SOME RESULTS CONCERNING PDES FROM MECHANICS

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Abstract

This short course deals with some nonlinear PDEs that describe important phenomena in mechanics. More precisely, we will consider the stationary and the time-dependent Navier-Stokes equations and also some other PDE systems that arise in nonlinear elasticity. We will recall the main known results (existence, conditional uniqueness, regularity, etc.) and will give a list of related open problems. Among others, we will use Galerkin approximation techniques, appropriate compactness results and elementary polyconvex analysis.

Extended Abstract

This short course deals with some nonlinear PDEs that describe important phenomena in mechanics. The main concepts, results and arguments are the following:

• The stationary Stokes system - Existence and uniqueness of a weak solution.

A linear non-scalar system for $\mathbf{u}$ (a velocity field) and p (a pressure). In the definition of the weak solution p disappears.

Th existence and uniqueness of a weak solution can be easily proved using Lax-Milgram Lemma. This gives $\mathbf{u}$. Then, p can be found using De Rham's Lemma.

• The stationary Navier-Stokes system - Existence and conditional uniqueness of a weak solution.

A similar nonlinear system. A positive constant ν appears multiplied to the second order term: it is the so called *viscosity coefficient.* The additional nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is due to "transport"; it can be neglected for *small velocity* (or large viscosity).

We mainly assume that the system is completed with homogeneous Dirichlet conditions.

The definition of weak solution is similar in this case. For the existence result, we use the so called *Galerkin compactness method:*

1. A family of Galerkin (finite dimensional) approximations $\mathbf{u}_m$ is introduced. We prove existence and uniform H^1 estimates of the $\mathbf{u}_m$.

2. A weakly convergent subsequence is extracted. From the compactness of Sobolev embeddings, we also get strong convergence in some L^p spaces and this allows to take limits and deduce that the limit is a weak solution.

For the uniqueness result, we look at the identities satisfied by the difference $\mathbf{w}=\mathbf{u}^1-\mathbf{u}^2$ of two weak solutions and get an estimate of the form $\nu\|\mathbf{w}\|^2_{H^1}\leq C\|\mathbf{w}\|^2_{H^1}$, where C only depends on the data of the problem. It is then easy to conclude that, if ν is large enough, uniquness holds.

When the Dirichlet conditions are not homogeneous, the results are more involved (and less satisfactory)

• Regularity results.

They are similar to the usual reults for elliptic systems: essentially, we gain two regularity orders with respect to the right-hand side (for instance, passing from L^2 to H^2).

For Stokes systems, the proof can be obtained (a) by first establishing uniform estimates of difference quotients or (b) by estimating the Galerkin approximations in strong spaces like H^2. For Navier-Stokes systems, it suffices to use Stokes regularity in a repeated feedback way and increase the regularity of the solution step by step.

• The relevance of Stokes and Navier-Stokes systems in control and inverse problems.

In fluid mechanics, control issues are of great relevance. For example, it makes sense to search for a right-hand side (or a boundary datum) such that an associated solution to the

Navier-Stokes system satisfies a desired property (for instance, the inequality $\|\mathbf{u}-\mathbf{u}_d\| \leq 10^{-5}$ for some $\mathbf{u}_d$).

Inverse problems have also a lot of interest. For instance, we can consider the Navier-Stokes equations in a domain of the form $\Omega \setminus \overline{D}$ for some *unknown* open set D and assume that the right-hand side vanishes, the boundary values $\mathbf{u}|_{\partial\Omega}$ and $\mathbf{u}|_{\partial D}$ are known and we also have access to the additional information $\beta := \mathbf{u}|_\omega$. Then the question os whether we can recover D from β.

• The time-dependent Navier-Stokes system - Existence of a weak solution and uniqueness for $N = 2$.

Now, we try to find a solution to the PDEs in a set of the form $Q := \Omega \times (0, T)$ and we complement the equations with boundary conditions on the lateral boundary $\Sigma := \partial\Omega \times (0, T)$ and initial conditions at $t = 0$.

Again, in the definition of a weak solution, the pressure disappears.

The existence of such a solution can be again established by applying a Galerkin compactness method (existence and uniform estimates of finite dimensional approximations, selection of convergent subsequences, etc.). Afterwards, the pressure can be once more obtained as a consequence of a suitable version of De Rham's Lemma.

For uniqueness, we consider again the function $\mathbf{w} = \mathbf{u}^1 - \mathbf{u}^2$ and we try to apply *Gronwall's Lemma* to $t \mapsto \|\mathbf{w}(\cdot\,, t)\|^2$. This can be done only when $N = 2$.

• Regularity results for $N = 2$ and $N = 3$ (for small data).

Appropriately modified arguments allow to get regularity results. In particular, we see that for $N = 2$ the solution depends continuously on the data. We can also prove results of the same kind when $N = 3$ and the data are small.

• Open problems concerning uniqueness and regularity.

We present a short summary of known results and open problems. In particular, we speak of *suitable weak solutions,* we mention several results concerning the non-uniqueness of a special kind of (very weak) solutions and we report some very recent achievements by OpenAI dealing with the famous *Navier-Stokes Millennium Problem.*

• Nonlinear elasticity - hyperelastic materials.

We give the definition, present some examples, explain the physical meaning of the variables and indicate the equations that, at least frmally, must be satisfied.

• Polyconvexity, existence results and open questions.

We introduce polyconvexity, a concept that allows to extend the Fundamental Principle of Calculus of Variations to a realistic framework in nonlinear Elasticity. We recall the main result by J. Ball an its proof and mention some related open questions.

Notation

• $\mathbf{R}$: The field of real numbers.

• $\mathbf{R}_+$: The set of positive real numbers.

• Ω, G, U, D: Open sets in $\mathbf{R}^N$.

• $\overline{\Omega}$ (resp. $\partial\Omega$): The closure (resp. the boundary) of Ω.

• $\mathbf{n}(\mathbf{x})$: The unit normal vector at $\mathbf{x}$ (a point on the boundary), outward directed.

• $\mathbf{a}, \mathbf{b}, \ldots$: N-dimensional vectors with components $a_i, b_j, \ldots$.

• $\mathbf{a} \otimes \mathbf{b}$: The $N \times N$ tensor with components $(\mathbf{ab})_{ij} = a_i b_j$.

• $\mathbb{1}_G$: The characteristic function of G, equal to 1 in G and 0 outside G.

• δ_{ij}: *Kronecker's symbol*, equal to 1 if $i = j$ and 0 otherwise.

• $C^0(U)$ and $C^0(\overline{U})$: The spaces of continuous functions $\varphi : U \mapsto \mathbf{R}$ and $\varphi : \overline{U} \mapsto \mathbf{R}$, respectively.

• $C^k(U)$: The space of functions with continuous partial derivatives of orders up to k in U.

• $C^\infty(U)$: The intersection of all $C^k(U)$ with $k \geq 1$.

• $C^k(\overline{U})$ (resp. $C^\infty(\overline{U})$): The space of the restrictions to $\overline{U}$ of the functions in $C^k(\mathbf{R}^N)$ (resp. $C^\infty(\mathbf{R}^N)$).

• Supp φ: The support of φ, i.e. the closure of the set where $\varphi \neq 0$.

• $C_c^k(\Omega)$ (resp. $C_c^k(\overline{\Omega})$): The subspace of $C^k(\Omega)$ (resp. $C^k(\overline{\Omega})$) formed by the functions with compact support in Ω (resp. $\overline{\Omega}$).

• $\mathcal{D}(\Omega)$ (resp. $\mathcal{D}(\overline{\Omega})$): The subspace of $C^\infty(\Omega)$ (resp. $C^\infty(\overline{\Omega})$) formed by the functions with compact support in Ω (resp. $\overline{\Omega}$).

• a.e.: "almost everywhere". It indicates that a property is satisfied at any point except possibly in a set of zero Lebesgue measure.

• H, G, ...: real Hilbert spaces; frequently, their norms and scalar products will be respectively denoted $\|\cdot\|_H$, $\|\cdot\|_G$, ... and $(\cdot\,,\cdot)_H$, $(\cdot\,,\cdot)_G$, ...

• X, Y, ...: real normed spaces with norms usually denoted $\|\cdot\|_X$, $\|\cdot\|_Y$, ...

• For any normed space X, we will denote by X' the corresponding dual space; also, $\langle\cdot\,,\cdot\rangle_{X,X'}$ or simply $\langle\cdot\,,\cdot\rangle$ will stand for the associated duality pairing.

• $C_w^0([0,T];X)$: the space of weakly continuous functions $f : [0,T] \mapsto X$, that is, the space of functions f such that, for any $h \in X'$, the real-valued function $t \mapsto \langle h, f(t)\rangle_{X',X}$ is continuous.

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Chapter 1

The Stationary Stokes and Navier-Stokes Equations

In this chapter, we will first consider the Dirichlet problem for the so called stationary Stokes system:

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}), \quad \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u} = \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \partial\Omega \end{cases} \tag{1.0.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded connected open set with Lipschitz-continuous boundary ($N = 2$ or $N = 3$), $\mathbf{f} \in L^2(\Omega)^N$ and $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$ (the trace of $\overline{\mathbf{u}}$) for some $\overline{\mathbf{u}} \in H^1(\Omega)^N$ with $\nabla \cdot \overline{\mathbf{u}} = 0$.

Then, we will consider the following similar nonlinear problem:

$$\begin{cases} -\nu\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}), \quad \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u} = \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \end{cases} \tag{1.0.2}$$

where Ω, $\mathbf{f}$ and $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$ are as before and $\nu > 0$. Note that $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is the field whose i-th component is given by $\mathbf{u} \cdot \nabla u_i = \sum_{j=1}^N u_j \partial_j u_i$. This is the stationary *Navier-Stokes* system.

1.1 Preliminaries

We will start by recalling an important technical result:

Lemma 1.1.1. *Let us set*

$$H_{\mathrm{div}}(\Omega) := \{\mathbf{v} \in L^2(\Omega)^N : \nabla \cdot \mathbf{v} \in L^2(\Omega)\}.$$

Then one has:

1. *$H_{\mathrm{div}}(\Omega)$ is a Hilbert space for the norm*

$$\|\mathbf{v}\|_{H_{\mathrm{div}}} := \left(\|\mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|^2|\right)^{1/2}.$$

2. *$C^1(\overline{\Omega})^N$ is dense in $H_{\mathrm{div}}(\Omega)$.*

3. *The mapping* $\mathbf{v} \in C^1(\overline{\Omega})^N \mapsto \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ *can be extended to a unique mapping* $\gamma_{\mathbf{n}} \in \mathcal{L}(H_{\rm div}(\Omega); H^{-1/2}(\partial\Omega))$ *(called the normal trace on* $\partial\Omega$*) such that*

$$\langle \gamma_{\mathbf{n}}\mathbf{v}, z|_{\partial\Omega}\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} = \int_{\partial\Omega} (\mathbf{v}\cdot\nabla z + (\nabla\cdot\mathbf{v})z)\, d\mathbf{x}$$

for any $v \in H_{\rm div}(\Omega)$ *and any* $z \in H^1(\Omega)$.

