begin{aligned} P_1 &= \frac{V^2}{R_1} = \frac{V^2}{R} \\ P_2 &= \frac{V^2}{R_2} = \frac{V^2}{2R} \ , \end{aligned}$$

and therefore

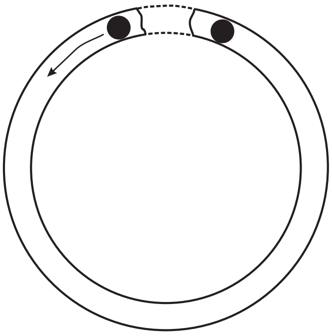
$$P_1 = 2P_2 \ .$$

Provided both conductors have the same potential difference, this means that the conductor with lower resistance will dissipate more power.

Try to find out what happens if both conductors carry the same current.

5.4 Electromotive force

Before discussing how to generate a steady electric current, we will analyze the case of a steady current of "mass". In the attached figure, small balls are moving inside a closed tube. The key question here is: can there be a steady current of mass flowing continuously in the above situation? Certainly, under the exclusive effect of the gravitational field, a ball that is released from a certain point cannot get to a point higher than the one from which it departed (in other words, the ball cannot get to a point of potential higher than the starting one). Moreover, if we take into account the unavoidable



existence of *friction*, there will be an additional loss of kinetic energy transformed into heat that will preclude the ball not even to reach the point of theoretical maximum height but one of lesser height. In brief, the ball in the previous device can never have a continuous circular motion if driven only by the gravitational field. Instead, it will undergo an oscillatory motion that will disappear after a few oscillations. Therefore, we can state that the gravitational field, which is *conservative*, is not able to maintain a steady current of mass. To achieve that steady current, it would be necessary to add a new element to the previous system that supplies the additional energy that the balls need to complete the circular motion. Clearly, this additional element has to produce a field of different nature than the gravitational one; i.e., a non-conservative field.

The same question may now be proposed respecting whether an electrostatic field can maintain a steady flow of electrical charges. For reasons similar to the ones given before, the answer is NO because of the *conservative* nature of the electrostatic field. In other words, the work per unit charge done by a electrostatic field, $\vec{E}_s$, in a circular path is null:

$$\frac{W}{q} = \oint \vec{E}_s \cdot d\vec{l} = 0$$

as discussed in relation to expression (2.19) (conservative nature of $\vec{E}_s$). Since there is always an energy loss coming from Joule effect in any real situation, a steady flow of charges requires the action of an external element to provide the required "external impulse" to the mobile charges that compensates their constant loss of power. Clearly, the agent of this external impulse to the charges cannot be an electrostatic field because this field would always provide zero energy per cycle.

Since the impulse provided to the mobile charge carriers may be located in a specific part of the circuit or distributed along the circuit, what actually matters is the overall effect of the force per unit charge, $\vec{f}$, that causes this impulse integrated along the whole circuit length. The circulation of this force per unit charge is called **electromotive force**, ξ, usually denoted as "emf":

Electromotive force (emf)

$$\xi = \oint_{\text{circuit}} \vec{f} \cdot d\vec{l}\,. \tag{5.22}$$

We can observe that the above integral represents the tangential force per unit charge integrated over the complete circuit length or, equivalently, the work (energy) per unit charge supplied by the external agent in each cycle. It should be noted that the denomination of electromotive "force" is rather unfortunate, since the units of ξ are "work per unit charge", which are precisely units of electric potential [remind that, according to (2.20), the potential difference is defined as the line integral of the electrostatic field]. Therefore, the units of emf are volts. However, it is key to notice that the electromotive force is NOT an electric potential difference,

Unit of emf : 1 volt (V)

$$\xi \neq \Delta V$$

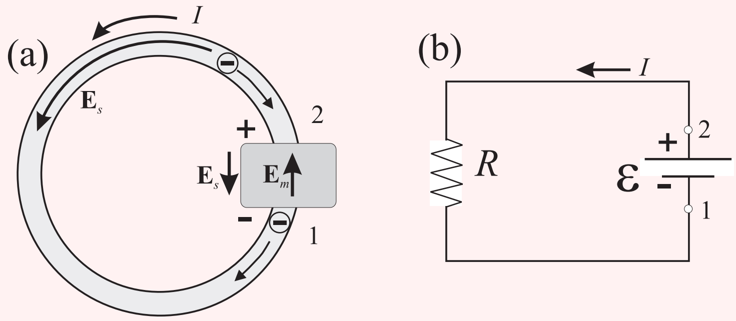


Figure 5.1: (a) Scheme of the action of a source of emf. (b) Circuital representation of the previous scheme.

since the agent of the emf can never be an electrostatic field, $\vec{E}_s$ (by definition, a field with null circulation) but an electric field of non-electrostatic nature, which will be denoted here as $\vec{E}_m$. The physical agent specifically responsible for this non-electrostatic electric field can be very diverse; for instance, forces of chemical origin in a battery, mechanical forces in a Van der Graaff generator, light in a photocell, mechanic pressure in a piezoelectric crystal, etc...

We can therefore conclude that the existence of an electric current in a circuit requires the action of an external agent called **source of emf** (or, more usually, **voltage source**), which provides the electric field necessary to "push" the positive/negative charges towards regions of increasing/decreasing potentials against the action of the electrostatic field. This fact is shown in Fig. 5.1(a), where the circulation of the total electric field,

$$\vec{E}_T = \vec{E}_s + \vec{E}_m \,, \tag{5.23}$$

due to the sum of the electrostatic and the non-electrostatic fields, is

$$\begin{aligned} \xi &= \oint \vec{E}_T \cdot d\vec{l} \\ &= \oint \vec{E}_s \cdot d\vec{l} + \oint \vec{E}_m \cdot d\vec{l} \,. \end{aligned} \tag{5.24}$$

Taking now into account that the electrostatic field always fulfills

$$\oint \vec{E}_s \cdot d\vec{l} = 0$$

and that $\vec{E}_m$ is bounded to the region between points 1 and 2, namely

$$\oint \vec{E}_m \cdot d\vec{l} = \int_1^2 \vec{E}_m \cdot d\vec{l}$$

it is finally found that

$$\xi = \int_1^2 \vec{E}_m \cdot d\vec{l} \,. \tag{5.25}$$

The circuital representation of the above situation is shown in Fig. 5.1(b).

Activity 5.6:

- If you have an elastic ball in your hand, what would you have to do if you want the ball to reach a higher position than your hand after bouncing on the ground? Now, explain this action in physical terms and identify the forces involved in the phenomenon as well as their nature (conservative or not).
- Can an electrostatic field produce an steady electric current? Justify your answer.
- An electrostatic field cannot move charges from one point to another. Right or False? Justify your answer.
- Find and explain whether the following statements are right or false.
  - The emf is the force produced inside the batteries.
  - The units of the electromotive force are newtons.
  - A non-electrostatic field always produce a voltage drop.
  - An electrostatic field can sometimes be the cause of a voltage drop.
  - There is no relation between the emf and the voltage drop in a given circuit.
  - A voltage source is not always necessary to have a direct current in a circuit.

5.4.1 Power and energy supplied by the emf source

The differential work done by the emf source (more specifically by its corresponding non-electrostatic field $\vec{E}_m$) to move a differential charge dq along the closed path of the circuit is given by

$$\mathrm{d}W = \mathrm{d}q \oint_{\mathrm{circ.}} \vec{E}_m \cdot \mathrm{d}\vec{l} = \mathrm{d}q\xi \,. \tag{5.26}$$

The total work that can be done by this source will be limited by the total amount of charge that can be set in motion, say Q_T. Thus after integrating (5.26) (noting that the emf is independent of the amount of charge it is acting on) it is obtained that

$$W_\xi = Q_T\xi \,. \tag{5.27}$$

Under ideal conditions, the total work done by the voltage source should be equal to the energy previously stored in the battery, U_ξ. Thus we can conclude that the energy stored in the battery is given by

$$\boxed{U_\xi = Q_T\xi} \,. \tag{5.28}$$

Taking now into account that the differential charge $\mathrm{d}q$ actually belongs to the flowing current in the circuit, namely $\mathrm{d}q = I\mathrm{d}t$, (5.26) can also be written as

$$\mathrm{d}W = I\xi\mathrm{d}t \,. \tag{5.29}$$

As the differential of work can be expressed in terms of the power as

$$\mathrm{d}W = \left(\frac{\mathrm{d}W}{\mathrm{d}t}\right)\mathrm{d}t = P\,\mathrm{d}t$$

it is readily deduced that the power, P_ξ, supplied by the voltage source is

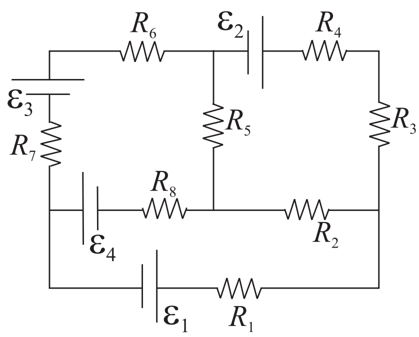
$$\boxed{P_\xi = I\xi} \,. \tag{5.30}$$

Power supplied by the emf source

Activity 5.7:

- Deduce that expression (5.26) actually corresponds to the differential work supplied by the voltage source to move a differential charge along the circuit closed path.
- Why is the emf independent of the amount of charge which is acting on?
- Find the amount of energy stored in the battery of your smartphone (take a look at the specifications written in the battery). Now deduce how long your smartphone can keep working (also written in the specifications of the battery).
- Which is the fundamental principle used to deduce (5.28)?

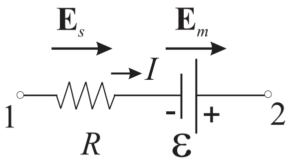
5.5 Kirchhoff's rules



In order to analyze circuits formed by voltage sources and resistors (see an example in the attached figure), we should come back to the basic electromagnetic theory studied in previous lessons to obtain some rules that can help us in this task. First we will define a few important terms commonly used in circuit theory:

- **Branch**: interconnection of elements in series; namely, with the same current flowing through them.
- **Node**: point where two or more branches are connected.
- **Network**: a set of interconnected branches.
- **Loop**: any closed path in the circuit resulting from going through a set of nodes without crossing any of the intermediate node twice.

5.5.1 Kirchhoff's voltage rule



If we compute the line integral of the total electric field, $\vec{E}_T = \vec{E}_s + \vec{E}_m$, from points 1 $\longrightarrow$ 2 along a circuit branch as the one shown in the attached figure, we find that

$$\int_1^2 \vec{E}_T \cdot \mathrm{d}\vec{l} = \int_1^2 \vec{E}_s \cdot \mathrm{d}\vec{l} + \int_1^2 \vec{E}_m \cdot \mathrm{d}\vec{l} \,. \tag{5.31}$$

According to (5.12), the first member of the above expression can be rewritten as

$$\int_1^2 \vec{E}_T \cdot \mathrm{d}\vec{l} = \int_1^2 \frac{\vec{J}}{\sigma} \cdot \mathrm{d}\vec{l} \,. \tag{5.32}$$

Assuming now that expression (5.16) is valid, after some derivations it is obtained

$$\int_1^2 \frac{\vec{J}}{\sigma} \cdot \mathrm{d}\vec{l} = \int_1^2 \frac{I}{\sigma S}\,\mathrm{d}l = IR \,. \tag{5.33}$$

It should be noted that, in the previous derivation, the current is assumed to be flowing along the same direction as the one from points 1 $\longrightarrow$ 2.

The first term of the RHS member in (5.31) is just the line integral of the electrostatic field from points 1 $\longrightarrow$ 2; that is, the potential difference (voltage drop, tension) between these two points:

$$\int_1^2 \vec{E}_s \cdot \mathrm{d}\vec{l} = V_{12} \,.$$

and the second term is, by definition, the electromotive force of the voltage source,

$$\int_1^2 \vec{E}_m \cdot \mathrm{d}\vec{l} = \xi \,.$$

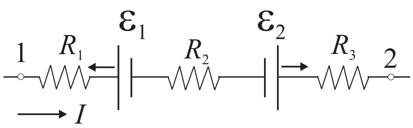
It means that expression (5.31) can be rewritten as

$$IR = V_{12} + \xi \tag{5.34}$$

or, equivalently,

$$\boxed{V_{12} = IR - \xi} \,. \tag{5.35}$$

It is worth noting that if there is only one emf source between points 1 and 2 and no resistors ($R = 0$), according to the above equation, the numerical value of the voltage drop $V_{21} (= -V_{12})$ is directly given by the value of the emf of the voltage source: $V_{21} = \xi$. This same identity would be found if no current was flowing through the branch (although $R \neq 0$).



If there are more than one emf source and/or resistor in the circuit branch from 1 $\longrightarrow$ 2, then the successive application of the previous procedure to the specific case shown in the side figure would have led us to

$$V_{12} = I(R_1 + R_2 + R_3) - (-\xi_1 + \xi_2)$$

expression that can be written in a more general way as

$$V_{12} = I \sum_j R_j - \sum \xi_i \,. \tag{5.36}$$

In the above expression the sign in the corresponding ξ_i is taken according to the following criterion:

$$\text{sign}(\xi) = \begin{cases} + & \text{if direction } \vec{E}_m \equiv \text{direction } 1 \to 2 \\ - & \text{otherwise} \quad . \end{cases}$$

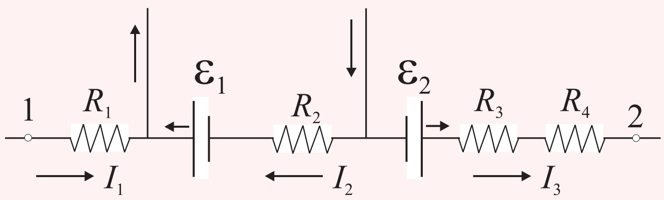


Figure 5.2: Different branches carry different currents.

In the more general case shown in Fig. 5.2, where the different circuit branches have different currents, the computation of the corresponding line integral from points $1 \to 2$ yields

$$V_{12} = \left[I_1R_1 - I_2R_2 + I_3(R_3 + R_4)\right] - (-\xi_1 + \xi_2)$$

where the sign of current, I_j, is taken according to the following criterion:

$$\text{sign}(I_j) = \begin{cases} + & \text{if direction } I_j \equiv \text{traveling direction } 1 \to 2 \\ - & \text{otherwise} . \end{cases}$$

For any number of resistors and voltage sources, the above expression can be expressed as

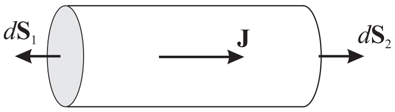
$$\boxed{V_{12} = \sum_j I_jR_j - \sum_i \xi_i} \qquad (5.37)$$

Kirchhoff's voltage rule

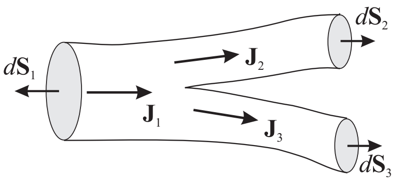
where R_j is the *total* resistance of the resistors in the j-th branch associated with current I_j. This final expression is known as **Kirchhoff's voltage rule**.