E 1.1. Prove Lemma 1.1.1.

In order to study (1.0.1), we need some particular function spaces:

$$\begin{array}{c} \mathcal{V} := \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^N : \nabla\cdot\boldsymbol{\varphi} = 0\}, \\ H := \{\mathbf{v} \in L^2(\Omega)^N : \nabla\cdot\mathbf{v} = 0, \quad \mathbf{v}\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega\}, \\ V := \{\mathbf{v} \in H_0^1(\Omega)^N : \nabla\cdot\mathbf{v} = 0\}. \end{array} \tag{1.1.1}$$

The spaces H and V coincide with the closures of $\mathcal{V}$ respectively in $L^2(\Omega)^N$ and $H_0^1(\Omega)^N$. These assertions are implied by the following result, that is known as *De Rham's Lemma:*

Lemma 1.1.2. *Assume that* $\mathbf{S} \in \mathcal{D}'(\Omega)^N$ *and*

$$\langle \mathbf{S}, \boldsymbol{\varphi}\rangle = 0 \quad \forall \boldsymbol{\varphi} \in \mathcal{V}. \tag{1.1.2}$$

Then there exists $q \in \mathcal{D}'(\Omega)$*, unique up to an additive constant, such that* $\mathbf{S} = \nabla q$*. Furthermore, if* $\mathbf{S} \in W^{m,r}(\Omega)^N$ *for* $m = -1$ *or* $m = 0$ *and* $r \in (1, +\infty)$*, then one has* $q \in W^{m+1,r}(\Omega)$ *and the mapping* $\mathbf{S} \mapsto q$ *can be defined as a continuous linear operator on* $W^{m,r}(\Omega)^N$.

The proof can be found in [J.-L. Lions(1969), Témam(2001)]. Note that the converse is a trivial assertion: indeed, if $\mathbf{S} = \nabla q$ for some $q \in \mathcal{D}'(\Omega)$, then (1.1.2) holds.

1.2 Existence and Uniqueness Results for the Stokes System

Let us solve (1.0.1) when $\mathbf{g} = 0$, that is, when $\mathbf{u}$ and p solve (in some sense) the system

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}), \quad \nabla\cdot\mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u} = \mathbf{0}, & \mathbf{x} \in \partial\Omega. \end{cases} \tag{1.2.1}$$

First, we introduce a *weak formulation* of the problem:

$$\begin{cases} \text{Find } \mathbf{u} \in V \text{ such that} \\ a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V, \end{cases} \tag{1.2.2}$$

where we have set

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x}, \tag{1.2.3}$$

that is, $a(\cdot\,,\cdot)$ is the usual scalar product in $H_0^1(\Omega)^N$ (and V).

Thanks to *Lax-Milgram's Theorem,* there exists exactly one solution to (1.2.2). This is by definition the weak solution to (1.0.1) for $\mathbf{g} = 0$.

Furthermore, if $\mathbf{u}$ solves (1.2.2), we can recover p from $\mathbf{u}$. Indeed, one has $-\Delta\mathbf{u} - \mathbf{f} \in H^{-1}(\Omega)^N$ and

$$\langle -\Delta\mathbf{u} - \mathbf{f}, \boldsymbol{\varphi}\rangle = a(\mathbf{u}, \boldsymbol{\varphi}) - (\mathbf{f}, \boldsymbol{\varphi}) = 0$$

for all $\boldsymbol{\varphi} \in \mathcal{V}$. Consequently, we see from De Rham's Lemma (Lemma 1.1.2) that there must exist $p \in L^2(\Omega)$ such that

$$-\Delta\mathbf{u} - \mathbf{f} = -\nabla p.$$

Thus, the following result has been proved:

Theorem 1.2.1. *Assume that* $\mathbf{f} \in L^2(\Omega)^N$ *and* $\mathbf{g} = 0$ *in* (1.0.1). *There exists exactly one solution* $(\mathbf{u}, p) \in H_0^1(\Omega)^N \times L^2(\Omega)$ *(*p *is unique up to an additive constant) in the following sense:*

1. $\mathbf{u}$ *is the weak solution to* (1.0.1).
2. $-\Delta\mathbf{u} + \nabla p = \mathbf{f}$ *in* $H^{-1}(\Omega)^N$, $\nabla \cdot \mathbf{u} = 0$ *in* $L^2(\Omega)$ *and* $\mathbf{u}|_{\partial\Omega} = 0$ *in* $H^{1/2}(\partial\Omega)^N$.

Remark 1.2.2. Several comments are in order:

- We can take $\mathbf{f} \in H^{-1}(\Omega)^N$ in (1.2.2). The same result holds. The unique modification is that in the second line in (1.2.2) we must write

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle_{H^{-1}, H_0^1} \quad \forall \mathbf{v} \in V.$$

- Let us assume that we impose to p the condition

$$\int_\Omega p\, d\mathbf{x} = 0.$$

  Then, the estimates implied by Lax-Milgram's Theorem and De Rham's Lemma, give at once

$$\|\mathbf{u}\|_{H_0^1} + \|p\| \leq C\|\mathbf{f}\|_{H^{-1}}, \tag{1.2.4}$$

  where C only depends on Ω.

- In a very similar way, other linear PDE systems can be solved. For instance, for any given $\mathbf{m} \in V$ and any $\mathbf{f} \in H^{-1}(\Omega)^N$, we can replace (1.0.1) by

$$\begin{cases} -\Delta\mathbf{u} + (\mathbf{m}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}), \quad \nabla\cdot\mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u} = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{1.2.5}$$

  where *linear transport* effects are considered. □

E 1.2. Using Lax-Milgram's Theorem and Lemma 1.1.2, prove (1.2.4).

E 1.3. Prove a result similar to Theorem 1.2.1 for (1.2.5). Also, prove that the solution satisfies an estimate of the kind (1.2.4).

Now, let us consider the general case, where $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$ is not necessarily zero.

Let us introduce the change of variable $\mathbf{u} = \mathbf{y} + \overline{\mathbf{u}}$. Then, (1.0.1) can be rewritten equivalently in the form

$$\begin{cases} -\Delta \mathbf{y} + \nabla p = \mathbf{f}(\mathbf{x}) + \Delta \overline{\mathbf{u}}, \quad \nabla \cdot \mathbf{y} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{y} = 0, & \mathbf{x} \in \partial\Omega \end{cases} \tag{1.2.6}$$

and, from Theorem 1.2.1 and the first comment in Remark 1.2.2, one has:

Theorem 1.2.3. *Assume that, in* (1.0.1), $\mathbf{f} \in H^{-1}(\Omega)^N$ *and* $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$, *where* $\overline{\mathbf{u}} \in H^1(\Omega)^N$ *and* $\nabla \cdot \overline{\mathbf{u}} = 0$ *in* Ω. *There exists exactly one couple* $(\mathbf{u}, p) \in H^1(\Omega)^N \times L^2(\Omega)$ *(*p *is unique up to an additive constant) that solves* (1.0.1) *in the following sense:*

1. $\mathbf{u}$ *is the weak solution to* (1.0.1).
2. $-\Delta \mathbf{u} + \nabla p = \mathbf{f}$ *in* $H^{-1}(\Omega)^N$, $\nabla \cdot \mathbf{u} = 0$ *in* $L^2(\Omega)$ *and* $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$ *in* $H^{1/2}(\partial\Omega)^N$.

E 1.4. Analyze the existence and uniqueness of solution to the following system:

$$\begin{cases} -\Delta \mathbf{u} + (\mathbf{m} \cdot \nabla)\mathbf{u} + \alpha \mathbf{u} + \nabla p = \theta \mathbf{k} + \mathbf{f}(\mathbf{x}), \quad \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ -\Delta \theta + \overline{\mathbf{m}} \cdot \nabla \theta + \beta \theta = h(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u} = \mathbf{g}(\mathbf{x}), \quad \theta = \eta(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \end{cases}$$

where $\mathbf{f}$ and $\mathbf{g}$ are as before, $\alpha, \beta \in \mathbf{R}_+$, $h \in L^2(\Omega)$, $\mathbf{k} \in \mathbf{R}^N$, $\eta \in H^{1/2}(\partial\Omega)$ (that is, $\eta = \overline{\theta}|_{\partial\Omega}$ for some $\overline{\theta} \in H^1(\Omega)$) and $\mathbf{m}, \overline{\mathbf{m}} \in H^1(\Omega)^N$ with $\nabla \cdot \mathbf{m} = \nabla \cdot \overline{\mathbf{m}} = 0$ in Ω.

1.3 Galerkin Approximations to Stokes Systems

This section deals with an alternative (constructive) proof of the existence of solution to (1.0.1). This will be achieved following a strategy based on a *Galerkin method.*

Thus, let us assume that $\mathbf{f} \in H^{-1}(\Omega)^N$ and, for simplicity, $\mathbf{g} = 0$. As already seen, after a change of variable the general case is reduced to this situation. The existence result will be obtained after the construction of finite dimensional approximations, the proof of appropriate uniform *(a priori)* estimates, the extraction of a convergent sequence and the passage to the limit.

Note that the Hilbert spaces H, V and V' satisfy

$$V \hookrightarrow H \cong H' \hookrightarrow V', \tag{1.3.1}$$

where the embeddings are dense and compact.

Lemma 1.3.1. *There exist functions* $\mathbf{w}_1, \mathbf{w}_2, \ldots$ *and positive numbers* $\lambda_1, \lambda_2, \ldots$ *with the following properties:*

1. *The system* $\{\mathbf{w}_1, \mathbf{w}_2, \dots\}$ *is an orthogonal basis in* V *and an orthonormal basis in* H.

2. *For every* $m \geq 1$, *one has*

$$(\mathbf{w}_m, \mathbf{v})_V = \lambda_m (\mathbf{w}_m, \mathbf{v})_H \quad \forall \mathbf{v} \in V. \tag{1.3.2}$$

3. *The sequence* $\{\lambda_m\}$ *is nondecreasing and the* $\lambda_m \to +\infty$ *monotonically as* $m \to +\infty$.