5.5.2 Kirchhoff's current rule

If expression (5.7) is applied to a section of a current-carrying wire as the one shown in the attached figure, we will obtain that



$$\oint_S \vec{J}\cdot d\vec{S} = \int_{S_1} \vec{J}\cdot d\vec{S} + \int_{S_2} \vec{J}\cdot d\vec{S}$$
$$= \vec{J}\cdot\vec{S}_1 + \vec{J}\cdot\vec{S}_2 = -I + I = 0\,.$$



In a more general case where three wires are connected to a junction (*node*), the application of (5.7) now yields

$$\oint_S \vec{J}\cdot d\vec{S} = \int_{S_1} \vec{J}\cdot d\vec{S} + \int_{S_2} \vec{J}\cdot d\vec{S} + \int_{S_3} \vec{J}\cdot d\vec{S}$$
$$= \vec{J}_1\cdot\vec{S}_1 + \vec{J}_2\cdot\vec{S}_2 + \vec{J}_3\cdot\vec{S}_3 = -I_1 + I_2 + I_3 = 0\,,$$

where the values of the different currents will be negative/positive if the currents are entering/leaving the region (namely, if the corresponding currents are directed inward/outward to the surface enclosing the node).

If the above expression is generalized to a node with N branches, the following **Kirchhoff's current rule** is obtained:

Kirchhoff's current rule

$$\boxed{\sum_{i=1}^{N} I_i = 0} \tag{5.38}$$

that is,

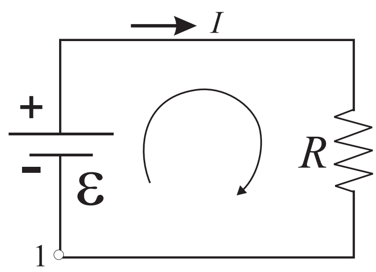
the sum of all the currents in a junction is null.

Activity 5.8:

- Have you noticed that we can choose different combinations of loops to "cover" completely any circuit?
- Can you explain the difference between a *rule* and a *law*?
- Can we study circuits with capacitors and/or inductors with the given Kirchhoff's rules? Justify your answer.
- Describe and explain which basic previously-reported laws have been used to derive Kirchhoff's rules Eq. (5.35) and Eq. (5.38).
- Why is it so important to remember the sign criteria employed in Eq. (5.35)?

5.6 Resolution of direct-current circuits

As already mentioned, a circuit of direct current (DC) is the interconnection of a set of resistors and voltage sources of direct current. This circuit can have any topology with multiple loops, being the simplest one the topology shown in the attached figure, which only involves one loop with one voltage source and one resistor. In general, the application of the two Kirchhoff's rules to the circuit under study will lead to a linear system of equations, whose solution will give us the values of the required unknown quantities.



In the simplified case of the attached figure, there is only one current to consider, I. The application of Kirchhoff's voltage rule (5.35) to that loop, when the traveling direction of the loop is chosen to be clockwise, yields

$$V_{11} = 0 = IR - \xi$$

and therefore the current is

$$I = \xi / R \, .$$

If, for instance, $R = 2\,k\Omega$ and $\xi = 6\,\text{V}$, the current flowing in this elementary circuit will be $I = 3\,\text{mA}$.

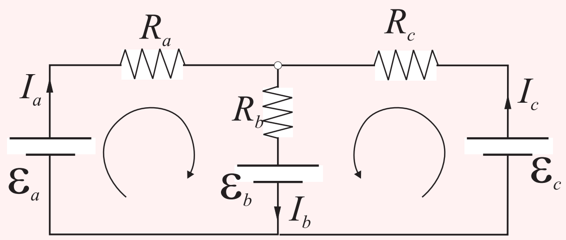


Figure 5.3: DC network with two loops.

For a more complex circuit with multiple loops as the one shown in Fig. 5.3, we take as unknowns the currents flowing through each branch: I_a, I_b and I_c. Kirchhoff's rules lead to the following linear system of three equations:

$$I_a R_a + I_b R_b = \xi_a - \xi_b \tag{5.39a}$$

$$I_c R_c + I_b R_b = \xi_c - \xi_b \tag{5.39b}$$

$$I_b = I_a + I_c \tag{5.39c}$$

which, after substituting I_b, reduces to

$$I_a(R_a + R_b) + I_c R_b = \xi_a - \xi_b \tag{5.40a}$$

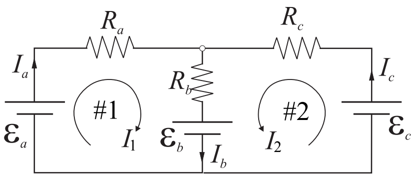
$$I_a R_a + I_c(R_b + R_c) = \xi_c - \xi_b \, . \tag{5.40b}$$

The solution of the above system for any of the well-known methods of resolution of linear systems of equations would allow us to obtain the values of the currents in any of the branches.

5.6.1 (*) Mesh current method

In textbooks we can find different network analysis techniques to deal with **linear circuits** (circuits whose elements have a linear relation between current and tension). In general, these techniques allow us to write, in a systematic way, a linear system of equations in terms of certain auxiliary circuit variables. One of the most common methods is the so-called *mesh current method* (also known as loop current method). The basis of this method is to "reorganize" the expressions resulting from Kirchhoff's rules for the branch currents in such a way that the unknowns are now the so-called **mesh currents**. The currents in the various branches of the circuit can easily be found later from these loop or mesh currents.

To apply the method, first we should identify the minimum number of meshes required to describe the network. If the network has a topology such that it can be laid out flat with no wires crossing over others, then the number of mesh currents required to describe a network equals the number of "inner" loops in the network (i.e., the loops that can directly be observed in the network).



In the case shown in the attached figure (the same as in Fig. 5.3), we have that the circuit is fully described by at least two meshes, with the most trivial choice being the loops #1 and #2 in the figure. For each of these meshes we define its corresponding mesh current (with a given arbitrary direction): I_1 and I_2. It is worth noting that the branch current I_b can be expressed in terms of the mesh currents as

$$I_b = I_1 + I_2 \ .$$

In general, the system to be solved in order to find the mesh currents, I_j, can be written as follows:

$$\xi_i = \sum_{j=1}^{N} R_{ij} I_j \qquad (i = 1, \dots, N) \tag{5.41}$$

where

- N is the number of required meshes to describe the circuit;
- ξ_i is the *total* emf of the i-th mesh, taking the sign of every partial emf ξ_{ij} positive if the non-electrostatic field associated with the source has the same direction as the mesh current, and negative otherwise;

$$\mathrm{sign}(\xi_{ij}) = \begin{cases} + & \text{if direction } \vec{E}_{m,ij} \equiv \text{direction } I_i \\ - & \text{otherwise .} \end{cases}$$

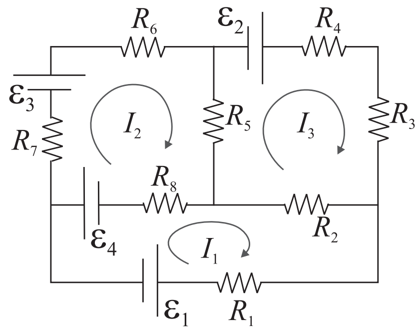
- R_{ij} is the total resistance that is *common* to mesh i and j, whose sign is given by

$$\mathrm{sign}(R_{ij}) = \begin{cases} + & \text{if direction } I_i \equiv \text{direction } I_j \\ - & \text{if direction } I_i \not\equiv \text{direction } I_j \text{ .} \end{cases}$$

Applying the present technique to the circuit in Fig. 5.3, we obtain the following linear system (written in matrix form):

$$\begin{bmatrix} \xi_a - \xi_b \\ \xi_c - \xi_b \end{bmatrix} = \begin{bmatrix} R_a + R_b & R_b \\ R_b & R_b + R_c \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{5.42}$$

This procedure is usually simpler than the direct application of Kirchhoff's rules when there are several meshes in the circuit. Also, this method can easily be implemented in a computer code that can straightforwardly solve DC circuits with an arbitrary number of meshes.



Example 5.3 *Find the system of equations for the mesh currents in the three-loop circuit shown in the figure.*

In the circuit of the attached figure we first define a current for each of the highlighted loops, taking the directions as shown in the figure. Following the criteria of signs already discussed for the resistances and electromotive forces, we find that the system of equations in matrix form that characterizes the circuit is as follows:

$$\begin{bmatrix} -\xi_1 - \xi_4 \\ \xi_3 + \xi_4 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} R_1 + R_2 + R_8 & -R_8 & -R_2 \\ -R_8 & R_5 + R_6 + R_7 + R_8 & -R_5 \\ -R_2 & -R_5 & R_2 + R_3 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Activity 5.9:

- What are the advantages of using the mesh current method?
- Do the exercises starting from 5.11.

5.6.2 (*) Superposition theorem

In linear electric circuits with more than one voltage source is often very useful the application of the superposition theorem. This theorem states that

> *the response (voltage or current) in any branch of a linear circuit having more than one independent voltage source is equal to the algebraic sum of the responses caused by each independent voltage source acting alone, with all the other independent voltage sources being replaced by their internal impedances.*

To prove this theorem we can start from the system of equations provided by the mesh current analysis,

$$[\xi] = [R][I] \tag{5.43}$$

or, equivalently,

$$[I] = [R]^{-1}[\xi] \,. \tag{5.44}$$

If we now carry out a decomposition of the emf sources, such that

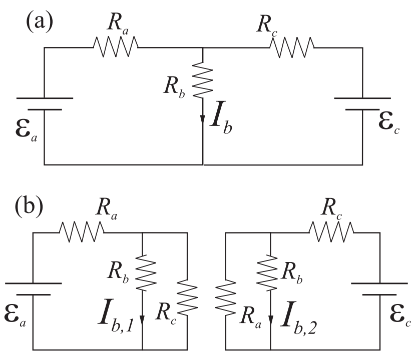
$$[\xi] = [\xi]_1 + [\xi]_2 \tag{5.45}$$

we will obtain a similar decomposition for the loop currents,

$$\begin{aligned}[I] &= [R]^{-1}[\xi] = [R]^{-1}[\xi]_1 + [R]^{-1}[\xi]_2 \\ &= [I]_1 + [I]_2 \,.\end{aligned} \tag{5.46}$$

The above expression shows that any linear combination of emf sources has a correspondence to a linear combination of currents.

EXAMPLE 5.4 *Apply the superposition theorem to find the current I_b in the circuit of part (a) of the figure.*



The obtaining of current I_b by means of the superposition theorem requires to consider the action of each emf source acting independently. This way,

$$I_b = I_{b,1} + I_{b,2}$$

and, therefore, we should solve the two simpler problems shown in part (b) of the figure. To compute $I_{b,1}$ we have to solve the following system:

$$\begin{aligned}\xi_a &= I_a R_a + I_{b,1} R_b \\ I_{b,1} R_b &= I_c R_c \\ I_a &= I_{b,1} + I_c \,.\end{aligned}$$

Similarly, the computation of $I_{b,2}$ requires to solve

$$\begin{aligned}\xi_c &= I_c R_c + I_{b,2} R_b \\ I_{b,2} R_b &= I_a R_a \\ I_c &= I_{b,2} + I_a \,.\end{aligned}$$

The final solution for I_b is then

$$I_b = \frac{\xi_a}{R_a + R_b + \dfrac{R_a R_b}{R_c}} + \frac{\xi_c}{R_c + R_b + \dfrac{R_c R_b}{R_a}} \,.$$

Although the above example does not clearly show any practical advantage in the resolution of the considered circuit, there are many other situations for which the application of this theorem can be very beneficial to simplify calculations. A very practical and relevant situation where this theorem finds its usefulness is for circuits having simultaneously direct-current and time-varying emf sources. An example of this situation will be consider further when discussing alternating-current circuits.

5.6.3 (*) Thevenin's theorem

Any linear electrical network with voltage sources and resistors can be replaced at any pair of terminals A-B by an equivalent voltage source ξ_{TH} connected in series with an equivalent resistance R_{TH}.
ξ_{TH} is the value of the voltage drop at terminals A-B of the network with terminals A-B open circuited.
R_{TH} is the resistance obtained from terminals A-B of the network with all its independent voltage sources short circuited.

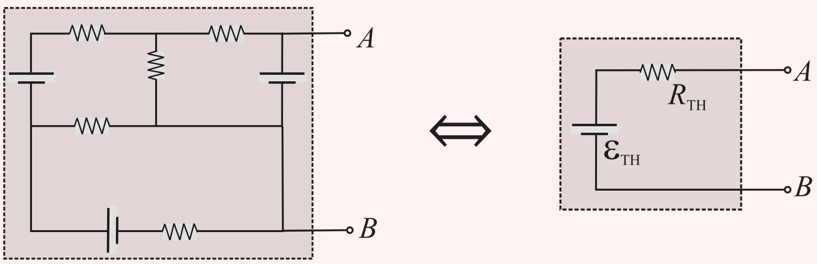
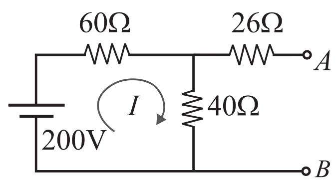


Figure 5.4: Left: network composed of multiple voltage source and resistors. Right: Thevenin equivalent circuit from terminals A-B.

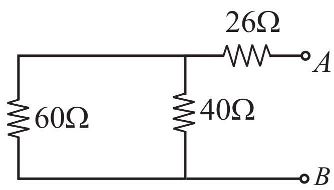
The above theorem simply states that any linear circuit with output terminals A-B can be substituted by a single emf source connected in series with a resistor (see Fig. 5.4). The specific values of the emf and the resistance can be obtained following the procedure described in the theorem.

EXAMPLE 5.5 *Find the Thevenin equivalent circuit of the one shown in the figure.*



To apply Thevenin's theorem we have to compute the value of the equivalent Thevenin resistance and emf. The Thevenin resistance R_{TH} is given by the equivalent resistance from terminals A-B when the voltage source is short circuited. First we obtain the parallel resistance, $R_{||}$, of the combination of the resistors with 60 Ω and 40 Ω:

$$\frac{1}{R_{||}} = \frac{1}{40} + \frac{1}{60} ,$$

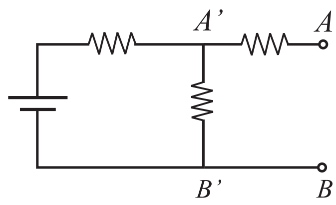


namely, $R_{||} = 24\Omega$. The Thevenin resistance will then be

$$R_{TH} = R_{||} + 26 = 50\Omega .$$

To obtain ξ_{TH} we have to compute the voltage drop at terminal A-B (since $\xi_{TH} = V_{AB}$). The current, I, flowing through the circuit will be

$$I = 200\ \mathrm{V}/(60\Omega + 40\Omega) = 2\ \mathrm{A} .$$



Taking into account that no current is flowing through branches A and B, we have that $V_{AB} = V_{A'B'}$ and therefore

$$\xi_{TH} = 40I = 80\ \mathrm{V} .$$

5.6.4 Power balance

In Secs. 5.3.3 and 5.4.1 we have studied the main effects related to the power dissipated and supplied in DC circuits. In a circuit with multiple emf sources and resistors, the principle of energy conservation states that the total power dissipated in all the resistors has to be same as the total power delivered by the set of emf sources. Thus, if we have N emf sources, each one supplying a power given by

$$P(\xi_n) = I_n \xi_n$$

(with I_n being the current flowing through the source with emf ξ_n) and there are M resistors, each one dissipating a power

$$P(R_m) = I_m V_m$$

(V_m and I_m are the voltage drop and current associated with the resistor of resistance R_m), then it must be satisfied that

Power supplied by all the voltage source must be equal to the power dissipated in all the resistors

$$\sum_{n=1}^{N} P(\xi_n) = \sum_{m=1}^{M} P(R_m) \tag{5.47}$$

or, equivalently,

$$\sum_{n=1}^{N} I_n \xi_n = \sum_{m=1}^{M} I_m V_m = \sum_{m=1}^{M} I_m^2 R_m = \sum_{m=1}^{M} V_m^2 / R_m \, . \tag{5.48}$$

Activity 5.10:

- Has the balance of power something to do with the principle of conservation of energy? Justify your answer.
- If we have two light bulbs of resistances R_1 and R_2, how should we connect them to a given voltage source in order to obtain the maximum luminescence?
- Does the current I increase/decrease when it flows through a resistor with a high resistance? And, what about if the current has to flow through two resistors in series instead of just one? Justify your answer.
- If the answer to the above question is NO, then which quantity is what changes in the resistors?