Note that $(\mathbf{w}_m, \lambda_m)$ solves (1.3.2) if and only if there exists $q_m \in L^2(\Omega)$ (unique up to an additive constant) such that

$$\begin{cases} -\Delta \mathbf{w}_m + \nabla q_m = \lambda_m \mathbf{w}_m, \quad \nabla \cdot \mathbf{w}_m = 0, & \mathbf{x} \in \Omega, \\ \mathbf{w}_m = 0, & \mathbf{x} \in \partial\Omega. \end{cases}$$

Let us come back to (1.0.1) and (1.2.2). For every $m \geq 1$, we can consider the m-th Galerkin approximated problem

$$\begin{cases} \mathbf{u}_m \in V_m, \\ a(\mathbf{u}_m, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_m, \end{cases} \tag{1.3.3}$$

where (again) $V_m := [\mathbf{w}_1, \dots, \mathbf{w}_m]$ is the m-dimensional subspace of V spanned by the first m eigenfunctions and $\langle \cdot, \cdot \rangle$ stands for the duality pairing in $H^{-1}(\Omega)^N \times H_0^1(\Omega)^N$.

In view of Lax-Milgram's Theorem applied to $a(\cdot, \cdot)$ in $V_m \times V_m$, there exists exactly one solution $\mathbf{u}_m$ to (1.3.3). Furthermore, one has

$$\|\mathbf{u}_m\|_{H_0^1}^2 = a(\mathbf{u}_m, \mathbf{u}_m) = \langle \mathbf{f}, \mathbf{u}_m \rangle \leq \|\mathbf{f}\|_{H^{-1}} \|\mathbf{u}_m\|_{H_0^1},$$

whence the $\mathbf{u}_m$ are uniformly bounded in V.

Consequently, there exist subsequences $\{\mathbf{u}_{m'}\}$ and functions $\mathbf{u} \in V$ such that

$$\mathbf{u}_{m'} \to \mathbf{u} \ \text{ weakly in } V \text{ and strongly in } H \text{ as } m' \to +\infty.$$

It is easy to see that $\mathbf{u}$ solves (1.2.2). More precisely, we first see from (1.3.3) that

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_0,$$

where we have introduced the linear space $V_0 := \cup_{m \geq 1} V_m$. Then, since V_0 is dense in V, we deduce that $\mathbf{u}$ is the solution to (1.2.2).

In fact, the whole sequence $\{\mathbf{u}_m\}$ converges weakly to $\mathbf{u}$. This is because the limit of any subsequence is a solution to (1.2.2) and, on the other hand, the solution to this problem is unique.

Furthermore, the convergence is strong in V. Indeed, one has weak convergence in V and, additionally,

$$\|\mathbf{u}_m\|_{H_0^1}^2 = a(\mathbf{u}_m, \mathbf{u}_m) = \langle \mathbf{f}, \mathbf{u}_m \rangle \to \langle \mathbf{f}, \mathbf{u} \rangle = a(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|_{H_0^1}^2,$$

that is, the norms of the $\mathbf{u}_m$ converge to the norm of $\mathbf{u}$.

Therefore, we have proved the following:

Theorem 1.3.2. *Let the* $(\mathbf{w}_m, \lambda_m)$ *be eigenfunction-eigenvalue couples for the Stokes problem* (1.0.1)*, with* $\|\mathbf{w}_m\| = 1$ *for all* m*. Then the solutions to the finite dimensional Galerkin approximations* (1.3.3) *converge strongly in* V *to the unique solution to* (1.0.1) *as* $m \to +\infty$*.*

Note that, in order to apply the Galerkin technique, we could have chosen any other basis of V.

E 1.5. Let $\{\mathbf{z}^1, \mathbf{z}^2, \dots\}$ be a (non necessarily orthogonal) base of V. Prove the existence of a weak solution to (1.0.1) by the Galerkin method using this base.

E 1.6. Use the Galerkin method to prove the existence of a solution to the system in Exercise 1.4.

1.4 A Technical Result

The goal of this section is to recall a technical lemma needed for the analysis of the Navier-Stokes problem (1.0.2). We will use again the spaces $\mathcal{V}$, H and V and the bilinear form $a(\cdot\,,\cdot)$ respectively defined in (1.1.1) and (1.2.3). We will also need the trilinear form $b(\cdot\,,\cdot\,,\cdot)$, with

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}\, d\mathbf{x}.$$

The following holds:

Lemma 1.4.1. *The trilinear form* $b(\cdot\,,\cdot\,,\cdot)$ *is well defined and continuous on* $H^1(\Omega)^N$*, that is,*

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C\|\mathbf{u}\|_{H^1}\|\mathbf{v}\|_{H^1}\|\mathbf{w}\|_{H^1} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^N. \tag{1.4.1}$$

Furthermore,

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\,b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u} \in V, \quad \forall \mathbf{v}, \mathbf{w} \in H^1(\Omega)^N. \tag{1.4.2}$$

The same happens if $\mathbf{u} \in H^1(\Omega)^N$*,* $\nabla \cdot \mathbf{u} = 0$ *and either* $\mathbf{v}$ *or* $\mathbf{w}$ *belongs to* $H^1_0(\Omega)^N$*.*

Remark 1.4.2. Let $\Gamma_0 \subset \partial\Omega$ be given. Note that we still have (1.4.2) when the functions $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^N$, $\nabla \cdot \mathbf{u} = 0$ in Ω and moreover

- $\mathbf{u} \cdot \mathbf{n} = 0$ on $\Gamma_0 \subset \partial\Omega$ and
- $\mathbf{v} = 0$ or $\mathbf{w} = 0$ on $\Gamma := \partial\Omega \setminus \Gamma_0$.

This will be used below. □

E 1.7. Prove that, under the conditions in Remark 1.4.2, (1.4.2) holds.

Other inequalities satisfied by the trilinear form $b(\cdot\,,\cdot\,,\cdot)$ in Sobolev spaces can be found in [Constantin and Foias(1988)].

1.5 Results for Navier-Stokes Systems

We will first consider (1.0.2) with $\mathbf{g} = 0$.

As in Section 1.2, given $\mathbf{f} \in H^{-1}(\Omega)^N$, it is convenient to introduce a weak formulation of the problem. It is the following:

$$\begin{cases} \mathbf{u} \in V, \\ \nu\, a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V, \end{cases} \tag{1.5.1}$$

where again $\langle \cdot\,, \cdot \rangle$ stands for the duality pairing associated with $H^{-1}(\Omega)^N \times H_0^1(\Omega)^N$.

If $\mathbf{u}$ solves (1.5.1), it is said that $\mathbf{u}$ is a *weak solution* to the Navier-Stokes system (1.0.2).

The following holds:

Theorem 1.5.1. *Assume that $\mathbf{f} \in H^{-1}(\Omega)^N$ and $\mathbf{g} = 0$ in (1.0.2). There exists at least one solution $(\mathbf{u}, p) \in H_0^1(\Omega)^N \times L^2(\Omega)$ in the following sense:*

1. *$\mathbf{u}$ is a weak solution to (1.0.2) with $\mathbf{g} = 0$.*
2. *$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}$ in $H^{-1}(\Omega)^N$, $\nabla\cdot\mathbf{u} = 0$ in $L^2(\Omega)$ and $\mathbf{u}|_{\partial\Omega} = 0$ in $H^{1/2}(\partial\Omega)^N$.*

Furthermore, there exists a constant C_0 that only depends on Ω such that, if

$$\nu \|\mathbf{f}\|_{H^{-1}}^{-1/2} > C_0,$$

the solution $(\mathbf{u}, p)$ is unique (p is unique up to an additive constant).

Proof. We will prove the existence of $\mathbf{u}$ by adapting the Galerkin method.

Thus, let us consider for every $m \geq 1$, the following m-th approximated problem:

$$\begin{cases} \mathbf{u}_m \in V_m, \\ \nu\, a(\mathbf{u}_m, \mathbf{v}) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_m. \end{cases} \tag{1.5.2}$$

Recall that $V_m = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ is the m-dimensional subspace of V spanned by the first m eigenfunctions. The existence of a solution to (1.5.2) for every m is ensured by the following result:

Lemma 1.5.2. *The finite dimensional problem (1.5.2) possesses at least one solution $\mathbf{u}_m$. Moreover, one has*

$$\|\mathbf{u}_m\|_{H_0^1} \leq \frac{1}{\nu} \|\mathbf{f}\|_{H^{-1}} \quad \forall m \geq 1. \tag{1.5.3}$$

The proof will be given below.

Assuming that this result holds, we get a uniformly bounded sequence $\{\mathbf{u}_m\}$ of approximated solutions that we are going to use to construct a weak solution to (1.0.2). Indeed, there exist subsequences $\{\mathbf{u}_{m'}\}$ and functions $\mathbf{u} \in V$ such that

$$\mathbf{u}_{m'} \to \mathbf{u} \ \text{ weakly in } V \text{ and strongly in } H \text{ as } m' \to +\infty.$$

Note that the $\mathbf{u}_{m'}$ must also converge strongly to $\mathbf{u}$ in all the spaces $L^r(\Omega)^N$ with $r < 6$.

Let us see that any of these functions $\mathbf{u}$ solves (1.2.2). Indeed, if $\mathbf{v} \in V_0 = \cup_{m\geq 1} V_m$, we have

$$\nu\, a(\mathbf{u}_{m'}, \mathbf{v}) + b(\mathbf{u}_{m'}, \mathbf{u}_{m'}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle$$

for all sufficiently large m'. Consequently, taking limits as $m' \to +\infty$, we find that

$$\nu\, a(\mathbf{u}, \mathbf{v}) + \lim_{m'\to+\infty} b(\mathbf{u}_{m'}, \mathbf{u}_{m'}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}.$$

On the other hand,

$$b(\mathbf{u}_{m'}, \mathbf{u}_{m'}, \mathbf{v}) = -b(\mathbf{u}_{m'}, \mathbf{v}, \mathbf{u}_{m'}) = -\sum_{i,j=1}^{N} \int_\Omega u_{m',i}\, \partial_i v_j\, u_{m',j}\, d\mathbf{x}$$

and in this sum we find integrals of products of functions that converge strongly in $L^2(\Omega)$ (the $u_{m',i}u_{m',j}$) times fixed functions in $L^2(\Omega)$ (the $\partial_i v_j$). Hence, we can take limits in these integrals and obtain that

$$\lim_{m'\to+\infty} b(\mathbf{u}_{m'}, \mathbf{u}_{m'}, \mathbf{v}) = -b(\mathbf{u}, \mathbf{v}, \mathbf{u}) = b(\mathbf{u}, \mathbf{u}, \mathbf{v}).$$

This shows that

$$\nu\, a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in V_0.$$

Since V_0 is dense in V, we deduce that $\mathbf{u}$ is a weak solution.

Again, the convergence is strong in V since, together with the weak convergence, one also has that the norms $\|\mathbf{u}_{m'}\|_{H_0^1}$ converge to $\|\mathbf{u}\|_{H_0^1}$. More precisely,

$$\nu\|\mathbf{u}_{m'}\|^2_{H_0^1} = a(\mathbf{u}_{m'}, \mathbf{u}_{m'}) = \langle \mathbf{f}, \mathbf{u}_{m'}\rangle_{H^{-1},H_0^1} \to \langle \mathbf{f}, \mathbf{u}\rangle_{H^{-1},H_0^1} = \nu\|\mathbf{u}\|^2_{H_0^1}.$$

Let us prove that, if ν is sufficiently large, the solution is unique. Thus, let $\mathbf{u}^1, \mathbf{u}^2$ be two weak solutions to (1.0.2) and let us set $\mathbf{w} := \mathbf{u}^1 - \mathbf{u}^2$. Then

$$\nu\, a(\mathbf{w}, \mathbf{v}) = -b(\mathbf{u}^1, \mathbf{u}^1, \mathbf{v}) + b(\mathbf{u}^2, \mathbf{u}^2, \mathbf{v}) = -b(\mathbf{w}, \mathbf{u}^1, \mathbf{v}) - b(\mathbf{u}^2, \mathbf{w}, \mathbf{v})$$

for all $\mathbf{v} \in V$. Therefore,

$$\nu\, a(\mathbf{w}, \mathbf{w}) = -b(\mathbf{w}, \mathbf{u}^1, \mathbf{w}) \leq C\|\mathbf{u}^1\|_{H_0^1}\|\mathbf{w}\|^2_{H_0^1} \tag{1.5.4}$$

for some C that only depends on Ω.

On the other hand, it is clear that $\mathbf{u}^1$ satisfies (1.5.3). Consequently, we deduce that

$$\|\mathbf{w}\|^2_{H_0^1} \leq \frac{C}{\nu^2}\|\mathbf{f}\|_{H^{-1}}\|\mathbf{w}\|^2_{H_0^1},$$

whence we see that the assertion on uniqueness holds good.

To complete the proof, it only remains to prove Lemma 1.5.2. □

PROOF OF LEMMA 1.5.2:

Let $S = (S_1, \ldots, S_m)$ be given, with

$$\begin{cases} S_j(\boldsymbol{\xi}) := \nu\, a(\mathbf{u}_m, \mathbf{w}_j) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{w}_j) - \langle \mathbf{f}, \mathbf{w}_j \rangle, \\ \text{where } \mathbf{u}_m = \displaystyle\sum_{i=1}^{m} \xi_i \mathbf{w}_i, \;\; \boldsymbol{\xi} \in \mathbf{R}^m. \end{cases}$$

The mapping S is obviously well defined and continuous in the whole $\mathbf{R}^m$. Moreover, if $\|\mathbf{u}_m\|_{H_0^1} \geq \frac{1}{\nu}\|\mathbf{f}\|_{H^{-1}}$, one has

$$S(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} = \nu\, a(\mathbf{u}_m, \mathbf{u}_m) - \langle \mathbf{f}, \mathbf{u}_m \rangle \geq \nu \|\mathbf{u}_m\|_{H_0^1}^2 - \|\mathbf{f}\|_{H^{-1}} \|\mathbf{u}_m\|_{H_0^1} \geq 0.$$

As a consequence of *Brouwer's Fixed-Point Theorem,* this guarantees the existence of a solution to (1.5.2).

Finally, (1.5.3) can be proved easily: we have from (1.5.2) that

$$\nu\, a(\mathbf{u}_m, \mathbf{u}_m) = \langle \mathbf{f}, \mathbf{u}_m \rangle_{H^{-1}, H_0^1} \leq \|\mathbf{f}\|_{H^{-1}} \|\mathbf{u}_m\|_{H_0^1}$$

and this directly gives the desired estimate (1.5.3). □

At present, uniqueness is unknown if in (1.0.2) $\mathbf{g} = 0$ and one has $0 < \nu \leq C_0 \|\mathbf{f}\|_{H^{-1}}^{1/2}$. Some partial results have been deduced, see [Témam(1995), Tsai(2018)]; but the situation is still not completely understood.

Now, let us consider the case of general boundary data $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$, where $\overline{\mathbf{u}} \in H^1(\Omega)^N$ and $\nabla \cdot \overline{\mathbf{u}} = 0$ in Ω. Note that, if this is the case, one must have

$$\int_{\partial\Omega} \overline{\mathbf{u}} \cdot \mathbf{n}\, d\Gamma = 0. \tag{1.5.5}$$

In fact, a slightly stronger assumption on $\overline{\mathbf{u}}$ will be imposed below.

This time, we must deal with the followng problem:

$$\begin{cases} \mathbf{u} \in \overline{\mathbf{u}} + V, \\ \nu\, a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}, H_0^1} \quad \forall \mathbf{v} \in V. \end{cases} \tag{1.5.6}$$

Again, if $\mathbf{u}$ solves (1.5.6), it is said that $\mathbf{u}$ is a weak solution to (1.0.2).

The following result holds:

Theorem 1.5.3. *Let us assume that $\mathbf{f} \in H^{-1}(\Omega)^N$ and $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$ for some $\overline{\mathbf{u}} \in H^1(\Omega)^N$ with $\nabla \cdot \overline{\mathbf{u}} = 0$ in Ω that satisfies*

$$\int_{\Gamma_i} \overline{\mathbf{u}} \cdot \mathbf{n}\, d\Gamma = 0 \tag{1.5.7}$$

for every connected component Γ_i of $\partial\Omega$. There exists at least one pair $(\mathbf{u}, p) \in H^1(\Omega)^N \times L^2(\Omega)$ that solves (1.0.2) in the following sense:

1. *$\mathbf{u}$ is a weak solution to (1.0.2).*
2. *$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}$ in $H^{-1}(\Omega)^N$, $\nabla\cdot\mathbf{u} = 0$ in $L^2(\Omega)$ and $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$ in $H^{1/2}(\partial\Omega)^N$.*

Remark 1.5.4. Obviously, if Ω is simply connected, we get from Theorem 1.5.3 an existence result for (1.0.2) under the assumption (1.5.5). A natural question is what can be said when Ω is not simply connected and (1.5.7) does not hold but we still have (1.5.5). This is open in the general case. However, for $N = 2$, a positive answer has been furnished in [Korobkov and others(2015)] when $\partial\Omega$ is (for instance) of class C^2, $\mathbf{f} \in H^1(\Omega)^2$ and $\mathbf{g}$ is the trace of a zero-divergence field $\overline{\mathbf{u}} \in H^2(\Omega)^2$; see [Tsai(2018)] for a complete proof and some additional explanations. □

For $\mathbf{g} \neq 0$, uniqueness fails even for large ν. Indeed, a counter-example to uniqueness was found by Yudovich in [Yudovich(1967)]; see also [Tsai(2018)].

1.6 On the Regularity of the Solutions

Both systems (1.0.1) and (1.0.2) possess regularity properties similar to those found for second-order elliptic PDEs. Roughly speaking, the solution $\mathbf{u}$ (resp. p) experiments a regularity gain of two units (resp. one unit) with respect to $\mathbf{f}$.

Let us present a result for the Stokes system:

Theorem 1.6.1. *Let us consider the system* (1.0.1)*, where $\partial\Omega$ is of class $W^{2,\infty}$, $\mathbf{f} \in L^r(\Omega)^N$ and $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$ for some $\overline{\mathbf{u}} \in W^{2,r}(\Omega)^N$ with $\nabla \cdot \overline{\mathbf{u}} = 0$; here, it is assumed that $r \in (1, +\infty)$. Then the unique solution to* (1.0.1) *satisfies*

$$\mathbf{u} \in W^{2,r}(\Omega)^N \quad \textit{and} \quad p \in W^{1,r}(\Omega). \tag{1.6.1}$$

Using a standard bootstrap (or feedback) technique, it is not difficult to deduce from Theorem 1.6.1 a similar result for the solutions to (1.0.2):

Theorem 1.6.2. *Let us consider the system* (1.0.2)*, where $\partial\Omega$, $\mathbf{f}$ and $\mathbf{g}$ are as in Theorem 1.6.1, with $r \in (1, +\infty)$. Then, any solution to* (1.0.2) *satisfies* (1.6.1)*.*

Proof. To fix ideas, let us assume that $N = 3$ and (for example) $3/2 < r \leq 3$. Let us write the first PDE in (1.0.2) in the form

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u}. \tag{1.6.2}$$

We know that $\mathbf{u} \in H^1(\Omega)^3$. Consequently, the components of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ are sums of products of functions in $L^6(\Omega)$ times functions in $L^2(\Omega)$ and therefore $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L^{3/2}(\Omega)^3$ and, since $r > 3/2$, the right-hand side of (1.6.2) belongs to $L^{3/2}(\Omega)^3$. In view of Theorem 1.6.1, we have $\mathbf{u} \in W^{2,3/2}(\Omega)^N \cap V$ and $p \in W^{1,3/2}(\Omega)$. In particular, $\mathbf{u} \in W^{1,3}(\Omega)^3$.

Then, in a second step, we use that for all finite $\sigma \geq 1$ the components of $(\mathbf{u} \cdot \nabla)\mathbf{u}$ are sums of products of functions in $L^\sigma(\Omega)$ times functions in $L^3(\Omega)$.

Therefore, $(\mathbf{u}\cdot\nabla)\mathbf{u} \in L^\beta(\Omega)^3$ for all $\beta \in [1, 3)$ and the right-hand side of the PDE belongs to $L^r(\Omega)^N$. This proves (1.6.1).

If we had $r \geq 3$, it would suffice to continue and perform one more step in this bootstrap argument. Clearly, this would give $(\mathbf{u} \cdot \nabla)\mathbf{u} \in L^\beta(\Omega)^3$ for all $\beta \in [1, +\infty)$, whence the regularity property (1.6.1) would follow. □

When $\mathbf{f} \in L^1(\Omega)^N$, the situation is much more complicated even for the Stokes system. The existence and regularity questions are not simple and deserve deeper analysis.

Chapter 2

The Time-Dependent Navier-Stokes Equations

In this chapter, we will recall the main results dealing with the theoretical analysis of the nonstationary Navier-Stokes equations and some important variants.