5.7 Problems

5.1: In a fluorescent tube of 3 cn of diameter, there are 2×10^{18} electrons and 0.5×10^{18} positive ions (with a charge $+q_e$) passing every second through a cross surface of the tube. Find the current in the tube.
Sol. 0.4 A.

5.2: In order to know the length of a wire wound to form a coil, it is measured the resistance of that wire, finding a value of 5.18 Ω. If the resistance of 200 cm of this same wire is 0.35 Ω, what is the length of the wound wire?
Sol.: l = 2960 cm.

5.3: **a)** Find the magnitude of the electric field inside a cooper wire of resistivity $\rho = 1.72\times 10^{-8}\,\Omega$m when this wire has a current density of $|\vec{J}| = 2.54\times10^6$ A/m². **b)** What is the potential difference between two points along the wire separated 100 m?
Sol.: **a)** $|\vec{E}|$ = 43.7 mV/m; **b)** ΔV = 4.37 V.

5.4: Certain device can move a charge of 1.5 C a distance of 20 cm in a region with an uniform electric field whose magnitude is $|\vec{E}| = 2\times10^3$ N/C. Find the emf of that device
Sol.: ξ = 400 V.

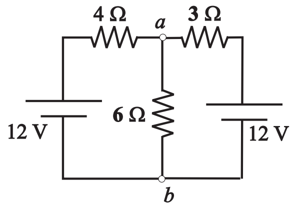
5.5: Find the heat dissipated in 5 min by an iron resistor that carries a current of 5 A under a voltage drop of 120 V?.
Sol. Heat $\approx 2.23\times10^5$ J.

5.6: Two wires of the same length but different cross section are connected in series as well as in parallel. Which wire of the two forming the connection would dissipate more heat if both connections have the same voltage drop?
Sol. Series: the wire with smaller cross section; Parallel: the wire with larger cross section.

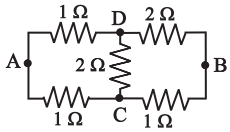
5.7: We have a battery whose fem is 2 V and total charge 2 A·h. **(a)** Find the value of the total energy that the battery can supply. **(b)** How long can the battery be working when connected to a resistor of 100 Ω? (neglect the inner resistance of the battery) **(c)** If the battery is connected to a circuit so that it has to supply a current of 250 mA, what power does it deliver before being discharged?
Sol: **(a)** 86400 J. **(b)** 16 hours and 40 minutes, **(c)** 3 W, 8 hours

5.8: An electric car works with batteries of 12 V. If the friction force is approximately 1.2 kN at 80 km/h, **(a)** find the power delivered by the motor to stand at this speed. **(b)** If every battery provides a total charge of 160 A·h before being discharged, find the total energy supplied by the batteries (express this value in kW·h). **(c)** What distance does the car undergo at 80 km/h without recharging the batteries?
Sol.: (a) 26,67 kW. (b) 69,12 MJ = 19,2 kW·h. (c) 57,6 km.

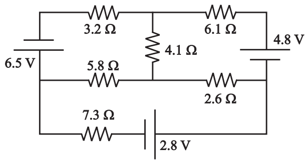
5.9: In the circuit of the attached figure, find **a)** the current in each resistor; **b)** the potential drop between points a and b; and **c)** the power delivered by each battery.
Sol.: **a)** $I_4 = 2/3$ A, I_3 = 8 A, $I_6 = 14/9$ A; **b)** $V_b - V_a = -28/3$ V; **c)** 8 W supplied by the LHS battery and 32/3 W by the other one.



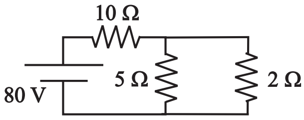
5.10: We have two batteries, one with ξ_1 = 9 V, r_1 = 0.8 Ω and another with ξ_2 = 3 V, r_2 = 0.4 Ω. **a)** How should we connect these two batteries in order to have the maximum current through a resistance R? **b)** Find the current for R = 0.2 Ω and R = 1.5 Ω.
Sol.: **a)** In parallel for the smallest R, in series for the largest R; **b)** $I_{0.2}$ = 10.7 A, $I_{1.5}$ = 4.44 A.



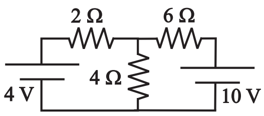
5.11: In the circuit shown in the attached figure a battery of emf 10 V and internal resistance 1 Ω is connected between points A and B. Find: **a)** the current supplied by the battery; **b)** the equivalent resistance between A and B; **c)** the potential difference between the plates of a capacitors which was connected between points C and D.
Sol.: **a)** 32/7 A; **b)** 1,18 Ω; **c)** 4/7 V.



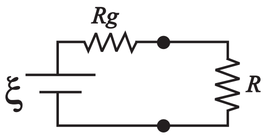
5.12: Find the currents in the circuit of the figure.
Sol.: 1.1 A, 0.87 A, 0.73 A, 0.36 A, 0.15 A and 0.22 A.



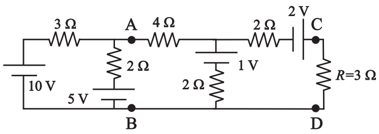
5.13: In the circuit of the figure: **a)** find the currents; **b)** check out the balance of power.
Sol.: **a)** 7 A, 2 A y 5 A; **b)** delivered power: 560 W; dissipated power: $P(R=10) = 490$ W, $P(R=5) = 20$ W, $P(R=2) = 50$ W.



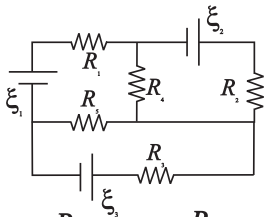
5.14: Find the current in $R = 6$ Ω by two methods: **a)** direct application of Kirchhoff's laws; **b)** making use of Thevenin equivalent.
Sol.: **a)** $i_{R=6} = 1$ A ; **b)** $V_{Th} = 22/3$ V and $R_{Th} = 4/3$ Ω, $i_{R=6} = 1$ A.



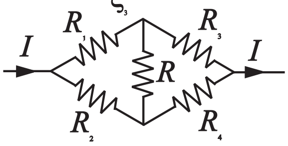
5.15: In the circuit of the figure, find the power dissipated by the resistor R and also the value of the resistance so that the dissipated power reaches a maximum. Plot the power as a function of R.
Sol.: $P(R) = \xi^2 R (R + R_g)^{-2}$; $P(R)$ is maximum if $R = R_g$.



5.16: In the circuit of the figure, find the current carried by resistor $R = 3$ Ω **a)** using Kirchhoff's laws; **b)** applying successively the Thevenin equivalent first between terminals A and B, and then between terminals C and D.
Sol.: **a)=b)** $i_{R=3} = 21/29$ A.



5.17: Write the equations derived from Kirchhoff's laws for the circuit in the figure. Now consider that $R_5 = R_3$ and, under this assumption, choose a possible set of values for the voltage sources so that the current delivered by source ξ_1 is null.
Sol.: A possible solution would be $\xi_1 = 1$ V, $\xi_2 = 0$ V and $\xi_3 = 2$ V. It should be noted that there are infinite solutions.



5.18: In the circuit of the figure, find the relations between resistances R_1, R_2, R_3 and R_4 so that the current carried by resistor R is null.
Sol.: $R_1 R_4 = R_2 R_3$.

Lesson 6

Transient Regime

6.1 RC circuit. Charging/discharging a capacitor.

A resistor-capacitor circuit (RC circuit) is a circuit composed of resistors and capacitors driven by voltage sources (only DC voltage sources will be considered in this chapter). The main difference with the DC circuits already studied in the previous lesson comes now from the fact that we are going to consider the charging and/or discharging processes in the capacitors. Thus, in the present lesson, the currents in the circuit will be time-varying during these charging/discharging processes. In particular we will study the **transient regime**; namely, the response of the circuit when it is changing from one steady-state regime to another steady-state regime. After this study of a RC circuit, a similar analysis will be carried out on the transient regime in resistor-inductor (RL) circuits.

6.1.1 Transient discharge in a capacitor

The simplest RC circuit is a capacitor connected directly to a resistor (no voltage source are considered here), as shown in Fig. 6.1. If we assume that the capacitor with capacitance C is charged with an initial charge Q_0, once the switch is closed (see Fig. 6.1), the capacitor will release its stored energy (or, equivalently, its charge) through the resistor. If the switch is closed at time $t = 0$, then the charge will start flowing from one plate of the capacitors to the other through the resistance R. This process will last until the capacitor is completely discharged; namely, until the voltage drop across the capacitor becomes null. If we apply Kirchhoff's voltage rule to the above circuit at any time, it is obtained that

$$V_C = V_R \, . \tag{6.1}$$

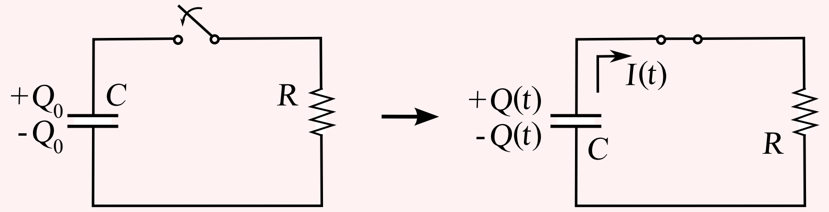


Figure 6.1: Transient Discharge of a capacitor through a resistor.

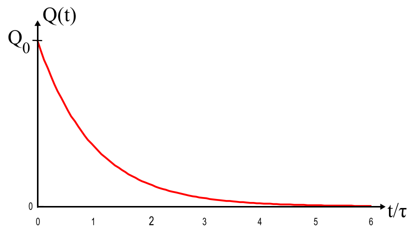
Taking now into account that $V_C = Q/C$ and that $V_R = RI = -R\mathrm{d}Q/\mathrm{d}t$,[1] the above equation can be rewritten as

$$\frac{Q}{C} = -R\frac{\mathrm{d}Q}{\mathrm{d}t} \tag{6.2}$$

or, equivalently,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{RC} = 0\,. \tag{6.3}$$

Now it should be noticed that the above equation is a *differential equation*, which means that the different terms of the equation actually relate certain function to its derivatives. In other words, we have to find the function $Q(t)$ whose derivative is equal to itself multiplied by $1/RC$. It is not difficult to realize that the only function whose derivative is proportional to itself is the *exponential* function. Thus, we can check that the solution to Eq. (6.3) is given by



$$Q(t) = Q_0 \mathrm{e}^{-t/RC} \tag{6.4}$$

where Q_0 is the value of the capacitor charge at $t = 0$ [$Q(0) = Q_0$].

The above expression shows that the capacitor charge is decreasing in time at an exponential rate determined by the factor $\tau = RC$, factor that is known as **time constant** of the system. For $t \gtrsim 4\tau$ we can check that the charge of the capacitor has disappeared almost completely (although it is not zero at this time, it will be considered null for practical purposes); namely, the capacitor is regarded "discharged" after $t \gtrsim 4\tau$.

The current flowing through the resistor can be computed by simply taking the time derivative in expression (6.4):

$$I(t) = -I_0 \mathrm{e}^{-t/RC} \tag{6.5}$$

where I_0 is the initial value of the current at $t = 0$, $I(0) = I_0 = Q_0/RC$.

[1] We have to write in this case $I = -\mathrm{d}Q/\mathrm{d}t$ (with minus sign) in order to account for the fact that a decreasing charge in the positively-charged capacitor plate implies an outward current from this capacitor, as initially assumed in our scheme of the RC circuit in Fig. 6.1. In other words, as $\mathrm{d}Q/\mathrm{d}t < 0$ in this case, the assumed direction of the current in Fig. 6.1 can only be achieved with the minus sign introduced.

Activity 6.1:

- Describe in simple terms why the charge of the capacitor starts to flow after the switch is closed in Fig. 6.1.
- Why is the voltage of a discharging capacitor going to zero?
- Explain succinctly the difference between an algebraic and a differential equation. In Physics we usually find differential equations (for instance, Newton's second law of motion), can you guess the reasons for that?
- Verify that for $t > 4\tau$ the charge of the capacitor has become almost null.
- In the discharging RC process, how could you obtain the total energy dissipated in the resistor? Does it have to be equal to the energy initially stored in the capacitor? Try to verify the above fact.

6.1.2 Charging a capacitor

The opposite process to discharging a capacitor is clearly the process of charging it. In this charging process, a DC emf source is required to supply the necessary external work. In the RC series circuit shown in Fig. 6.2,

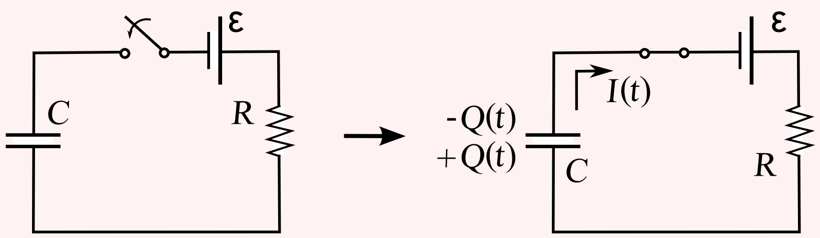


Figure 6.2: Charging a capacitor in a circuit with a resistor R and a battery of electromotive force ξ.

with the capacitor assumed to be initially uncharged, the switch is closed at $t = 0$. Then, for $t > 0$, the emf source will carry an amount of charge from one capacitor plate to the other until the voltage difference across the capacitor equals the value of the imposed emf ξ. Applying Kirchhoff's voltage rule to this circuit, it is obtained

$$\xi = V_C + V_R \tag{6.6}$$

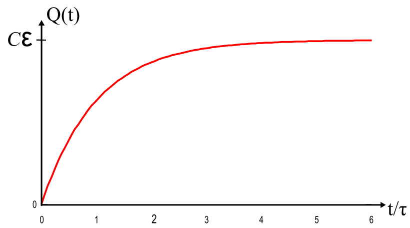
which can be rewritten as (note that now $I = +\mathrm{d}Q/\mathrm{d}t$)

$$\xi = \frac{Q}{C} + R\frac{\mathrm{d}Q}{\mathrm{d}t} \quad \Longrightarrow \quad \frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{RC} = \frac{\xi}{R} \,. \tag{6.7}$$

This differential equation is very similar to (6.3) except that the RHS is not zero now. The solution to this equation is very similar to the one found for (6.3), although another term has to be added to account for the non-zero RHS. The solution to Eq. (6.7) can then be expressed as

$$Q(t) = C\xi + Q'\mathrm{e}^{-t/RC} \,. \tag{6.8}$$

The coefficient Q' has to be obtained from the so-called *initial condition* for the charge. In our present case, this initial condition is $Q(t=0)=0$. Applying this condition to (6.8), it is obtained

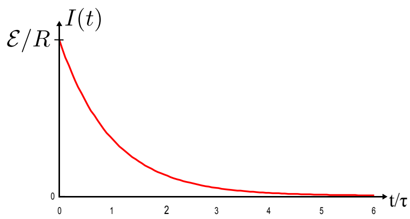


$$C\xi + Q = 0 \quad \Longrightarrow \quad Q' = -C\xi \,,$$

and thus we can finally write

$$Q(t) = C\xi\left(1 - \mathrm{e}^{-t/RC}\right) \,. \tag{6.9}$$

It should be noted that the charging process is characterized by a monotonically increasing function, in such a way that the transient regime takes approximately an interval time $t \approx 4\tau$. Depending on the given values of R and C, this time interval (similar to the discharging process) can be very small (a few nanoseconds) or take a few seconds.