2.1 Introduction. Formulation of the Problem and Results

As before, it will be assumed that $N = 2$ or $N = 3$, $\Omega \subset \mathbf{R}^N$ is a non-empty bounded connected open set with boundary of class $W^{1,\infty}$ and, unless otherwise stated, $0 < T < +\infty$.

We will be mainly concerned with the nonlinear problem

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, & (\mathbf{x},t) \in Q, \\ \mathbf{u}(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Sigma, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{2.1.1}$$

where $\nu > 0$, $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$ and $\mathbf{u}_0 = \mathbf{u}_0(\mathbf{x})$ are given. This serves to model the behavior of a *Newtonian viscous incompressible fluid* whose particles fill the spatial domain Ω during the time interval $(0,T)$.

Note that the first PDE in Q can be written in the form

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\cdot\sigma = \mathbf{f}(\mathbf{x},t),$$

where the tensor σ is given by the so called *Newton law*

$$\sigma = -p\,\mathrm{Id.} + 2\nu D\mathbf{u}, \quad \text{with} \quad D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T). \tag{2.1.2}$$

It is usual to say that σ is the *stress tensor*. It must be interpreted as a measure of the internal forces among particles. Thus, it is reasonable to accept that σ can be decomposed as the sum of *frontal* presssure forces and *oblique* viscous or friction forces and, also, that the later depend linearly on the spatial derivatives of the components of $\mathbf{u}$.

We have complemented these equations with boundary conditions on Σ that express that the particles adhere to the wall (and therefore do not slip) and initial conditions at time $t = 0$.

Together with (2.1.1), we will consider later several variants:

- The *Ladyzhenskaja model,* where instead of (2.1.2) one has

$$\sigma = -p\,\mathrm{Id.} + 2\nu_r(|D\mathbf{u}|^2)D\mathbf{u} \tag{2.1.3}$$

and $\nu_r : \mathbf{R}_+ \mapsto \mathbf{R}$ is given by

$$\nu_r(s) = \nu_0 + \nu_1 s^{(r-2)/2} \quad \forall s > 0 \tag{2.1.4}$$

for some $\nu_0, \nu_1 \geq 0$ and $r > 1$, see [Ladyzhenskaja(1967)]. As in Chapter 1, it is said that the fluid under study is quasi-Newtonian and (2.1.3) is called a *power law.*

- The *Ladyzhenskaja-Smagorinsky model,* where the constitutive law is nonlocal in space and takes the form

$$\sigma(\mathbf{x},t) = -p(\mathbf{x},t)\,\mathrm{Id.} + 2\nu_r(\|D\mathbf{u}(\cdot\,,t)\|)D\mathbf{u}(\mathbf{x},t) \ \text{ a.e. in } \ Q \tag{2.1.5}$$

for the same function ν_r. This law allows to desribe the behaviour of some turbulent flows; see to this respect [Smagorinsky(1963)] and [Germano and others(1991)].

Let us indicate the kind of solution we are looking for:

Definition 2.1.1. *It will be said that* $\mathbf{u}$ *is a weak solution to* (2.1.1) *if the following holds:*

1. $\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ *and* $\mathbf{u}_t \in L^\sigma(0,T;V')$*, where* $\sigma = 2$ *if* $N = 2$ *and* $\sigma = 4/3$ *if* $N = 3$.

2. *The PDEs in* (2.1.1) *are satisfied in the following sense:*

$$\langle \mathbf{u}_t, \mathbf{v}\rangle + \nu\, a(\mathbf{u},\mathbf{v}) + b(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f}(\cdot\,,t),\mathbf{v}) \quad \forall \mathbf{v} \in V, \quad \textit{a.e. in } (0,T). \tag{2.1.6}$$

3. $\mathbf{u}|_{t=0} = \mathbf{u}_0$.

It will be seen very soon that, if $\mathbf{u}$ belongs to $L^2(0,T;V) \cap L^\infty(0,T;H)$ and satisfies (2.1.6), then $\mathbf{u}_t \in L^\sigma(0,T;V')$. Under these circumstances, it is clear that $\mathbf{u} \in C^0([0,T];H)$ if $N = 2$ and $\mathbf{u} \in C^0_w([0,T];H)$ if $N = 3$. Therefore, the initial condition in Definition 2.1.1 makes always sense as an equality in H.

Let us introduce the Stokes operator $A : V \mapsto V'$, with

$$\langle A\mathbf{u}, \mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in V.$$

Also, let the bilinear operator $B : V \times V \mapsto V'$ be defined by the identities

$$\langle B(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in V.$$

Obviously, if $\mathbf{u} \in L^2(0,T;V)$, one has $A\mathbf{u} \in L^2(0,T;V')$. We will also see later that, whenever $\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;H)$, the function $B(\mathbf{u},\mathbf{u})$ belongs to $L^\sigma(0,T;V')$, where σ is as above.

Let $P : L^2(\Omega)^N \mapsto H$ be the usual orthogonal projector. Then, (2.1.6) can be rewritten as an equation in $L^\sigma(0,T;V')$:

$$\mathbf{u}_t + \nu A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \overline{\mathbf{f}}(t) \ \text{ a.e. in } (0,T),$$

where we have set $\overline{\mathbf{f}}(t) := P\mathbf{f}(\cdot\,,t)$.

The main results concerning the existence and uniqueness of a weak solution to (2.1.1) are the following:

Theorem 2.1.2. *Assume that $\mathbf{f} \in L^2(Q)^N$ and $\mathbf{u}_0 \in H$ are given. There exists at least one couple $(\mathbf{u},p)$ that solves (2.1.1) in the following sense:*

1. *$\mathbf{u}$ is a weak solution to (2.1.1).*
2. *$p \in W^{-1,\infty}(0,T;L^2(\Omega))$ (that is, for some $q \in L^\infty(0,T;L^2(\Omega))$ one has $p = q_t$) and*
   - *$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}$ in $\mathcal{D}'(Q)^N$,*
   - *$\nabla\cdot\mathbf{u} = 0$ in $L^2(Q)$,*
   - *$\mathbf{u}|_\Sigma = 0$ in $L^2(0,T;H^{1/2}(\partial\Omega)^N)$ and*
   - *$\mathbf{u}|_{t=0} = \mathbf{u}_0$ in H.*

Theorem 2.1.3. *Assume that $N = 2$, $\mathbf{f} \in L^2(Q)^2$ and $\mathbf{u}_0 \in H$ are given. Then the weak solution to (2.1.1) is unique. The associated p is unique up to an additive distribution in $W^{-1,\infty}(0,T)$.*

2.2 Proof of Existence

We will need the following two results:

Lemma 2.2.1. *Assume that $N = 2$. Then there exists a constant $C > 0$ such that*

$$\|v\|_{L^4} \le C\|v\|^{1/2}\|\nabla v\|^{1/2} \quad \forall v \in H^1(\Omega). \tag{2.2.1}$$

Lemma 2.2.2. *Assume that $N = 3$. Then there exist constants $C > 0$ such that*

$$\|v\|_{L^4} \le C\|v\|^{1/4}\|\nabla v\|^{3/4} \quad \forall v \in H^1(\Omega) \tag{2.2.2}$$

and

$$\|v\|_{L^3} \le C\|v\|^{1/2}\|\nabla v\|^{1/2} \quad \forall v \in H^1(\Omega).$$

In fact, Lemma 2.2.1 and Lemma 2.2.2 are particular cases of the so called Galiardo-Nirenberg inequalities; see for instance [Fiorenza and others(2021)].

Let $\{\mathbf{w}_1, \mathbf{w}_2, \dots\}$ be the special basis of V introduced in Chapter 1. As before, we denote by V_m the space spanned by the $\mathbf{w}_j$ with $1 \le j \le m$ and $P_m : H \mapsto V_m$ and $\tilde{P}_m : V' \mapsto V_m$ stand for the usual orthogonal projectors. Recall that

$$\|\tilde{P}_m\|_{\mathcal{L}(V';V')} = 1 \quad \forall m \ge 1. \tag{2.2.3}$$

Let $\{\overline{\mathbf{f}}_m\}$ be a sequence in $C^0([0,T];H)$ satisfying

$$\overline{\mathbf{f}}_m \to \overline{\mathbf{f}} \text{ strongly in } L^2(0,T;H) \text{ as } m \to \infty$$

and let us introduce the $\mathbf{u}_{0m} := P_m \mathbf{u}_0$. x For each $m \geq 1$, we consider the following finite dimensional problem:

$$\begin{cases} \mathbf{u}_m : [0,T] \mapsto V_m, \\ (\mathbf{u}'_m, \mathbf{v}) + \nu a(\mathbf{u}_m, \mathbf{v}) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) = (\overline{\mathbf{f}}_m(t), \mathbf{v}) \quad \forall \mathbf{v} \in V_m, \quad t \in (0,T), \\ \mathbf{u}_m(0) = \mathbf{u}_{0m}. \end{cases} \tag{2.2.4}$$

This is the m-th Galerkin approximation to (2.1.1).

Once more, if we put

$$\mathbf{u}_m(t) = \sum_{i=1}^{m} \xi_{im}(t) \mathbf{w}_i, \tag{2.2.5}$$

then (2.2.4) can be regarded as a Cauchy problem for a nonlinear first order ODE where the unknowns are the functions ξ_{im}. Thus, an equivalent formulation of (2.2.4) is the following:

$$\begin{cases} \xi'_{jm} + \nu \lambda_j \xi_{jm} + \displaystyle\sum_{i,k=1}^{m} b(\mathbf{w}_i, \mathbf{w}_k, \mathbf{w}_j)\, \xi_{im} \xi_{km} = (\overline{\mathbf{f}}_m(t), \mathbf{w}_j), \quad t \in (0,T), \\ \xi_{jm}(0) = (\mathbf{u}_0, \mathbf{w}_j), \quad j = 1, \ldots, m. \end{cases} \tag{2.2.6}$$

The classical existence and uniqueness theory can be applied to (2.2.4).

Thus, we can affirm that, for every $m \geq 1$, there exist a maximal time $t_m > 0$ and a unique local (classical) solution $\mathbf{u}_m : [0, t_m) \mapsto V_m$. Furthermore, for every m, the following alternative holds:

$$\text{Either } t_m = T \text{ or } \limsup_{t \to t_m} \|\mathbf{u}_m(t)\| = +\infty.$$

As in other previous situations, the estimates that follow show that only the first option can hold.