In order to obtain the current flowing in the above circuit, we have to differentiate the charge given by expression (6.9) with respect to t; namely,

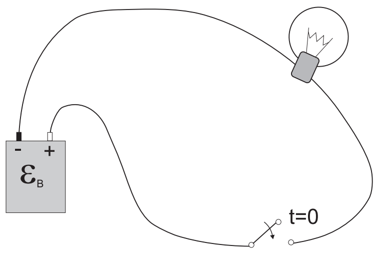
$$I(t) = \frac{\xi}{R}\mathrm{e}^{-t/RC} \tag{6.10}$$

whose behavior is plotted in the attached figure. In this figure it can be observed that at $t = 0$ (uncharged capacitor) the current is equal to the one found if the capacitor were substituted by a short-circuit (a wire). On the other hand, at $t \to \infty$ (fully charged capacitor), the current is null as if the capacitor were now substituted by an open-circuit.

Activity 6.2:

- Why do we need a voltage source in the charging process of a capacitor? Was it necessary in the discharging process? Explain the differences.
- Try to guess the reasons for which we need to add an extra term in the solution of Eq. (6.7) to take into account the non-zero RHS in the differential equation.
- Find $Q(t)$ in the charging process if the capacitor was initially charged with half the final charge.
- Find $Q(t)$ in the charging process if we have two capacitors in series with capacitances C_1 and C_2.
- Find $Q(t)$ in the charging process if we have two capacitors in parallel with capacitances C_1 and C_2. Also find $Q_1(t)$ and $Q_2(t)$.

6.2 Transient analysis in *RL* circuits



A situation similar to that studied in the previous section is also found in the circuit shown in the attached figure, which is composed of a DC voltage source (a battery) of emf ξ_B that feeds through a switch a light bulb (or a similar resistance-type device) . From a circuital point of view, the light bulb can be modeled as a resistor with resistance R. In the present situation the only magnetic flux passing through the circuit is the *self-flux* (see Sec. 4.3.2). Applying Kirchhoff's voltage rule to the above configuration we will find that the sum of the existing emf's in the circuit has to equal the voltage drop in the resistor. As there are two emf's: the emf supplied by the battery (ξ_B) and the *induced* emf caused by the time rate of change of the flux linkage ($\xi_{\rm ind}$), Kirchhoff's voltage rule leads to

$$\xi_B + \xi_{\rm ind} = V_R (= Ri) \; . \tag{6.11}$$

As discussed in Sec. 4.3.2, the effect of a changing flux linkage can be modeled by means of a new element in the circuit, an inductor characterized by an inductance L, in such a way that the voltage drop in this element can be expressed as

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t} \; .$$

Thus, the corresponding circuit that we have to consider is the one shown in Fig. 6.3, with the switch in position (1). The current $i(t)$ flowing in this circuit

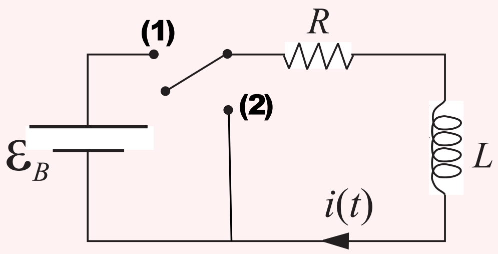


Figure 6.3: Circuit with an inductor L in series with a resistor R and a battery ξ.

is then the solution to

$$\xi_B = V_R + V_L$$

which can be written as the following differential equation for $i(t)$:

$$\xi_B = Ri + L\frac{\mathrm{d}i}{\mathrm{d}t}$$

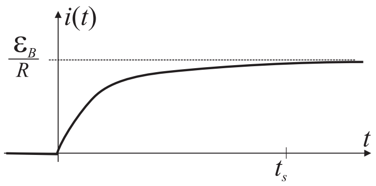
or, equivalently,

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{R}{L}i = \frac{\xi_B}{L} \; . \tag{6.12}$$

The function $i(t)$ solution to this equation is found to be

$$i(t) = I_0 \mathrm{e}^{-\frac{R}{L}t} + \frac{\xi_B}{R}$$

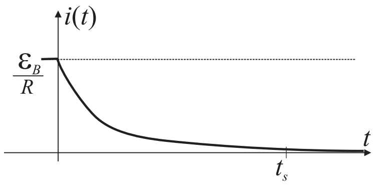
where I_0 comes determined by the initial condition; namely, the value of $i(t)$ at $t = 0$. In the present case, as $i(0) = 0$ [the current was null before setting the switch to position (1)], it is found that $I_0 = -\xi_B/R$ and therefore



$$i(t) = \frac{\xi_B}{R}\left(1 - e^{-\frac{R}{L}t}\right) . \tag{6.13}$$

The behavior shown by the plot of $i(t)$ clearly points out that this current does not vary suddenly (as it would have happened if there was only a resistor and no inductor). On the contrary, the final current value ξ_B/R is reached after approximately an interval given by $t_s \approx 4\tau$, with the constant time of this problem now being $\tau = L/R$. If the inductance L is high (that is, if the flux linkage is large), the steady-state current is only reached after a certain time. The plot also shows that the current is null at $t = 0$ ("uncharged" inductor), as if the inductor was open-circuited. On the other hand, at $t \to \infty$ (fully "charged" inductor), the current is now ξ_B/R, as if the inductor was short-circuited (the inductor behaves like a plain wire).

If we now consider that the switch in Fig. 6.3 is set to position (2) at a later time (again taken as $t = 0$), the initial current flowing through the circuit at $t = 0$ is then $i(0) = \xi_B/R$. As the second member of Eq. (6.12) is now null, the solution for $i(t)$ becomes



$$i(t) = i(0)e^{-\frac{R}{L}t} \tag{6.14}$$

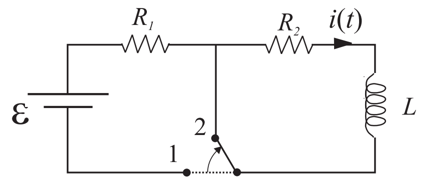
$$= \frac{\xi_B}{R}\,e^{-\frac{R}{L}t} . \tag{6.15}$$

In this case we can observe that the current does not go to zero abruptly but after a time interval given again by $t_s = 4L/R$.

Activity 6.3:

- Explain the main differences that we would observe in the setup with the battery and the light bulb when we do/don't neglect the flux linkage through the circuit.
- Why the induced emf is better regarded as a voltage drop in an inductor?
- Find $I(t)$ in the transient process if the inductor was initially carrying a current of half its final value.
- Find $I(t)$ in the transient process if we have two inductors in series/parallel with inductances L_1 and L_2.

EXAMPLE 6.1 *Find the amount of heat dissipated in the resistor R_2 after the switch is commuted from position (1) to (2).*



Assuming that the switch was in (1) for long time [see expression (6.13)] and that the commutation is carried out at $t = 0$, we have that the value of the current at this instant is given by

$$I_0 = \frac{\xi}{R_1 + R_2} ,$$

According to (6.14), for $t > 0$ the current flowing through the resistor is

$$i(t) = I_0 e^{-\frac{R_2}{L}t} .$$

Since the heat dissipated in the resistor R_2 per unit time is given by

$$P_{R_2} = \frac{dW}{dt} = i^2 R_2 ,$$

the total heat $\mathcal{Q}$ dissipated in this resistor can be computed from

$$\mathcal{Q} = \int_0^\infty P_{R_2} dt = \int_0^\infty i^2 R_2 dt = \int_0^\infty I_0^2 e^{-\frac{2R_2}{L}t} R_2 dt = I_0^2 R_2 \int_0^\infty e^{-\frac{2R_2}{L}t} dt .$$

If we introduce the following change of variable in the integral:

$$t = \frac{L}{2R_2}\alpha$$

it is obtained that

$$\mathcal{Q} = I_0^2 R_2 \left(\frac{L}{2R_2} \int_0^\infty e^{-\alpha} d\alpha \right) = \frac{1}{2} L I_0^2 .$$

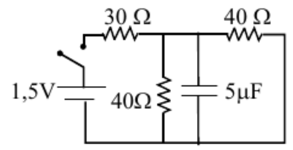
The total heat dissipated in the resistor is just the amount of magnetic energy stored in the inductor [see expression (4.37)].

6.3 Problems

6.1: We have a capacitor of capacitance C connected in series to a resistor R. **a**) Find the time required for the charge to reach 50% of its final value when the series RC connection is driven by a voltage source. **b**) Once the capacitor is charged with a final charge Q_0, we proceed to discharge this capacitor by substituting the voltage source by a short circuit. In that case, find the amount of energy delivered by the capacitor in that process for a time t. Which is this energy transformed into?

Sol.: **a**) $t = RC \ln 2$; **b**) $\Delta U = \frac{Q_0^2}{2C}\left[1 - e^{-2t/(RC)}\right]$, energy is transformed into heat dissipated in the the resistor.

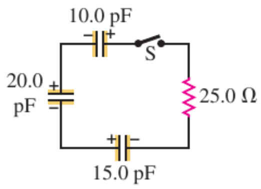
6.2: Taking into account the behavior of a capacitor at the initial time ($t = 0$) of a charging process and once it is completely charged, **a**) find the current supplied by the source of the circuit shown in the figure at $t = 0$ and once the capacitor is charged. **b**) If the above process is repeated but now with a different voltage source so that the final charge in the capacitor is 9 μC, find the emf of the voltage source, and also the currents at the initial and final times of the charging process.



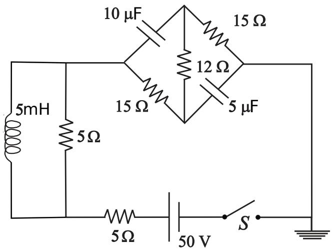
Sol.: **a**) 50 mA, 30 mA, 3 μC; **b**) 4, 5 V, 150 mA; 90 mA.

6.3: The current through a coil of inductance L varies according to $i(t) = I_0(1 - e^{-t/\tau})$, where τ is a constant. Find: **a**) the initial current at ($t = 0$) as well as the final current ($t = \infty$) in the coil; **b**) the instantaneous expressions of the magnetic energy in the coil and the power supplied to it; **c**) the time, t, at which the power is maximum, **d**) the final energy stored in the coil (that is, for $t = \infty$).

Sol.: **a**) $i(0) = 0$, $i(\infty) = I_0$; **b**) $U_m(t) = \dfrac{Li^2(t)}{2}$, $P(t) = \dfrac{LI_0 e^{-t/\tau} i(t)}{\tau}$; **c**) $t = \tau \ln 2$; **d**) $U_m = \dfrac{LI_0^2}{2}$.

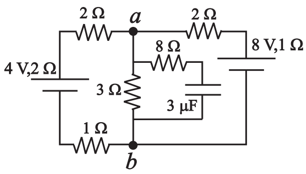


6.4: In the circuit illustrated in the figure, each capacitor has an initial charge of 3.50 nC. After the switch S is closed, what will be the current in the circuit the instant the capacitors have lost 80% of their initially stored energy?



6.5: The capacitors of the circuit in the figure are initially uncharged. Find the value of the current supplied by the battery **(a)** just after the switch S is closed and **(b)** after a long time. **(c)** Also find the final charges of the capacitors and the energy stored in the inductor.

Sol.: **a)** 3.42 A; **b)** 1.064 A; **c)** Q_{10} = 287.2 μC, Q_5 = 143.6 μC, U_L = 2.83 mJ.



6.6: In the circuit of the figure, find: **a)** the current carried by each branch, **b)** the potential drop between a and b calculated for all the possible paths, **c)** the charge of the capacitor, **d)** the power supplied by the sources and the one dissipated by the resistors.

Sol.: **a)** 0 A, 4/3 A, 4/3 A; **b)** 4 V; **c)** 12 μC; **d)** delivered powers: $P(\xi = 4V) = 0$ W, $P(\xi = 8V) =$ 10.67 W; dissipated power: P = 10.76 W.

Lesson 7

Circuits of Alternating Current

7.1 Alternating current

Among the possible time dependences that the current $I(t)$ in a circuit can take, in this lesson we will study the so-called **time-harmonic** dependence; namely, the one given by sine/cosine functions, and which can be written in general as

$$I(t) = I_0 \cos(\omega t + \varphi) \tag{7.1}$$

(see Sec. 1.10 for a mathematical description of harmonic functions), where I_0 is the current amplitude, $\omega = 2\pi/T$ the angular frequency, and φ the initial phase.

There are two main reasons to study this type of time-varying current, known as **alternating current**, or simply **AC**:

1. **Technological relevance.**
   From a technological standpoint, the use of the alternating current is very convenient because it is very easy to generate and its transport can easily be performed at high voltage (and small current), thus minimizing Joule losses (subsequently, the alternating current can easily be transformed to the usual working voltage values). Since the late XIX century, these features as well as its straightforward application in electric motors have made the AC be the most usual type of electric current for household and industrial applications. As a consequence, the electrical technology has been widely developed around this form of current (in Europe, the frequency of the alternating current is 50 Hz). An additional and key feature of this current is that its harmonic behavior is preserved when it is modified by the effect of linear elements; namely, resistors, capacitors, inductors, and ideal transformers.

The time-harmonic feature of AC is preserved by the effect of resistors, capacitors, inductors, and ideal transformers

2. **Mathematical relevance.**
   Since any periodic function can be expressed as a series of different harmonics (according to Fourier's theorem), the study of the alternating current constitutes the basis for the analysis of general time-varying signals in linear networks.