Let us take $\mathbf{v} = \mathbf{u}_m(t)$ in (2.2.4), with $t \in [0, t_m)$. This gives the identity

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|^2 + \nu \, \|\nabla \mathbf{u}_m(t)\|^2 = (\overline{\mathbf{f}}_m(t), \mathbf{u}_m(t)). \tag{2.2.7}$$

Integrating in time, we also have

$$\frac{1}{2} \|\mathbf{u}_m(t)\|^2 + \nu \int_\tau^t \|\nabla \mathbf{u}_m(s)\|^2 \, ds = \frac{1}{2} \|\mathbf{u}_m(\tau)\|^2 + \int_\tau^t (\overline{\mathbf{f}}_m(s), \mathbf{u}_m(s)) \, ds \tag{2.2.8}$$

for all $\tau, t \in [0, t_m)$ and, in particular,

$$\frac{1}{2} \|\mathbf{u}_m(t)\|^2 + \nu \int_0^t \|\nabla \mathbf{u}_m(s)\|^2 \, ds = \frac{1}{2} \|\mathbf{u}_{0m}\|^2 + \int_0^t (\overline{\mathbf{f}}_m(s), \mathbf{u}_m(s)) \, ds \tag{2.2.9}$$

for all t. Consequently,

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_m(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}_m(s)\|^2\,ds \\ \leq \frac{1}{2}\|\mathbf{u}_0\|^2 + \frac{\nu}{2}\int_0^t \|\nabla\mathbf{u}_m(s)\|^2\,ds + C\int_0^t \|\bar{\mathbf{f}}(s)\|^2\,ds\end{aligned} \tag{2.2.10}$$

and

$$\|\mathbf{u}_m(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}_m(s)\|^2\,ds \leq \|\mathbf{u}_0\|^2 + C\|\mathbf{f}\|^2_{L^2(Q)}. \tag{2.2.11}$$

This must hold for all $t \in [0, t_m)$. Consequently, we have $t_m = T$ and

$$\|\mathbf{u}_m\|_{L^\infty(0,T;H)} + \|\mathbf{u}_m\|_{L^2(0,T;V)} \leq C(\|\mathbf{u}_0\| + \|\mathbf{f}\|_{L^2(Q)}), \tag{2.2.12}$$

where C depends on Ω, T and ν but is independent of m.

The estimates (2.2.12) do not suffice to pass to the limit in (2.2.4), due to presence of the nonlinear term $b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v})$. We also need some uniform estimates of $\mathbf{u}'_m$.

To this purpose, we observe that

$$\mathbf{u}'_m(t) = \tilde{P}_m(\bar{\mathbf{f}}_m(t) - \nu A\mathbf{u}_m(t) - B(\mathbf{u}_m(t), \mathbf{u}_m(t))) \quad \forall t \in (0,T),$$

whence

$$\begin{aligned}\|\mathbf{u}'_m(t)\|_* &\leq \|\bar{\mathbf{f}}_m(t)\|_* + \|B(\mathbf{u}_m(t), \mathbf{u}_m(t))\|_* + \nu\|A\mathbf{u}_m(t)\|_* \\ &\leq C\left(\|\bar{\mathbf{f}}(t)\| + \|\nabla\mathbf{u}_m(t)\|\right) + \|B(\mathbf{u}_m(t), \mathbf{u}_m(t))\|_*.\end{aligned} \tag{2.2.13}$$

When $N = 2$, we can apply Lemma 2.2.1 and get

$$\|B(\mathbf{u}_m(t), \mathbf{u}_m(t))\|_* \leq \|\mathbf{u}_m(t)\|^2_{L^4} \leq C\|\mathbf{u}_m(t)\|\,\|\nabla\mathbf{u}_m(t)\|.$$

Otherwise, when $N = 3$ we can apply Lemma 2.2.2; this gives

$$\|B(\mathbf{u}_m(t), \mathbf{u}_m(t))\|_* \leq \|\mathbf{u}_m(t)\|^2_{L^4} \leq C\|\mathbf{u}_m(t)\|^{1/2}\,\|\nabla\mathbf{u}_m(t)\|^{3/2}.$$

Consequently, $\mathbf{u}'_m$ is uniformly bounded in $L^\sigma(0,T;V')$ where $\sigma = 2$ if $N = 2$ and $\sigma = 4/3$ if $N = 3$.

We deduce that there exists a subsequence $\{\mathbf{u}_{m'}\}$ satisfying

$$\begin{aligned}&\mathbf{u}_{m'} \to \mathbf{u} \text{ weakly-}* \text{ in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;V),\\ &\mathbf{u}'_{m'} \to \mathbf{u}_t \text{ weakly in } L^\sigma(0,T;V') \text{ and}\\ &\mathbf{u}_{m'} \to \mathbf{u} \text{ strongly in } L^2(0,T;H) \text{ and a.e.}\end{aligned} \tag{2.2.14}$$

Note that the strong convergence in $L^2(0,T;H)$ is implied by well known compact embedding results.

A crucial point is that the third assertion in (2.2.14) implies

$$B(\mathbf{u}_{m'}, \mathbf{u}_{m'}) \to B(\mathbf{u}, \mathbf{u}) \text{ weakly in } L^\sigma(0,T;V'). \tag{2.2.15}$$

Indeed, the $B(\mathbf{u}_{m'}, \mathbf{u}_{m'})$ are uniformly bounded in $L^\sigma(0,T;V')$. Consequently, at least for a new subsequence (again indexed by m') one has weak convergence to some $\tilde{B}$ in this space. For any $\boldsymbol{\varphi} \in \mathcal{V}$ and any $\psi \in \mathcal{D}(0,T)$, we then have

$$\begin{aligned}\int_0^T \langle \tilde{B}(t), \boldsymbol{\varphi}\rangle\, \psi(t)\, dt &= \lim_{m'\to\infty} \int_0^T \langle B(\mathbf{u}_{m'}(t), \mathbf{u}_{m'}(t)), \boldsymbol{\varphi}\rangle\, \psi(t)\, dt \\ &= \lim_{m'\to\infty} \iint_Q (\mathbf{u}_{m'}\cdot\nabla)\mathbf{u}_{m'}\cdot\boldsymbol{\varphi}\,\psi\, d\mathbf{x}\, dt = -\lim_{m'\to\infty} \iint_Q (\mathbf{u}_{m'}\cdot\nabla)(\boldsymbol{\varphi}\psi)\cdot\mathbf{u}_{m'}\, d\mathbf{x}\, dt.\end{aligned}$$

But this coincides with

$$-\iint_Q (\mathbf{u}\cdot\nabla)(\boldsymbol{\varphi}\psi)\cdot\mathbf{u}\, d\mathbf{x}\, dt = \iint_Q (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\varphi}\,\psi\, d\mathbf{x}\, dt,$$

because all the functions $u_{m',i}u_{m',j}$ converge strongly (for example) in $L^1(Q)$.

We thus find that

$$\int_0^T \langle \tilde{B}(t), \boldsymbol{\varphi}\rangle\, \psi(t)\, dt = \int_0^T \langle B(\mathbf{u}(\cdot\,,t), \mathbf{u}(\cdot\,,t)), \boldsymbol{\varphi}\rangle\, \psi(t)\, dt$$

for all $\boldsymbol{\varphi} \in \mathcal{V}$ and $\psi \in \mathcal{D}(0,T)$.

Obviously, this implies that $\tilde{B} = B(\mathbf{u},\mathbf{u})$. Since this argument can be applied to any subsequence converging weakly in $L^\sigma(0,T;V')$, we get (2.2.15).

Now, arguing as in the case of a semilinear heat PDE, it is not difficult to prove after taking limits in (2.2.4) that $\mathbf{u}$ is a weak solution to (2.1.1).

Remark 2.2.3. Taking into account (2.2.8) (that must be satisfied for any $\tau, t \in [0,T]$), it is not difficult to prove that the solution we have found satisfies the inequalities

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}(\cdot\,,t)\|^2 + \nu\int_\tau^t \|\nabla\mathbf{u}(\cdot\,,s)\|^2\, ds \\ \leq \frac{1}{2}\|\mathbf{u}(\cdot\,,\tau)\|^2 + \int_\tau^t (\mathbf{f}(\cdot\,,s), \mathbf{u}(\cdot\,,s))\, ds\end{aligned} \tag{2.2.16}$$

for all $\tau, t \in [0,T]$ with $\tau \leq t$. However, for $N = 3$ it is unknown in general whether the equality holds. It is frequent to say that (2.2.16) is the *energy inequality* for $\mathbf{u}$. It can be interpreted as follows:

> *The sum of the instantaneous kinetic energy of the fluid particles at time t and the energy that has been dissipated (or lost) as a consequence of the action of viscosity forces during the time interval (τ, t) is less or equal to the sum of the kinetic energy at time τ and the mechanical work carried out by external forces during (τ, t).*

More information on these inequalities can be found in [P.-L. Lions(1996)]. □

Remark 2.2.4. When $N = 2$, since $\mathbf{u}_t \in L^2(0,T;V')$, we can take $\mathbf{v} = \mathbf{u}_t(\cdot\,,t)$ in (2.2.4) for almost all $t \in (0,T)$ and deduce that

$$\frac{1}{2}\|\mathbf{u}(\cdot\,,t)\|^2 + \nu\int_\tau^t \|\nabla\mathbf{u}(\cdot\,,s)\|^2\,ds = \frac{1}{2}\|\mathbf{u}(\cdot\,,\tau)\|^2 + \int_\tau^t (\mathbf{f}(\cdot\,,s),\mathbf{u}(\cdot\,,s))\,ds$$

for all $\tau, t \in [0,T]$ with $\tau \le t$. That is, the energy inequalities are in fact energy identities. In particular, we have

$$\frac{1}{2}\|\mathbf{u}(\cdot\,,t)\|^2 + \nu\int_0^t \|\nabla\mathbf{u}(\cdot\,,s)\|^2\,ds = \frac{1}{2}\|\mathbf{u}_0\|^2 + \int_0^t (\mathbf{f}(\cdot\,,s),\mathbf{u}(\cdot\,,s))\,ds \tag{2.2.17}$$

for all $t \in [0,T]$. A consequence of this property is that the Galerkin approximations converge strongly in $L^2(0,T;V)$ and, for every $\bar{t} \in [0,T]$, the $\mathbf{u}_m(\bar{t})$ converge strongly in H. Indeed, in the product space $H \times L^2(0,\bar{t};V)$, the couples $(\mathbf{u}_{m'}(\bar{t}),\mathbf{u}_{m'})$ converge weakly to $(\mathbf{u}(\cdot\,,\bar{t}),\mathbf{u})$ and the squares of their norms

$$\frac{1}{2}\|\mathbf{u}_{m'}(\bar{t})\|^2 + \nu\int_0^{\bar{t}} \|\nabla\mathbf{u}_{m'}(s)\|^2\,ds$$

converge to the left hand side of (2.2.17) for $t = \bar{t}$. Since the weak solution is unique (recall Theorem 2.1.3), the whole sequence $\{\mathbf{u}_m\}$ converges. □

To end the proof, let us see that there exists $p \in W^{-1,\infty}(0,T;L^2(\Omega))$ satisfying

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \ \text{ in } \mathcal{D}'(Q)^N. \tag{2.2.18}$$

We will use the following Banach-valued version of De Rham's Lemma, whose proof can be found in [Simon(1999)]:

Lemma 2.2.5. *Let E be a Banach space and let $\mathbf{S} \in H^{-1}(\Omega;E)^N$ be given,*[1] *with*

$$\langle \mathbf{S}, \boldsymbol{\varphi}\rangle = 0 \quad \forall \boldsymbol{\varphi} \in \mathcal{V}.$$

Then there exists $q \in L^2(\Omega;E)$, unique up to an additive element in E, such that $\mathbf{S} = \nabla q$. Furthermore, the mapping $\mathbf{S} \mapsto q$ can be defined as a continuous linear operator from $H^{-1}(\Omega;E)^N$ into $L^2(\Omega;E)$.

Let $\mathbf{u}$ be a weak solution to (2.1.1) and let us set

$$\mathbf{S} = \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f}.$$

It is then clear that

$$\mathbf{S} \in W^{-1,\infty}(0,T;H) + L^\sigma(0,T;H^{-1}(\Omega)^N) \subset W^{-1,\infty}(0,T;H^{-1}(\Omega)^N).$$

But this last Banach space is isomorphic to $H^{-1}(\Omega;W^{-1,\infty}(0,T))^N$.

Therefore, we can apply Lemma 2.2.5 with $E = W^{-1,\infty}(0,T)$ (observe that $\langle \mathbf{S}, \boldsymbol{\varphi}\rangle = 0$ for all $\boldsymbol{\varphi} \in \mathcal{V}$). The conclusion is that there exists $p \in L^2(\Omega;W^{-1,\infty}(0,T))$ such that $\mathbf{S} = -\nabla p$, Since the spaces $L^2(\Omega;W^{-1,\infty}(0,T))$ and $W^{-1,\infty}(0,T;L^2(\Omega))$ are also isomorphic, one has (2.2.18) for some $p \in W^{-1,\infty}(0,T;L^2(\Omega))$, unique up to an additive distribution in $W^{-1,\infty}(0,T)$.

This ends the proof of Theorem 2.1.2.

[1] This means that each component of $\mathbf{S}$ is a sum of first-order partial derivatives of functions in $L^2(\Omega;E)$.

Remark 2.2.6. With similar arguments, the existence of a weak solution can be proved under slightly more general assumptions. Thus, $\Omega \subset \mathbf{R}^N$ can be an arbitrary connected open set (not necessarily bounded), the right-hand side $\mathbf{f}$ can belong to a larger space like $L^2(0,T;H^{-1}(\Omega)^N)$, etc. □

E 2.1. Consider the system (2.1.1) with $N=3$, $\mathbf{u}_0 \in H$ and

$$\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^3) + L^1(0,T;L^2(\Omega)^3).$$

1. Define appropriately the concept of weak solution.
2. Prove a result similar to Theorem 2.1.2 in this case.

E 2.2. Same statement for the system

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, & (\mathbf{x},t) \in Q, \\ \mathbf{u}(\mathbf{x},t) = \mathbf{g}(\mathbf{x}), & (\mathbf{x},t) \in \Sigma, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

where $\mathbf{f} \in L^2(Q)^N$, $\mathbf{u}_0 \in H$ and $\mathbf{g} = \overline{\mathbf{u}}|_{\partial\Omega}$ for some $\overline{\mathbf{u}} \in H^1(\Omega)^N$ with $\nabla\cdot\overline{\mathbf{u}} = 0$.

E 2.3. Same statement for the reduced Boussinesq system

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \theta\mathbf{k} + \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, & (\mathbf{x},t) \in Q, \\ \theta_t - \kappa\Delta\theta + \mathbf{u}\cdot\nabla\theta = h(\mathbf{x},t), & (\mathbf{x},t) \in Q, \\ \mathbf{u}(\mathbf{x},t) = \mathbf{g}(\mathbf{x}), \quad \theta(\mathbf{x},t) = \zeta(\mathbf{x}), & (\mathbf{x},t) \in \Sigma, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \quad \theta(\mathbf{x},0) = \theta_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases}$$

where $\mathbf{f}$, $\mathbf{u}_0$ and $\mathbf{g}$ are as in Exercise 2.2, $\mathbf{k} \in \mathbf{R}^N$, $\kappa > 0$, $h \in L^2(Q)$, $\zeta \in H^{1/2}(\partial\Omega)$ and $\theta_0 \in L^2(\Omega)$.

2.3 Proof of Uniqueness when $N = 2$

In this section, we will prove that, when $N=2$, there exists at most one weak solution to (2.1.1). We will use the following consequence of Lemma 2.2.1:

Lemma 2.3.1. *Assume that $N=2$. For any $\mathbf{v} \in L^2(0,T;V)\cap L^\infty(0,T;H)$ one has $B(\mathbf{v},\mathbf{v}) \in L^2(0,T;V')$. Furthermore, there exists a constant $C>0$ such that*

$$\begin{cases} \|B(\mathbf{v},\mathbf{v}))\|_{L^2(0,T;V')} \le C\|\mathbf{v}\|_{L^\infty(0,T;H)}\|\mathbf{v}\|_{L^2(0,T;V)} \\ \forall \mathbf{v} \in L^2(0,T;V)\cap L^\infty(0,T;H). \end{cases} \tag{2.3.1}$$

Proof. Let $\mathbf{v}$ be given in $L^2(0,T;V)\cap L^\infty(0,T;H)$. For every $\mathbf{w} \in V$ we have:

$$\begin{aligned} \langle B(\mathbf{v}(\cdot\,,t),\mathbf{v}(\cdot\,,t)),\mathbf{w}\rangle &= \int_\Omega (\mathbf{v}(\mathbf{x},t)\cdot\nabla)\mathbf{v}(\mathbf{x},t)\cdot\mathbf{w}(\mathbf{x})\,d\mathbf{x} \\ &= -\int_\Omega (\mathbf{v}(\mathbf{x},t)\cdot\nabla)\mathbf{w}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x},t)\,d\mathbf{x} \\ &\le C\,\|\mathbf{v}(\cdot\,,t)\|_{L^4}^2\|\nabla\mathbf{w}\| \\ &\le C\,\|\mathbf{v}(\cdot\,,t)\|\,\|\nabla\mathbf{v}(\cdot\,,t)\|\,\|\nabla\mathbf{w}\|. \end{aligned}$$

Hence,

$$\|B(\mathbf{v}(\cdot\,,t),\mathbf{v}(\cdot\,,t))\|_* \le C\,\|\mathbf{v}(\cdot\,,t)\|\,\|\nabla\mathbf{v}(\cdot\,,t)\| \quad \text{a.e. in } (0,T),$$

whence we obviously deduce that $B(\mathbf{v},\mathbf{v})$ belongs to $L^2(0,T;V')$ and (2.3.1) holds. □

Now, let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two weak solutions to (2.1.1), where the right-hand side $\mathbf{f} \in L^2(Q)^2$ and the initial velocity $\mathbf{u}_0 \in H$ are given.

Note that, in this case, $\mathbf{u}^1_t$ and $\mathbf{u}^2_t$ belong to $L^2(0,T;V')$.

Indeed, we have

$$\mathbf{u}^i_t = \overline{\mathbf{f}} - B(\mathbf{u}^i,\mathbf{u}^i) - \nu A\mathbf{u}^i \quad \text{for} \quad i = 1,2.$$

Let us set $\mathbf{w} := \mathbf{u}^1 - \mathbf{u}^2$. Then $\mathbf{w} \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\mathbf{w}_t \in L^2(0,T;V')$,

$$\mathbf{w}_t + \nu A\mathbf{w} = -B(\mathbf{u}^1,\mathbf{u}^1) + B(\mathbf{u}^2,\mathbf{u}^2) = -B(\mathbf{u}^1,\mathbf{w}) - B(\mathbf{w},\mathbf{u}^2) \tag{2.3.2}$$

and $\mathbf{w}|_{t=0} = 0$.

One has $\mathbf{w} \in C^0([0,T];H)$ and, moreover,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{w}(\cdot\,,t)\|^2 = \langle \mathbf{w}_t(\cdot\,,t),\mathbf{w}(\cdot\,,t)\rangle \quad \text{a.e. in } (0,T).$$

This, together with (2.3.2), yields the following:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}(\cdot\,,t)\|^2 + \nu\|\nabla\mathbf{w}(\cdot\,,t)\|^2 &= -b(\mathbf{w}(\cdot\,,t),\mathbf{u}^2(\cdot\,,t),\mathbf{w}(\cdot\,,t))\\ &\le C\,\|\mathbf{w}(\cdot\,,t)\|^2_{L^4}\,\|\nabla\mathbf{u}^2(\cdot\,,t)\|\\ &\le C\,\|\mathbf{w}(\cdot\,,t)\|\,\|\nabla\mathbf{w}(\cdot\,,t)\|\,\|\nabla\mathbf{u}^2(\cdot\,,t)\|\\ &\le \frac{\nu}{2}\|\nabla\mathbf{w}(\cdot\,,t)\|^2 + C\,\|\nabla\mathbf{u}^2(\cdot\,,t)\|^2\|\mathbf{w}(\cdot\,,t)\|^2.\end{aligned}$$

After integration in time, we see that

$$\|\mathbf{w}(\cdot\,,t)\|^2 \le C\int_0^t \|\nabla\mathbf{u}^2(\cdot\,,s)\|^2\|\mathbf{w}(\cdot\,,s)\|^2\,ds \quad \forall t \in [0,T] \tag{2.3.3}$$

and, from Gronwall's Lemma, we find that $\mathbf{w} \equiv \mathbf{0}$ and $\mathbf{u}^1$ and $\mathbf{u}^2$ must coincide.