Time-harmonic functios are the basis for the study of general time-varying functions

7.1.1 AC generator

Previously it has been mentioned that one of the most relevant properties of the AC in regards to its practical use comes from its easy generation. A very simple source (or generator) of AC is based on a direct application of Faraday's law of magnetic induction [see Sec. 4.2.2] by transforming mechanical energy into electromagnetic energy; in an opposite way as the electric motor does [see Sec. 3.3.2]. A basic scheme of this AC generator, also called

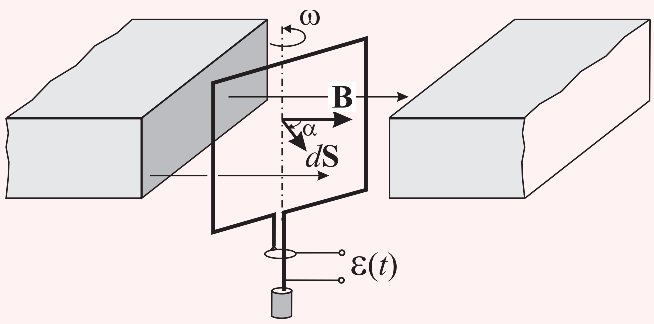


Figure 7.1: Basic scheme of a source of alternating emf.

alternator, is shown in Fig. 7.1, where it can be observed that the magnetic flux through the rotating loop is given by

$$\Phi(t) = \int_S \vec{B} \cdot d\vec{S} = |\vec{B}| S \cos\alpha(t) \tag{7.2}$$

where it has been assumed that the magnetic field is uniform in the loop region and $S = \int_S dS$ is the area of the loop surface.

If the loop is rotating *uniformly* with a constant angular velocity ω (this uniform circular motion can be caused by a constant steam jet that drives a turbine), the rotation angle can be written as $\alpha(t) = \omega t + \alpha_0$, and then the magnetic flux through the loop can be written as

$$\Phi(t) = |\vec{B}| S \cos(\omega t + \alpha_0) \,. \tag{7.3}$$

Applying Faraday's induction law (4.9), the induced emf $\xi(t)$ in a N-turn loop is found to be

$$\xi(t) = -N \frac{d\Phi}{dt} = N|\vec{B}| S\omega \sin(\omega t + \alpha_0) \tag{7.4}$$

namely, an alternating emf has been generated. The emf of this AC voltage source can be expressed in general as

$$\xi(t) = \xi_0 \cos(\omega t + \varphi) \tag{7.5}$$

with $\xi_0 = N|\vec{B}|S\omega$ and $\varphi = \alpha_0 - \pi/2$ in the present situation.

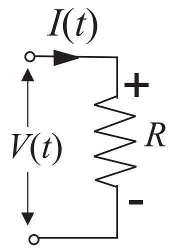
Activity 7.1:

- Describe other types of time-varying functions that are not of harmonic type.
- Why any AC can always be expressed as $I(t) = I_0 \cos(\omega t + \varphi)$?
- Why is it required that the loop in the alternator spins with a constant angular velocity in order to generate an AC?
- What are the advantages of using a N-turn loop in the alternator?

7.1.2 Relation $I \leftrightarrow V$ for resistors, capacitors, and inductors

- **Resistor**

  As it was already discussed in Sec. 5.3.2, Ohm's law provides the relationship between voltage V and current I in steady-state regime; namely, $V = RI$. Experimentally it can be checked that Ohm's law is still valid for time-varying current/voltage and, therefore, it can be written that

  $$\boxed{I(t) = \frac{V(t)}{R}} . \tag{7.6}$$

  It should be noted that the sign +/— in the resistor and other elements in AC circuits stands for the point of high/low voltage when the current is flowing in the direction shown in the corresponding figure.

- **Capacitor**

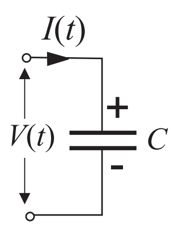
  In expression (2.34) the capacitance was defined as the ratio between the charge Q on the capacitor plates and the potential difference V between them; namely,

  $$C = \frac{Q}{V} . \tag{7.7}$$

  This relationship is also satisfied for AC, and therefore the time-varying charge, $Q(t)$, can be written as

  $$Q(t) = CV(t) . \tag{7.8}$$

  If we take the time derivative of the above expression it is found that the current $I(t)$ and the voltage $V(t)$ in a capacitor are related through

  $$\boxed{I(t) = C\frac{dV(t)}{dt}} . \tag{7.9}$$

  This relationship states that the *time derivative* of the voltage drop in the capacitor plates is linearly related through parameter C to the current flowing to the capacitor.

- **Inductor**
  As stated in (4.27), the effect of the magnetic flux linkage in an inductor with inductance L with a carrying-current $I(t)$ can be modeled as a voltage drop in the inductor, $V(t)$, given by

$$\boxed{V(t) = L\frac{\mathrm{d}I(t)}{\mathrm{d}t}}\ . \tag{7.10}$$

  The inductor can then be regarded as a circuit element characterized by the inductance L, which linearly relates the time derivative of the flowing current in the inductor to the corresponding voltage drop in that element.

Activity 7.2:

- What is the main difference between the relation $I \leftrightarrow V$ in a resistor and the one in capacitors and inductors?
- What are the main differences between the relations $I \leftrightarrow V$ in a capacitor and an inductor?
- Try to figure out what are the main mathematical implications of the fact that the relation $I \leftrightarrow V$ in capacitors/inductors is given in terms of the time derivative of the voltage/current.

7.2 Phasor analysis of AC circuits

Given that the study of AC circuits implies to deal with harmonic functions, the use of **phasors** associated with these functions will greatly simplify the required mathematical computations. As explained in the review of phasors in Sec. 1.11, every harmonic function of the type $I(t) = I_0 \cos(\omega t + \varphi)$ can be associated with a phasor $\tilde{I}$,

$$I(t) \leftrightarrow \tilde{I}$$

written as

Phasor $\tilde{I}$ associated with $I(t) = I_0 \cos(\omega t + \varphi)$

$$\boxed{\tilde{I} = I_0 \mathrm{e}^{\mathrm{j}\varphi}} \tag{7.11}$$

in such a way that the correspondence between them is given by

$$I(t) = \mathrm{Re}\left(\tilde{I}\mathrm{e}^{\mathrm{j}\omega t}\right)\ . \tag{7.12}$$

From the above expression it can be found the following additional and crucial correspondence:

$$\frac{\mathrm{d}I(t)}{\mathrm{d}t} \leftrightarrow \mathrm{j}\omega\tilde{I}\ . \tag{7.13}$$

This property allows us to turn all the operations “derivative with respect to time” of harmonic functions into “multiplication by jω” when using phasors.

7.2.1 Phasor voltage/current relation for resistors, capacitors, and inductors

Making use of the appropriate phasor relationships it is then possible to write the fundamental relationships for resistor, capacitors, and inductors in the following way:

- **Resistor**
  Relationship (7.6) can be expressed in terms of phasors as

$$\tilde{I} = \frac{\tilde{V}}{R} \tag{7.14}$$

  or, equivalently,

$$\boxed{\tilde{V} = R\tilde{I}}\ . \tag{7.15}$$

- **Capacitor.**
  Making use of (7.13), relationship (7.9) can be expressed as

$$\tilde{I} = \mathrm{j}\omega C\tilde{V} \tag{7.16}$$

  or, equivalently,

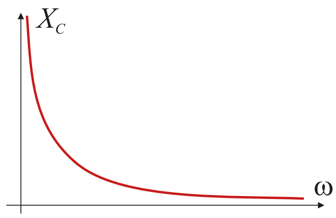
$$\boxed{\tilde{V} = \frac{1}{\mathrm{j}\omega C}\tilde{I}}\ . \tag{7.17}$$

  The above expression can also be written as

$$\tilde{V} = -\mathrm{j}X_C\tilde{I} \tag{7.18}$$

  where

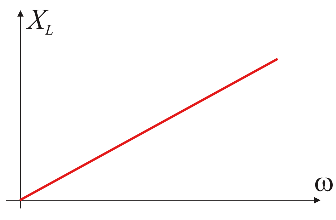
$$X_C = \frac{1}{\omega C} \tag{7.19}$$



  is called **capacitive reactance** with its SI units being ohms (Ω). This quantity *depends* on the operation frequency and tends to zero for very high frequencies and to infinity for very low frequencies. This fact implies that, at low frequencies, the capacitor behaves like an element similar to an open circuit (opposing to the flow of current) whereas, at high frequencies, it behaves in a similar way as a short circuit.

- **Inductor.**
  Relationship (7.13) for the inductor can be expressed in phasor form as

$$\boxed{\tilde{V} = \mathrm{j}\omega L\tilde{I}}\ . \tag{7.20}$$



  If we define the **inductive reactance**, X_L, as

$$X_L = \omega L \tag{7.21}$$

  the phasor expression (7.20) can also be written as

$$\tilde{V} = \mathrm{j}X_L\tilde{I}\ . \tag{7.22}$$

where the units of the inductive reactance are also ohms. This parameter *depends* linearly on the operation frequency, in such a way that it tends to zero for low frequencies and to infinity for high frequencies. We can then conclude that the inductor is an element that will oppose increasingly to the current flow as the frequency gets higher. It means that it behaves like a short circuit for low frequencies and like an open circuit for high ones.

It is very interesting to note that the relationships voltage/current for the capacitor and the inductor, which were expressed in Sec. 7.1.2 as *differential* expressions, have now been rewritten as simple *algebraic* expressions after making use of their associated phasors. Moreover, it has been found that phasor $\tilde{V}$ can be linearly related to phasor $\tilde{I}$ through a generic parameter Z known as **impedance**:

$$\boxed{\tilde{V} = Z\tilde{I}}\ . \tag{7.23}$$

In general, the impedance is a complex number that characterizes each element in the following way:

Impedance of a resistor, capacitor, and inductor

$$Z = \begin{cases} R & \text{Resistor} \\ -\mathrm{j}X_C & \text{Capacitor} \\ \mathrm{j}X_L & \text{Inductor}\ . \end{cases} \tag{7.24}$$

It should be noted that the impedance is not a phasor since no harmonic function is associated with this parameter.

Activity 7.3:

- Give the main advantages provided by the use of phasors when dealing with AC circuits.
- What are the SI units of impedance? What are the SI units of ωL and ωC?
- Why does an inductor behave as an open circuit at high frequencies? Try to relate this fact to the value of the impedance and give a physical explanation of it.
- Why does a capacitor behave as a short circuit at high frequencies? Try to relate this fact to the value of the impedance and give a physical explanation of it.
- Is the concept of impedance useful in DC circuits?
- And what about in circuits where the voltage source supplies a square-type signal.

7.2.2 Kirchhoff's rules for CA circuits

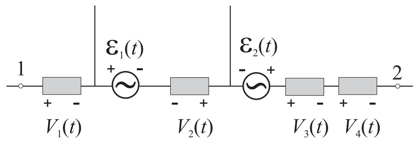
Kirchhoff's rules together with the voltage/current relationships for the different elements in the circuit will allow us to find the behavior of the quantities involved in AC circuits. Kirchhoff's rules were introduced in Sec. 5.5 for DC circuits, where it was assumed a stationary regime with voltages and currents constant in time. The experiments show that these rules can still be considered valid for circuits with a time-varying current,[1] and then they will be considered valid for AC circuits. Thus, we can express Kirchhoff's rules for the instantaneous values of time-harmonic currents and voltages in the following way:

▶ KIRCHHOFF'S RULE FOR VOLTAGE

$$V_{12}(t) = \sum_j V_j(t) - \sum_i \xi_i(t) \tag{7.25}$$

where $V_j(t)$ is the instantaneous voltage drop in the j-th element and $\xi_i(t)$ is the instantaneous value of the i-th voltage source along the path $1 \rightarrow 2$ (the criterion of signs is the same as explained in Sec. 5.5). In the example shown in the attached figure, rule (7.25) leads to

$$V_{12}(t) = \left[V_1(t) - V_2(t) + V_3(t) + V_4(t)\right] - \left[-\xi_1(t) + \xi_2(t)\right] .$$



▶ KIRCHHOFF'S RULE FOR CURRENTS

$$\sum_{i=1}^{N} I_i(t) = 0 \tag{7.26}$$

namely, at each instant, the sum of all the currents entering/leaving a node is zero.

The above rules for the instantaneous values of time-harmonic currents and voltages can alternatively be expressed in terms of their associated phasors as follows.

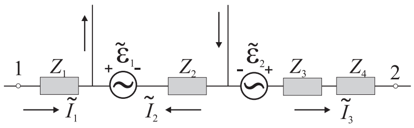
- **Phasor Kirchhoff's rule for voltage**
  Expression (7.25) in the time domain can be associated with the following relation between their associated phasors:

$$\tilde{V}_{12} = \sum_j \tilde{V}_j - \sum_i \tilde{\xi}_i \tag{7.27}$$

or, equivalently,

$$\boxed{\tilde{V}_{12} = \sum_j Z_j \tilde{I}_j - \sum_i \tilde{\xi}_i} \tag{7.28}$$

[1] It should be noted that there can be restrictions to this validity, although its analysis is beyond the scope of the present notes.



where Z_j is the impedance of the j-th element flown through by current phasor $\tilde{I}_j$. In the example of the figure (following the criteria of signs already discussed for DC circuits), the application of (7.28) leads to

$$\tilde{V}_{12} = Z_1\tilde{I}_1 - Z_2\tilde{I}_2 + (Z_3 + Z_4)\tilde{I}_3 - \left[-\tilde{\xi}_1 + \tilde{\xi}_2\right] .$$

- **Phasor Kirchhoff's rule for currents**
  Expression (7.26) in the time domain can be written for their associated phasors as

$$\boxed{\sum_{i=1}^{N} \tilde{I}_i = 0} \tag{7.29}$$

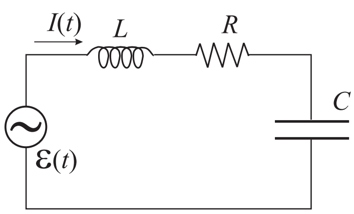
namely, the sum of all phasor currents in a node is null.

Activity 7.4:

- At the light of Kirchhoff's rules expressed in the time domain and in phasor form, describe again the main advantages provided by the use of phasors when dealing with AC circuits.
- If we had both DC and AC voltage sources in the same circuit, would Kirchhoff's rules still be valid in this situtation? Justify your answer.
- Can you see any relation between Kirchhoff's rules (5.37),(5.38) as presented in Sec. 5.5 and these rules now expressed in phasor form in (7.28),(7.29)?

7.2.3 Series RLC circuit

It should be noted that Kirchhoff's rules as written in (7.28) and (7.29) are formally identical to rules (5.37) and (5.38) deduced for DC circuits. The only difference is that there appear phasors and impedances in (7.28) and (7.29) instead of real numbers and resistances in (5.37) and (5.38). As an easy application example of phasor-like Kirchhoff's rules, next it will be considered an AC series RLC circuit.



If the AC voltage source supplies an instantaneous emf given by

$$\xi(t) = \xi_0 \cos(\omega t + \theta) \tag{7.30}$$

whose associated phasor is

$$\tilde{\xi} = \xi_0 e^{j\theta} \tag{7.31}$$

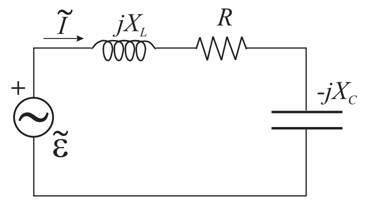
the application of (7.25) to the circuit shown in the figure leads to

$$\xi(t) = V_R(t) + V_C(t) + V_L(t) \tag{7.32}$$

or, in terms of phasors,

$$\tilde{\xi} = \tilde{V}_R + \tilde{V}_C + \tilde{V}_L \,. \tag{7.33}$$

Taking now into account the phasor expressions (7.15), (7.18) and (7.22), it is obtained



$$\tilde{\xi} = \left[R + \mathrm{j}(X_L - X_C)\right]\tilde{I} \tag{7.34}$$
$$= Z\tilde{I} \tag{7.35}$$

where the impedance, Z, of the series RLC circuit is given by

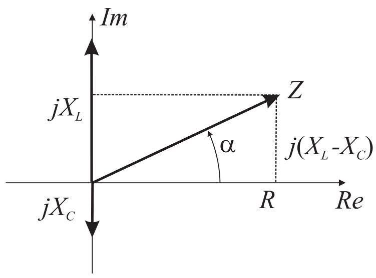
$$Z = R + \mathrm{j}(X_L - X_C) \tag{7.36}$$

Impedance of a series RLC circuit

namely, the sum of the impedances of every element in the series circuit.

This impedance can alternatively be expressed in terms of its magnitude and phase as

$$Z = |Z|\mathrm{e}^{\mathrm{j}\alpha} \tag{7.37}$$



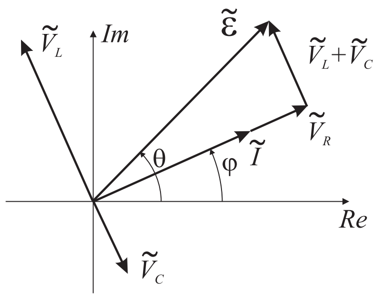
where

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2} \tag{7.38}$$

and

$$\alpha = \arctan\left(\frac{X_L - X_C}{R}\right) \,. \tag{7.39}$$

From expression (7.35), the current phasor $\tilde{I} = I_0\mathrm{e}^{\mathrm{j}\varphi}$ can be obtained as



$$\tilde{I} = \frac{\tilde{\xi}}{Z} \,. \tag{7.40}$$

Introducing now (7.31) and (7.37) into the above expression, phasor $\tilde{I}$ can be rewritten as

$$\tilde{I} = \frac{\xi_0}{|Z|}\,\mathrm{e}^{\mathrm{j}(\theta-\alpha)}$$

and, then, the magnitude and phase of the current phasor are given by

$$I_0 = \frac{\xi_0}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{7.41}$$

and

$$\varphi = \theta - \arctan\left(\frac{X_L - X_C}{R}\right) \,. \tag{7.42}$$

Finally, the time-harmonic expression for the current can be obtained after substituting the above expressions for I_0 and φ into $I(t) = I_0\cos(\omega t + \varphi)$.

Activity 7.5:

- Try to find the differential equation for the current in the RLC series circuit. At the light of this differential equation, can you see the clear advantages of using phasors and impedances?
- Are Kirchhoff's rules valid for any circuit and time-varying current? Justify your answer. [Maybe you have seen somewhere that there are some limits of validity for Kirchhoff's rules; however, these cases are beyond the scope of the present lesson.]
- In the RLC series circuit, what are the direct consequences of having $X_L = X_C$?

7.2.4 Resonance

If the magnitude of the current phasor (amplitude of the time-harmonic current) for the series RLC circuit given by (7.41) is explicitly expressed in terms of the frequency, it is obtained

$$I_0(\omega) = \frac{\xi_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{7.43}$$

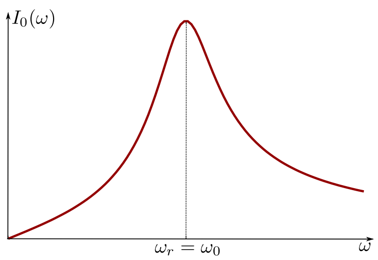
or, equivalently,

$$I_0(\omega) = \frac{\xi_0}{\sqrt{R^2 + \frac{L^2}{\omega^2}\left(\omega^2 - \frac{1}{LC}\right)^2}} \,. \tag{7.44}$$

Defining now the frequency ω_0 as

$$\omega_0^2 = \frac{1}{LC} \tag{7.45}$$

we can rewrite (7.44) as



$$I_0(\omega) = \frac{\omega \xi_0}{\sqrt{\omega^2 R^2 + L^2\left(\omega^2 - \omega_0^2\right)^2}} \tag{7.46}$$

where it can be observed that the amplitude of the time-harmonic current in the series RLC circuit clearly depends on the operation frequency and has a maximum at $\omega = \omega_0$. This phenomenon of having a maximum response of the system at an specific frequency when driven by an external time-harmonic agent is known in general as **resonance** and appears in many practical situations that involve *forced oscillations*.[2]

[2] Forced oscillations appear in systems in which some physical quantity (position, charge, mass,..) undergoes oscillations due to the action of an external time-harmonic agent.

The frequency, ω_r, corresponding to the maximum of amplitude is known as **resonance frequency** and, for the series RLC circuit, is found to be

$$\omega_r = \omega_0 \equiv \frac{1}{\sqrt{LC}} . \tag{7.47}$$

At this frequency it is also found that

$$X_L = X_C \tag{7.48}$$

namely, the two reactances cancel out each other, thereby making the series LC combination to behave as a short circuit. It means that, according to (7.39), the impedance at resonance in an AC RLC series circuit is purely real and given by R.

The resonance phenomena have multiple practical applications; for instance, if the series RLC circuit is employed as the tuning element in a radio apparatus, the capacitance can be varied in order to change the resonance frequency of the series RLC circuit so that it coincides with the emitting frequency of any particular radio station.

Activity 7.6:

- Can you find any reason to explain that the amplitude of the current in the series RLC circuit depends on the frequency of the time-harmonic voltage source?
- Plot I_0 versus ω according to Eq. (7.46). Explain what would happen if $R \to 0$?
- Prove that at resonance the voltage amplitude across the inductor or the capacitor can be much greater than the ampliutde of the emf source.
- Find the amplitude of the current if the voltage source is now in parallel with the capacitor and also in parallel with the series connection of the resistor and the inductor. Is there resonance in this case?

7.2.5 (*) Loop current method in AC circuits

The resolution of the series RLC circuit in AC has been an example on how the use of phasors and impedances makes the analysis of the AC circuit completely equivalent to a DC circuit in which the current and voltage quantities are now phasors and the resistances are substituted by impedances. It was a clear indication that all the techniques introduced in Sec. 5 for solving DC circuits can now be applied to solving AC circuits after considering the above equivalences.

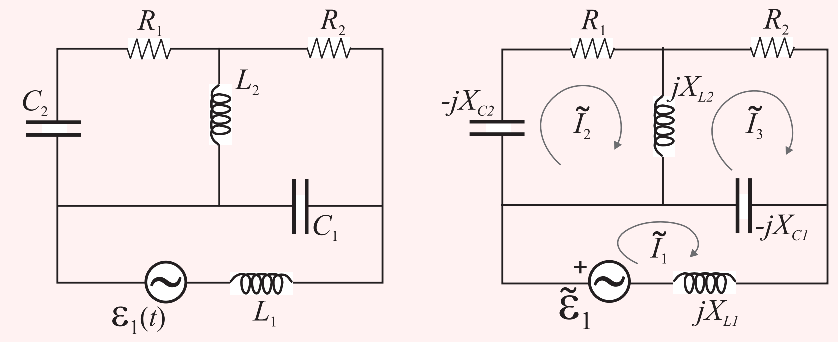


Figure 7.2: AC circuit with three loops.

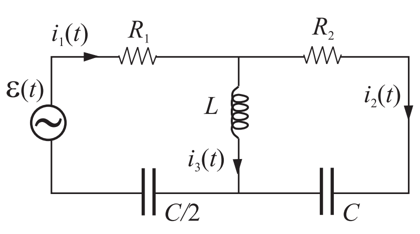
As an example, the circuit shown in Fig. 7.2 will be solved by following the loop current procedure described in Sec. 5.6.1. Thus, if we define a loop-current phasor in each of the three loops in the circuit and take into account the corresponding impedances of the different elements in the network, the matrix equation to be solved is

$$\begin{bmatrix} \tilde{\xi}_1 \\ 0 \\ 0 \end{bmatrix} = \left[Z_{ij}\right] \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ \tilde{I}_3 \end{bmatrix} ,$$

where the impedance matrix is given by

$$\left[Z_{ij}\right] = \begin{bmatrix} j(X_{L1} - X_{C1}) & 0 & jX_{C1} \\ 0 & R_1 + j(X_{L2} - X_{C2}) & -jX_{L2} \\ jX_{C1} & -jX_{L2} & R_2 + j(X_{L2} - X_{C1}) \end{bmatrix} .$$

In the practical resolution of circuit with several loops it is convenient to work directly, if possible, with numerical values rather than with formal expressions. Operations with complex numbers always reduce to another complex number but symbolic operations with complex quantities can usually give rise to very long expressions.



EXAMPLE 7.1 *Find the phasor and time-harmonic currents in the circuit shown in the figure; namely, the quantities $\tilde{I}_1,\tilde{I}_2,\tilde{I}_3$, and $i_1(t), i_2(t), i_3(t)$.*
Data: $\xi(t) = 20\sin(4\times 10^4 t)$ V, $R_1 = 8\,\Omega$, $R_2 = 4\,\Omega$, $L = 0.2$ mH, $C = 3.125\,\mu$F.

First we have to obtain the emf phasor and the impedance of each element. As the voltage source suppliess a harmonic emf given by

$$\xi(t) = 20\sin(4\times 10^4 t)\ \text{V} = 20\cos(4\times 10^4 t - \pi/2)\ \text{V}$$

it means that the operation angular frequency is

$$\omega = 4\times 10^4\ \text{rad/s}$$

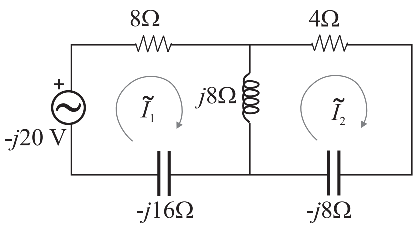
and the corresponding emf phasor is then

$$\tilde{\xi} = 20e^{-j\pi/2} = -j20\ \text{V} .$$

The inductive and capacitance reactances associated with the inductor and the capacitor respectively are

$$X_L = \omega L = 4\times 10^4 \cdot 2\times 10^{-4} = 8\Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{4\times 10^4 \cdot 3.125\times 10^{-6}} = 8\Omega$$

and, thus, the resulting "phasor-like" circuit to be solved is the one shown in the attached figure.



The equations for the loop-current phasors, $\tilde{I}_1$ and $\tilde{I}_2$, are

$$\begin{bmatrix} -j20 \\ 0 \end{bmatrix} = \begin{bmatrix} 8-j8 & -j8 \\ -j8 & 4 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}$$

or, equivalently (after dividing by 4),

$$\begin{bmatrix} -j5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-j2 & -j2 \\ -j2 & 1 \end{bmatrix} \begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix} .$$

The loop-current phasors can now be obtained after considering that

$$\tilde{I}_2 = 2j\tilde{I}_1$$

and being introduced in the first equation,

$$-j5 = (2-j2)\tilde{I}_1 - j2j2\tilde{I}_1 = (2-j2+4)\tilde{I}_1 = (6-j2)\tilde{I}_1 .$$

Simplifying it is obtained

$$\tilde{I}_1 = \frac{-j5}{6-j2} = \frac{-j5(6+j2)}{(6-j2)(6+j2)} = \frac{-j5(6+j2)}{5\cdot 8} = \frac{-j6+2}{8} = \frac{1-j3}{4}$$

and also

$$\tilde{I}_2 = \frac{2j(1-j3)}{2\cdot 2} = \frac{3+j}{2} .$$

The phasor $\tilde{I}_3$ is calculated after noting that

$$\tilde{I}_3 = \tilde{I}_1 - \tilde{I}_2$$

and therefore

$$\tilde{I}_3 = \frac{1-j3}{4} - \frac{3+j}{2} = \frac{1-j3-6-j2}{4} = \frac{-5-j5}{4} .$$

Before writing the time-harmonic expressions for the currents it is convenient to express the above phasors (which were written in rectangular form) in polar form:

$$\tilde{I}_1 = \frac{\sqrt{10}}{4}\, e^{j\arctan(-3)} = \frac{\sqrt{10}}{4}\, e^{-j1.249}$$
$$\tilde{I}_2 = \frac{\sqrt{10}}{2}\, e^{j\arctan(1/3)} = \frac{\sqrt{10}}{2}\, e^{j0.291}$$
$$\tilde{I}_3 = \frac{5\sqrt{2}}{4}\, e^{j\arctan(-1/-1)} = \frac{5\sqrt{2}}{4}\, e^{j5\pi/4} .$$

(Note that $\tilde{I}_3$ is located in the third quadrant of the complex plane and, therefore, its phase will be $\pi + \pi/4 = 5\pi/4$).

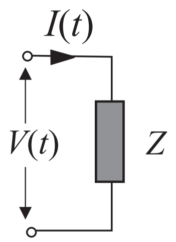
Finally, the time-harmonic currents are given by

$$
\begin{aligned}
i_1(t) &= \frac{\sqrt{10}}{4}\cos(4\times 10^4 t - 1.249)\ \mathrm{A}\\
i_2(t) &= \frac{\sqrt{10}}{2}\cos(4\times 10^4 t + 0.291)\ \mathrm{A}\\
i_3(t) &= \frac{5\sqrt{2}}{4}\cos(4\times 10^4 t + 5\pi/4)\ \mathrm{A}\,.
\end{aligned}
$$

Activity 7.7:

- The only known effective way to familiarize yourself with the analysis of AC circuits is to solve as many AC-circuit exercises as possible. Some proposed problems can be found in Sec. 7.4 and many more in other textbooks and elsewhere.

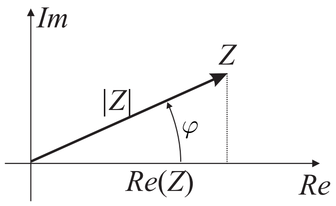
7.3 Power in AC circuits



7.3.1 Average power

Let us consider a branch of an AC circuit characterized by a load of impedance Z, which stands either for a single element or for any network of resistors, capacitors, and inductors. If the impedance of this load is written as

$$Z = |Z|\mathrm{e}^{\mathrm{j}\varphi}$$



and the phase of the voltage phasor, $\tilde{V}$, is set equal to zero

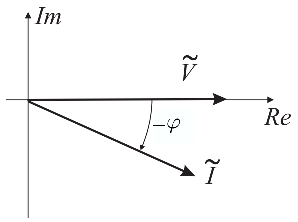
$$\tilde{V} = V_0 \tag{7.49}$$

(this simplifying assumption will not affect the conclusions and results of this section), we will find that the current phasor is

$$
\begin{aligned}
\tilde{I} &= \frac{\tilde{V}}{Z} = \frac{V_0}{|Z|\mathrm{e}^{\mathrm{j}\varphi}} = \frac{V_0}{|Z|}\mathrm{e}^{-\mathrm{j}\varphi}\\
&= I_0\mathrm{e}^{-\mathrm{j}\varphi}
\end{aligned} \tag{7.50}
$$

namely, the magnitude of the current phasor is given by

$$I_0 = \frac{V_0}{|Z|} \tag{7.51}$$



with its argument being $-\varphi$.

The time-harmonic expressions for the voltage and current associated with the above phasors are then

$$
\begin{aligned}
V(t) &= V_0\cos(\omega t)\\
I(t) &= I_0\cos(\omega t - \varphi)
\end{aligned}
$$

which leads to the following expression for the **time-dependent power**, $P(t)$, dissipated in the load:

$$P(t) = I(t)V(t) = I_0 V_0 \cos(\omega t)\cos(\omega t - \varphi) \,. \tag{7.52}$$

Taking into account that

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right] \,,$$

expression (7.52) can be expressed as

$$P(t) = \frac{1}{2} I_0 V_0 \cos\varphi + \frac{1}{2} I_0 V_0 \cos(2\omega t - \varphi) \,. \tag{7.53}$$

It can be observed that this power is a time *periodic* function whose period is $T = \pi/\omega$. Due to this periodic nature, the most convenient quantity to to measure the "power dissipated in the load" is the **average power**, $P_{\rm avg}$, dissipated in a period, whose expression is given by

$$P_{\rm avg} = \langle P(t) \rangle = \frac{1}{T}\int_0^T P(t)\mathrm{d}t \,. \tag{7.54}$$

This average power is the usual value given when referring to the amount of power dissipated by any electric AC device. This quantity allows us to calculate the energy, $\Delta\mathcal{E}$, consumed by the device in a given time interval $\Delta t \gg T$ as

$$\Delta\mathcal{E} \simeq P_{\rm avg}\,\Delta t \,.$$

After substituting (7.52) into (7.54) to obtain the average power, it is obtained

$$\begin{aligned} P_{\rm avg} &= \frac{1}{2T} I_0 V_0 \cos\varphi \int_0^T \mathrm{d}t + \frac{1}{2T} I_0 V_0 \int_0^T \cos(2\omega t - \varphi)\,\mathrm{d}t \\ &= \frac{1}{2T} I_0 V_0 \cos\varphi\, T + 0 \end{aligned} \tag{7.55}$$

given that the second integral above is null. It can finally be concluded that

Dissipated average power

$$\boxed{P_{\rm avg} = \frac{1}{2} I_0 V_0 \cos\varphi = I_{\rm rms} V_{\rm rms} \cos\varphi} \,. \tag{7.56}$$

As mentioned in Sec. 1.10.1, the root-mean-square value of the current, $I_{\rm rms}$ is given by $I_{\rm rms} = I_0/\sqrt{2}$.

From an practical point of view, it is worth noting that the average power could also have been defined as

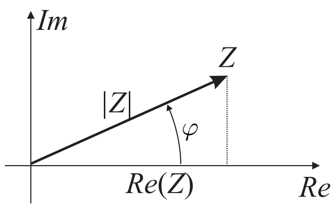
$$P_{\rm avg} = \frac{1}{2}\mathrm{Re}\left(\tilde{V}\tilde{I}^*\right) = \frac{1}{2}\mathrm{Re}\left(\tilde{V}^*\tilde{I}\right) \,. \tag{7.57}$$

Activity 7.8:

- Is the instantaneous power $P(t)$ given in (7.53) a time-harmonic function?
- Can we associate a phasor $\tilde{P}$ with the instantaneous power $P(t)$.
- Explain the reasons for the convenience of employing the average power instead of the time-dependent expression of the power.
- Carry out the steps to get Eq. (7.56) from Eq. (7.55).
- Show that Eq. (7.57) for computing the average power is equivalent to Eq. (7.56).
- Can you deduce the conditions that make the average power become null? Can you find physical reasons for that?

7.3.2 Power factor

In the expression (7.56) for the average power dissipated in load Z, it should be pointed out the appearance of factor $\cos\varphi$, known as **power factor**. This fact can easily be related to the load impedance Z through the following expression:



$$\cos\varphi = \frac{\text{Re}(Z)}{|Z|} . \tag{7.58}$$

This power factor is very relevant to determine the power consumption of the electric device since it can have any value within the range [0, 1]; negative values are excluded because the real part of the impedance of a passive element cannot be negative. For instance, in a network composed only of resistors (or any other network also with capacitors and inductors working at resonance) where the phase shift between the current and voltage is found to be null, the power factor is 1 and, consequently, the power consumption is maximum. On the contrary, if the phase shift is $\pm\pi/2$, the power consumption will be null. This last condition is found in reactive elements, inductors and/or capacitors, since no Joule's effect is taking place in these elements (it should be reminded that these elements only store energy rather than dissipate it).

There is no power consumption in inductors and capacitors

Taking into account (7.58), the average power could also be written as

$$P_{\text{avg}} = I_{\text{rms}}V_{\text{rms}}\cos\varphi = I_{\text{rms}}|Z|I_{\text{rms}}\frac{\text{Re}(Z)}{|Z|} = I_{\text{rms}}^2\,\text{Re}(Z) \tag{7.59}$$

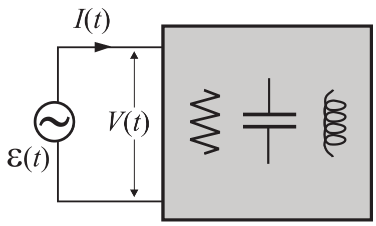
or with an equivalent expression in term of V_{rms}.

Activity 7.9:

- When is the maximum power consumption found in a series RLC circuit? Justify your answer.
- When is the maximum power consumption found in a parallel RLC circuit? (all the elements in parallel). Justify your answer.
- Can we have a network of resistors, capacitors, and inductors where the power consumption becomes null? Justify your answer. Show that an example of that is a parallel LC in series with a resistor.

7.3.3 Power balance in AC circuits

Expression (7.59) points out that the average power dissipated in the load can directly be related to the real part of the impedance of that load. If the load was composed of elements connected in series, then the real part of the impedance will simply be given by the sum of the involved resistances. However, in general, the reactive elements (inductors and capacitors) would affect the real part of the impedance. We have already discussed that the power consumption only takes place in the resistors and NOT in inductors and/or capacitors. However, it does not mean that the reactive elements do not have any influence on the power consumption. We should better say that power is only dissipated in the resistors but the presence of capacitors and/or inductors determines the specific amount of power dissipated in the resistors.



In the example of the circuit shown in the attached figure, where a voltage source is feeding a circuit, an analysis similar to the one carried out in Sec. 7.3.1 would say that the instantaneous power supplied by the generator of emf $\xi(t)$, which provides a current $I(t)$, is given by

$$P(t) = \xi(t)I(t)\,. \tag{7.60}$$

The average power supplied by this generator will then be

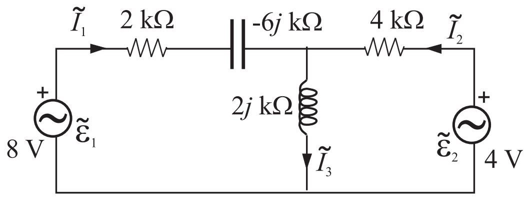
Average power supplied by a voltage source

$$P^{\text{gen}}_{\text{avg}} = \frac{1}{T}\int_0^T \xi(t)I(t)\mathrm{d}t = \frac{1}{2}\mathrm{Re}\left(\tilde{\xi}\tilde{I}^*\right)\,. \tag{7.61}$$

As the average powers given in (7.61) and (7.56) stand respectively for the energy per period supplied by the source and the one dissipated in the circuit, it must be satisfied that

the sum of the average powers supplied by the generators must equal the sum of the average powers dissipated in the resistors.

EXAMPLE 7.2 *In the circuit shown in the figure, check that the average power supplied by the voltage source is equal to the sum of the average powers dissipated in the resistors.*



Taking into account that $\tilde{\xi}_1 = 8$ and $\tilde{\xi}_2 = 4$, after solving the circuit equation to obtain the phasor currents in each branch, it is found that

$$\begin{aligned}\tilde{I}_1 &= 1 + \mathrm{j}\ \mathrm{mA} = \sqrt{2}\,\mathrm{e}^{\mathrm{j}\pi/4}\ \mathrm{mA}\\ \tilde{I}_2 &= 1 - \mathrm{j}\ \mathrm{mA} = \sqrt{2}\,\mathrm{e}^{-\mathrm{j}\pi/4}\ \mathrm{mA}\\ \tilde{I}_3 &= 2\ \mathrm{mA}\,.\end{aligned}$$

The voltage phasors in the resistors can be obtained by simply multiplying the corresponding current phasors by the value of the resistance, in such a way that

$$\begin{aligned}\tilde{V}_{2\mathrm{k}\Omega} &= 2\sqrt{2}\,\mathrm{e}^{\mathrm{j}\pi/4}\ \mathrm{V}\\ \tilde{V}_{4\mathrm{k}\Omega} &= 4\sqrt{2}\,\mathrm{e}^{-\mathrm{j}\pi/4}\ \mathrm{V}\,.\end{aligned}$$

The average power, P_{avg}, dissipated in each of the resistors can be obtained according to (7.57) to give

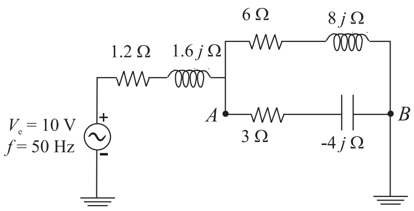
$$\begin{aligned}P_{\mathrm{avg}}(R = 2\mathrm{k}\Omega) &= 2\ \mathrm{mW}\\ P_{\mathrm{avg}}(R = 4\mathrm{k}\Omega) &= 4\ \mathrm{mW}\,.\end{aligned}$$

Analogously, the average power supplied by each voltage source will be

$$\begin{aligned}P_{\mathrm{avg}}(\xi_1) &= \frac{1}{2}\mathrm{Re}\left(\tilde{I}_1\tilde{\xi}_1^*\right) = 4\ \mathrm{mW}\\ P_{\mathrm{avg}}(\xi_2) &= \frac{1}{2}\mathrm{Re}\left(\tilde{I}_2\tilde{\xi}_2^*\right) = 2\ \mathrm{mW}\,.\end{aligned}$$

Finally it is obtained that the total average power supplied by the sources coincides with the total average power dissipated in the resistors.

EXAMPLE 7.3 *In the circuit of the attached figure, find: (a) the instantaneous and rms currents in the voltage source; (b) the average power dissipated in the circuit; (c) the Thevenin equivalent between terminals A and B; and (d) the energy stored in the inductor of inductive reactance $X_L = 1.6\,\Omega$ at any time t.*



(a) To compute the current phasor, $\tilde{I}$, flowing through the voltage source, we can first compute the impedance, Z, in series with the source. For that, we should note that

$$\frac{1}{Z_{AB}} = \frac{1}{6 + \mathrm{j}8} + \frac{1}{3 - \mathrm{j}4} = 0.18 + \mathrm{j}0.08 = 0.2\,\mathrm{e}^{\mathrm{j}0.418}$$

and therefore

$$Z_{AB} = 4.6 - j2 = 5\,e^{-j0.418}$$

which leads to

$$Z = (1.2 + j1.6) + (4.6 - j2) = 5.8 - j0.4 = 5.8\,e^{-j0.069}\ .$$

Now we can compute the current phasor from

$$\tilde{I} = \frac{\tilde{\xi}}{Z} = \frac{10}{5.8\,e^{-j0.069}} = 1.72\,e^{j0.069}$$

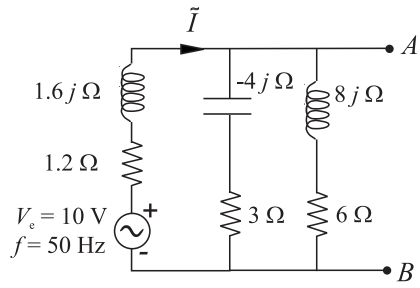
to obtain that

$$I_{\rm rms} = 1.72\,\mathrm{A}$$
$$I(t) = 2.43\cos(100\pi t + 0.069)\,\mathrm{A}\ .$$

[It should be reminded that the amplitude, I_0, of the instantaneous current is given by $I_0 = I_{\rm rms}\sqrt{2}$.]

(b) Taking into account that the average power dissipated in the circuit will equal the one supplied by the voltage source, making use of expression (7.61), it is obtained

$$P_{\rm avg} = \frac{1}{2}\mathrm{Re}\left(\tilde{\xi}\tilde{I}^*\right) = 10 \times 1.72 \times \cos(0.069) = 17.16\,\mathrm{W}\ .$$



(c) To compute the Thevenin equivalent, it is convenient to draw the original circuit in the most convenient way shown in the attached figure. Thus, to compute the Thevenin impedance, $Z_{\rm TH}$, we have to calculate the equivalent impedance associated with the three branches connected in parallel that result after short-circuiting the voltage source; namely,

$$\frac{1}{Z_{\rm TH}} = \frac{1}{4.6 - j2} + \frac{1}{1.2 + j1.6}$$

which, after some operations, results in

$$Z_{\rm TH} = 1.43 + j0.95 = 1.72\,e^{j0.588}\ .$$

To compute the Thevenin voltage phasor, $\tilde{V}_{\rm TH}$, we should note that the three branches are shunt connected, and so

$$\tilde{V}_{\rm TH} = \tilde{V}_{AB} = Z_{AB}\tilde{I} = 8.6\,e^{-j0.349}\ ,$$

result that could have also been obtained by considering that

$$\tilde{V}_{\rm TH} = \tilde{\xi} - (1.2 + j1.6)\tilde{I}\ .$$

(d) The instantaneous value of the stored energy in the inductor can be computed using

$$U_m(t) = \frac{1}{2}LI^2(t)$$

which, after operating, is found to be

$$U_m(t) = \frac{1}{2}\frac{1.6}{100\pi}[1.72\sqrt{2}\cos(100\pi t + 0.069)]^2$$
$$= 0.015\cos^2(100\pi t + 0.069)\,\mathrm{J}\ .$$

7.4 Problems

7.1: A coil of 200 turns has a cross section of $4\,\mathrm{cm}^2$ and it is rotating in a magnetic field. Find the value of the magnitude of this magnetic field to generate an emf with amplitude of 10 V at 60 Hz?
Sol. 0,332 T.

7.2: Find the rms and the amplitude values of the current of a laundry dryer that provided an average power of 5.0 kW when connected to a network of rms voltage **a)** 240 V and **b)** 120 V.
Sol.: a) $I_{\mathrm{rms}} = 20.8$ A, $I_0 = 29.5$ A ; **b)** $I_{\mathrm{rms}} = 41.7$ A, $I_0 = 58.9$ A.

7.3: An electric appliance carries a current of 10 A rms and consumes an average power of 720 W when connected to a network of 120 V rms at 60 Hz. **a)** Find the absolute value of the impedance of this appliance. **b)** Fins the equivalent series connection of resistance and reactance. **c)** If the instantaneous current leads the emf, is the reactance capacitive or inductive?
Sol.: a) $|Z| = 12\,\Omega$; **b)** $R = 7,2\,\Omega$, $X = 9.6\,\Omega$;**c)** Capacitive.

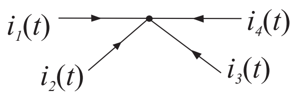
7.4: Find the amplitude, period, and initial phase of the harmonic function given by $f(t) = 7.32\cos(3.8\pi t + \pi/6)$ and plot it.

7.5: Write the dual rectangula/polar representation of the following complex numbers: $z_1 = -3j$, $z_2 = 3 + j4$, $z_3 = \mathrm{e}^{-j\pi/2}$, $z_4 = 4.6\mathrm{e}^{j\pi/3}$

7.6: Carry out the following operations with the complex numbers of the above problem: $z_1 + z_3$, $z_2 z_4$, z_2/z_4, z_2^3, e^{z_2}, $z_1^{z_3}$.

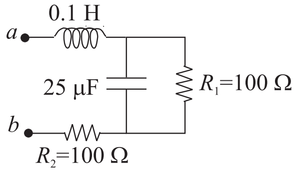
7.7: Using phasors, find the sum of $u(t) = u_1(t) + u_2(t)$ if $u_1(t) = 3\sin(2\pi t)$ y $u_2(t) = -2\cos(2\pi t)$.

7.8: Write the phasor associated to the derivative of the following harmonic function: $f(t) = 3.2\cos(2.5t + \pi/4)$.

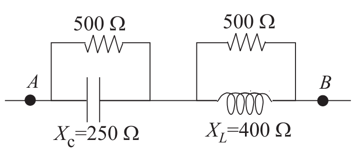


7.9: A given node of a circuit is the junction of four branches. The currents carried by three of them are $i_1(t) = 3\cos(\omega t)$ A, $i_2(t) = 4\cos(\omega t + \pi/4)$ A, and $i_3(t) = 2\cos(\omega t + 5\pi/4)$ A. Using phasor analysis, find the current $i_4(t)$ in the remaining branch.
Sol.: $i_4(t) = 4.414\cos(\omega t + 0.31)$ A.

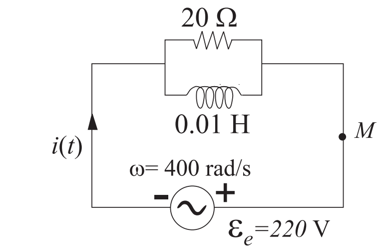
7.10: A resistor of $10\,\Omega$ and a load of impedance Z are series connected. The rms voltage drops in the resistor and the load are 10 V and 5 V respectively. The rms voltage drop in the above series connection is 12 V. If the operation frequency is 100 Hz, find the value of the load impedance and which series-connected elements would be required to build that load.
Sol.: $Z = 0.95 + 4.9j\,\Omega$, the load can be composed of resistor of $0.95\,\Omega$ in series with an inductor of $L \approx 7.8$ mH.



7.11: In the circuit of the figure, find the voltage difference in the resistor R_2 when it is connected between terminals a and b: **a)** a DC voltage source of 100 V; **b)** an AC voltage source of rms emf 100 V and frequency $f = 400/\pi$ Hz.
Sol.: a) 50 V; **b)** $V(t) = 79.05\sqrt{2}\cos(800t - 0.3217)$ V.



7.12: In the circuit of the figure there is an AC voltage source connected at terminals A and B with rms emf of 500 V and frequency 50 Hz. Find: **a)** the total impedance between terminals A and B; **b)** the current, $i(t)$, driven by the voltage source; **c)** the capacitance of the capacitor and the inductance of the coil; **d)** the average power dissipated in the circuit.
Sol.: a) $Z_{AB} = (100/41)(121 + 18j)\,\Omega$; **b)** $i(t) = 2.37\cos(100\pi t - 0.1477)$ A;
c) $C = 12.73\,\mu$F, $L = 1.273$ H; **d)** $P = 828.8$ W.

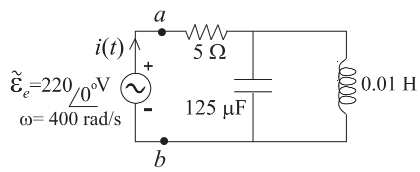


7.13: In the circuit of the figure, find: **a**) the impedance of each element and the equivalent admittance of the network; **b**) the current $i(t)$ driven by the source; **c**) the current phasors in the branches of the resistor and the inductor. Also plot the phasor diagram for the currents; **d**) the value of the capacitance, C, which series-connected at point M will make that the current driven by the source is in phase with voltage of that source.
Sol.: **a**) $R = 20\ \Omega$, $Z_L = 4j\ \Omega$; **b**) $i(t) = 56.09\sqrt{2}\cos(\omega t - 1.3734)$ A;
c) $\widetilde{I}_R = 11\sqrt{2}$ A, $\widetilde{I}_L = -55\sqrt{2}j$ A; **d**) $C = 650\ \mu$F.

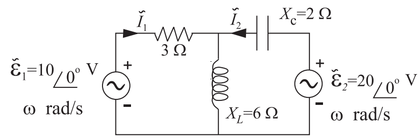
7.14: A coil of 0.1 H is series-connected to a resistor of 10 Ω and a capacitor. The capacitor is chosen so that the circuit is in resonance when is driven by an AC voltage source of 100 V (maximum voltage) and 60Hz. Find the value of the capacitance and the voltage difference in the capacitor ($V_C(t)$) and in the coil ($V_L(t)$).
Sol.: $C = 70.4\ \mu$F, $V_C(t) = 120\pi\ \cos(120\pi t + \pi/2)$V, $V_L(t) = 120\pi\ \cos(120\pi t - \pi/2)$V.

7.15: A radio receptor is tuned to receive the signal emitted by a radio station. The tuning circuit (modeled by a series RLC connection) has a capacitor of 32.3 pF and a coil of 0.25 mH. Find the emission frequency of the radio station.
Sol.: 1.77 MHz.

7.16: A method to measure inductances consists in connecting the coil in series with a capacitor and a resistor of known values of capacitance and resistance, respectively. The frequency of a driving AC generator is changed until detecting a maximum in the current carried by the circuit. If $C = 10\ \mu$F, $\xi_{\rm max} = 10$ V, $R = 100\ \Omega$, with the current being maximum at $\omega = 5000$ rad/s, find the values of L and $I_{\rm max}$.
Sol. $L = 4$ mH, $I_{\rm max} = 100$ mA.

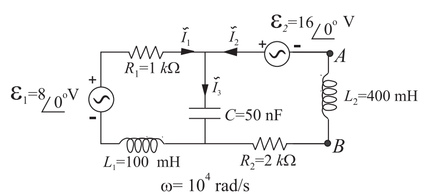


7.17: In the circuit of the attached figure, find: **a**) the impedance Z_{ab}; **b**) the instantaneous current, $i(t)$, flowing through the emf source; **c**) the average power supplied by the source and the one dissipated by the resistor (verifying the power balance); **d**) the element that should be connected between points a and b so that the voltage and the current in the source are in phase.
Sol.: **a**) $Z_{ab} = 5 + 5j\ \Omega$; **b**) $i(t) = 44\ \cos(400t - \pi/4)$ A; **c**) $P_{\rm act} = P_R = 4840$W; **d**) a capacitor of 250 μF.



7.18: In the circuit shown in the figure, find: **a**) the currents (instantaneous and phasor expressions) and plot their corresponding phasor diagram; **b**) the average power supplied by the source and dissipated in the circuit.
Sol.: **a**) $\widetilde{I}_1 = -10(1+{\rm j})/3$ A, $\widetilde{I}_2 = 5$ A, $i_1(t) = 10\sqrt{2}/3\cos(\omega t - 3\pi/4)$ A, $i_2(t) = 5\cos(\omega t)$ A;
b) source (1) supplies -50/3 W, source (2) supplies 50 W, power dissipated in the resistor 100/3 W.

7.19: We wish to design a series RLC device to be connected to an AC voltage source with inner resistance R_s and operation frequency ω. Find the values of R, L and C (in terms of ω and R_s) so that the circuit satisfy the following specs: 1) the rms voltage at the resistor terminal is equal to the one in the inductor; 2) the device should be globally resistive (namely, it should be equivalent to a resistor); 3) the power dissipated in R_s should be equal to the one dissipated in the RLC device. Also find the current supplied by the source if the maximum voltage amplitude of that source is V_0.
Sol.: $R = R_s$, $L = R_s/\omega$ y $C = 1/(\omega R_s)$; $i(t) = \dfrac{V_0}{2R_s}\cos(\omega t)$



7.20: In the circuit shown in the figure, **a**) find the instantaneous and phasor currents, and plot its corresponding phasor diagram; **b**) calculate the average powers supplied by the sources and dissipated in the circuit; **c**) find the Thevenin equivalent from terminals A and B, giving also the current that would flow through these terminal assuming that a capacitor of 50 nF was connected between them.
Sol.: **a**) $\widetilde{I}_1 = 2 + 6j$ mA, $\widetilde{I}_2 = 2$ mA, $\widetilde{I}_3 = 4 + 6j$ mA,
$i_1(t) = \sqrt{40}\cos(10^4 t + \arctan(3))$ mA, $i_2(t) = 2\cos(10^4 t)$ mA,
$i_3(t) = \sqrt{52}\cos(10^4 t + \arctan(3/2))$ mA;
b) Power supplied: $P_1 = 8$ mW, $P_2 = 16$ mW, Power dissipated: $P_{R_1} = 20$ mW, $P_{R_2} = 4$ mW;
c) $\widetilde{V}_{\rm Th} = 8j$, $Z_{\rm Th} = (2+2j)$ kΩ, $i_C(t) = 4\cos(10^4 t - \pi/2)$ mA.

Part IV

APPENDICES

Appendix A

Gauss's law

Gauss's law ($\sim$ 1867) states that the flux of the electric field produced by a given charge distribution through a closed surface S is equal to $1/\epsilon_0$ times the total charge, Q_{int}, enclosed by the surface; namely,

$$\boxed{\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}} \tag{A.1}$$

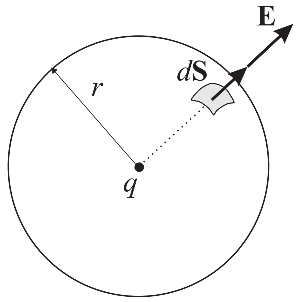
Gauss's law

In order to justify (not to prove it) Gauss's law, let us consider the electric field produced by a point charge:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} .$$

Now it should be noted that expression (2.7) says that the electric field at the points of a spherical surface of radius r centered in the position of the charge q can be written as

$$\vec{E} = |\vec{E}(r)| \hat{\mathbf{r}} . \tag{A.2}$$



It means the magnitude of the field only depends on the radius of the sphere (rather than on the exact position over this sphere) and its direction coincides with the outward normal to this sphere at each point (this field then has spherical symmetry).

If the following integral is performed (see Sec. 1.9.4):

$$\oint_{\text{surf}} \vec{E} \cdot d\vec{S} , \tag{A.3}$$

[known as **flux of the electric field**, Φ] for the field produced by the point charge q over a spherical surface of radius r centered at the charge position, it is obtained that

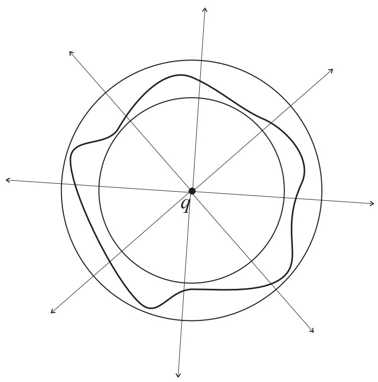
$$\Phi = \oint_{\text{surf}} \vec{E} \cdot d\vec{S} = |\vec{E}(r)| \oint_{\text{surf}} \hat{\mathbf{r}} \cdot d\vec{S} \tag{A.4}$$

since $|\vec{E}(r)|$ is *constant* over the spherical surface. Taking now into account that

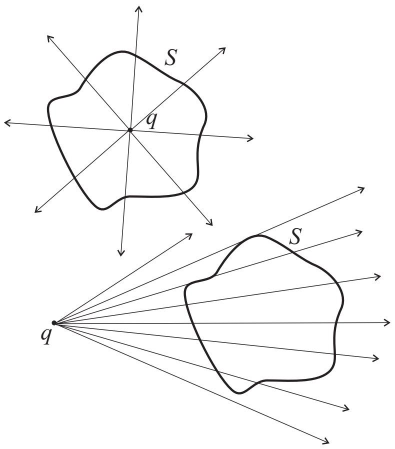
$$\hat{\mathbf{r}} \cdot d\vec{S} = dS \qquad (\hat{\mathbf{r}} \parallel d\vec{S})$$

the integral (A.3) can be rewritten in the present case as

$$\begin{aligned}\Phi &= |\vec{E}(r)| \oint_{\text{surf}} dS = |\vec{E}(r)| \times (\text{Area of the sphere}) \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(4\pi r^2) = \frac{q}{\epsilon_0}\,. \end{aligned} \tag{A.5}$$



It is worthy to note that the flux Φ *does not depend* on the radius of the sphere and its value is given by the charge enclosed by the surface divided by ϵ_0. Had we considered another sphere centered at the same point but with different radius, the flux would have been the same. It seems then reasonable to suppose that the flux through any other surface bounded by both two previous spherical surfaces would also be q/ϵ_0.



Since the number of field lines passing through any of the previous surfaces is the same, the flux of the electric field through these surfaces could be interpreted as a "measure" of the number of field lines passing through them. In this way, if the number of field lines going through a closed surface is zero (that is, as many lines enter as they come out), the flux of the electric field through this surface turns out to be zero too. Thereby, we can write that the flux of the electric field produced by a point charge through an arbitrary closed surface, S, is

$$\Phi = \oint_S \vec{E}\cdot d\vec{S} = \begin{cases} \dfrac{q}{\epsilon_o} & \text{if } \; q \subset S \\ 0 & \text{otherwise}\,. \end{cases} \tag{A.6}$$

If we now have a distribution of point charges, the application of the superposition principle would lead to

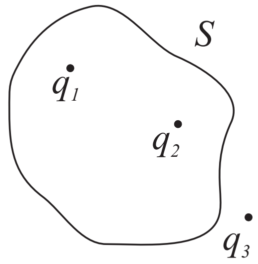
$$\Phi = \oint_S \vec{E}\cdot d\vec{S} = \oint_S \left(\sum_i \vec{E}_i\right)\cdot d\vec{S} = \sum_i \oint_S \vec{E}_i \cdot d\vec{S} = \sum_i \Phi_i \tag{A.7}$$

that is, the flux of the electric field produced by the charge distribution through surface S is the sum of the fluxes associated with each charge individually. As the flux of one point charge was already obtained in (A.6), it can be then conclude that

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{\text{int}}}{\epsilon_0}$$

where Q_{int} stands for the total charge enclosed by surface S. The above expression would also hold for continuous distributions of charge.

Example A.1 *Application of Gauss's law*
Find the flux of $\vec{E}$ through S for the case shown in the attached figure.



In the situation shown in the figure, the charge inside the surface S is just

$$Q_{\rm int} = q_1 + q_2$$

and, therefore, the flux through this surface according to (2.11) will be

$$\begin{aligned}\Phi &= \oint_S \vec{E}\cdot d\vec{S} \\ &= \frac{q_1+q_2}{\epsilon_0}\,.\end{aligned}$$

Although Gauss's law (2.11) is **valid** for any charge distribution and surface, this law is mostly employed to obtain the electric field in situation of *high symmetry*. These situations are usually found when there exists a Gauss surface, S_G, such that in those part where the flux is different from zero (this part of the surface will be denoted as S'_G), the flux integral can be performed taking advantage of the fact that the magnitude of the electric field is *constant* over such surface. In that case we could write

Gauss's law is useful to compute the field produced by high-symmetry charge distributions

$$\Phi = \oint_{S_G} \vec{E}\cdot d\vec{S} = |\vec{E}| \oint_{S'_G} dS\,. \tag{A.8}$$