This ends the proof of uniqueness. □

Remark 2.3.2. With a very similar argument, it can also be proved that the weak solution depends continuously of the data. For details, see [Témam(2001)]; see also Exercise 2.4. □

Remark 2.3.3. In general, for $N = 3$ and not necessarily small data $\mathbf{f}$ and $\mathbf{u}_0$, the uniqueness of weak solution is unknown. Actually, this is a major open problem; however, several positive and negative partial answers can be given. To this respect, see Section 2.4 [2]. □

[2] Another major problem for $N = 3$, dealing with the regularity of weak solutions, has apparently been solved very recently; see Section 2.5.

E 2.4. Assume that $N = 2$, $\mathbf{f}^0 \in L^2(Q)^2$, $\mathbf{u}_0^0 \in H$ and $\|\mathbf{f}^0\|_{L^2(Q)^2} + \|\mathbf{u}_0^0\| \leq M$. Let the right-hand sides and initial velocity fields in (2.1.1) respectively satisfy $\mathbf{f}^i \in L^2(Q)^2$ and $\mathbf{u}_0^i \in H$ for $i = 1, 2$, with $\|\mathbf{f}^i - \mathbf{f}^0\|_{L^2(Q)^2} + \|\mathbf{u}_0^i - \mathbf{u}_0^0\| \leq r$. Prove that there exists a constant K that depends on Ω, T, ν, M and r such that the following estimate holds for the associated weak solutions $\mathbf{u}^i$:

$$\|\mathbf{u}^1 - \mathbf{u}^2\|_{L^2(0,T;V)} + \|\mathbf{u}_t^1 - \mathbf{u}_t^2\|_{L^2(0,T;V')} \leq K(\|\mathbf{f}^1 - \mathbf{f}^2\|_{L^2(Q)^2} + \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|).$$

As a consequence, we see that, if $N = 2$, the weak solution to (2.1.1) depends continuously on the data.

2.4 The Uniqueness Problem for $N = 3$

When $N = 3$, we have the uniqueness of regular solution. Thus, for instance, if a weak solution to (2.1.1) satisfies

$$\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap L^s(0,T;L^r(\Omega)^3)$$

with $2/s + 3/r \leq 1$ and $r > 3$, then it is unique in this class.

This is a result from [Prodi(1959)] (see also [J.-L. Lions(1969)]) that can be proved as follows:

• It is sufficient to consider the case in which $2/s + 3/r = 1$ and $r > 3$. Thus, let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two weak solutions with the previous regularity and let us set $\mathbf{w} = \mathbf{u}^1 - \mathbf{u}^2$. Then, it is not difficult to check that both $\mathbf{u}_t^1$ and $\mathbf{u}_t^2$ belong to $L^2(0,T;V')$.

• Therefore, proceeding as in the proof of Theorem 2.1.3, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{w}(\cdot\,,t)\|^2 + \nu\|\nabla\mathbf{w}(\cdot\,,t)\|^2 &= -b(\mathbf{w}(\cdot\,,t), \mathbf{u}^2(\cdot\,,t), \mathbf{w}(\cdot\,,t)) \\ &\leq \frac{\nu}{2}\|\nabla\mathbf{w}(\cdot\,,t)\|^2 + \psi(t)\|\mathbf{w}(\cdot\,,t)\|^2\end{aligned}$$

a.e. in $(0,T)$, for some $\psi \in L^1(0,T)$.

The consequence is that $\mathbf{w} \equiv \mathbf{0}$, i.e. $\mathbf{u}^1 = \mathbf{u}^2$.

E 2.5. Give a detailed proof of the uniqueness of regular solution for $N = 3$.

More recently, under the assumption $\mathbf{u} \in L^2(0,T;V) \cap L^\infty(0,T;H) \cap C^0([0,T];L^3(\Omega)^3)$, the uniqueness of weak solution has been established, see [P.-L. Lions and Masmoudi(2001)]; see also [Furioli(2000)].

At this point, it is worth mentioning the more recent works [Buckmaster and Vicol(2019)] and [Buckmaster and others(2022)], where the non-uniqueness of a more general kind of weak solutions on the torus is established.

More precisely, the authors consider functions $\mathbf{u} \in C^0([0,T);L^2(\mathbb{T}^3)^3)$ such that

$$(\nabla \cdot \mathbf{u})(\cdot\,,t) = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \mathbf{u}(\mathbf{x},t)\,d\mathbf{x} = 0 \quad \text{a.e. in } (0,T) \tag{2.4.1}$$

and the Navier-Stokes system is satisfied in the following sense:

$$\begin{cases} \displaystyle\iint_{\mathbb{T}^3\times(0,T)} \mathbf{u}\cdot(\boldsymbol{\varphi}_t+(\mathbf{u}\cdot\nabla)\boldsymbol{\varphi}-\nu\Delta\boldsymbol{\varphi})\,d\mathbf{x}\,dt=-\int_{\mathbb{T}^3}\mathbf{u}_0(\mathbf{x})\cdot\boldsymbol{\varphi}(\mathbf{x},0)\,d\mathbf{x} \\ \text{for any } \boldsymbol{\varphi}\in\mathcal{D}(\mathbb{T}^3\times[0,T))^3 \text{ with } \nabla\cdot\boldsymbol{\varphi}\equiv 0, \end{cases} \tag{2.4.2}$$

where the initial field $\mathbf{u}_0$ belongs to $L^2(\mathbb{T}^3)^3$ and satisfies $\nabla\cdot\mathbf{u}_0=0$. A continuous function $\mathbf{u}$ satisfying (2.4.1)–(2.4.2) is called a *B-weak solution* in $\mathbb{T}^3\times(0,T)$.

It will be said that $(\mathbf{u},p)$ is a *strong solution in* $\mathbb{T}^3\times(0,T)$ if

$$\mathbf{u},\mathbf{u}_t,\partial_i\mathbf{u},\partial_i\partial_j\mathbf{u},p,\partial_i p\in L^2(\mathbb{T}^3\times(0,T))^3$$

and the PDEs in (2.1.1) are satisfied a.e. It will be said that two strong solutions $(\mathbf{u}^{(1)},p^{(1)})$ and $(\mathbf{u}^{(2)},p^{(2)})$ to the Navier-Stokes equations in $\mathbb{T}^3\times(0,T)$ are *connected* if there exists a B-weak solution $\mathbf{u}$ such that

$$\mathbf{u}^{(1)}=\mathbf{u} \text{ in } \mathbb{T}\times(0,T/3) \text{ and } \mathbf{u}^{(2)}=\mathbf{u} \text{ in } \mathbb{T}\times(2T/3,T).$$

The following result is proved in [Buckmaster and others(2022)]:

Theorem 2.4.1. *There exists $\beta>0$ with the following two properties:*

- *For any $T>0$, any two strong solutions $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ to the Navier-Stokes equations in $\mathbb{T}^3\times(0,T)$ with zero mean initial data are connected through a B-weak solution satisfying*
$$\mathbf{u}\in C^0([0,T);W^{1,1+\beta}(\mathbb{T}^3)^3).$$
- *Moreover, for any such $\mathbf{u}$ there exists a zero Lebesgue measure set $Z_T\subset[0,T]$ of Hausdorff dimension $<1-\beta$ such that*
$$\mathbf{u}\in C^\infty(\mathbb{T}^3\times([0,T]\setminus Z_T))^3.$$

In particular, this implies the non-uniqueness of B-weak solutions in the torus for any strong initial datum. It would be interesting to know whether a similar property holds when we consider solutions in a set $Q=\Omega\times(0,T)$ complemented with Dirichlet conditions on $\Sigma=\partial\Omega\times(0,T)$.

E 2.6. Deduce the non-uniqueness of *B*-weak solutions in the torus from Theorem 2.4.1.

Remark 2.4.2. Other non-uniqueness results can be found in [Albritton and others(2022), Albritton and others(2023), Cheskidov and Luo(2022), Luo(2019)]. □

Let us end this section with a very interesting open question.

In a celebrated paper, Caffarelli, Kohn and Nirenberg [Caffarelli and others(1982)] proved the existence of a kind of weak solution $\mathbf{u}$ such that the set

$$\mathcal{S}:=\{(\overline{\mathbf{x}},\bar{t}):\mathbf{u}\notin L^\infty(B((\overline{\mathbf{x}},\bar{t});r))^3 \ \ \forall r>0\}$$

is *small* in an appropriate sense (and, accordingly, $\mathbf{u}$ is of class C^∞ in a very large set).

These are the so called *suitable weak solutions.*

By definition, they are the weak solutions in $L^2(0,T;V)\cap L^\infty(0,T;H)$ that satisfy the following local energy inequalities:

$$\left(\int_\Omega |\mathbf{u}|^2\varphi\,d\mathbf{x}\right)(t)+\nu\iint_{\Omega\times(0,t)}|\nabla\mathbf{u}|^2\varphi\,d\mathbf{x}\,ds$$
$$\leq\left(\int_\Omega|\mathbf{u}_0|^2\varphi\,d\mathbf{x}\right)(0)+\iint_{\Omega\times(0,t)}(|\nabla\mathbf{u}|^2(\varphi_t+\nu\Delta\varphi)+(|\mathbf{u}|^2+2p)\mathbf{u}\cdot\nabla\varphi)\,d\mathbf{x}\,ds$$

for any $t\geq 0$ and any $\varphi\in\mathcal{D}(\Omega\times[0,T))$ with $\varphi\geq 0$.

Among other things, it is proved in [Caffarelli and others(1982)] that the Haussdorf dimension of the corresponding singular set $\mathcal{S}$ is ≤ 1, that is, $\mathcal{S}$ is at most as "large" as a curve in $\mathbf{R}^4$; see also [Lin(1998)].

The existence of suitable weak solutions can be established by several methods; this is shown in [Beirao da Vega(1985), Guermond(2007)]. An interesting question is whether or not suitable weak solutions are unique. This is expectable, at least from a physical viewpoint. However, the answer is unknown.

2.5 Some Regularity Results

Besides existence and uniqueness, it is also interesting to investigate the *regularity properties* of the solutions to (2.1.1).

We will use in the sequel that the mapping $\mathbf{v}\mapsto\|A\mathbf{v}\|$ is a Hilbertian norm in $H^2(\Omega)^N\cap V$, equivalent to the usual norm in $H^2(\Omega)^N$. In fact, this assertion is equivalent to the H^2-regularity property of the Stokes problem (recall Theorem 1.6.1 in Chapter 1).

It is reasonable to expect that, if the open set Ω and the data $\mathbf{f}$ and $\mathbf{u}_0$ are more regular than in Theorem 2.1.2, the same happens to the associated solutions.

When $N=2$, this can be established rigorously. Thus, we have among others the following result:

Theorem 2.5.1. *Assume that $N=2$. Also, assume that $\Omega\subset\mathbf{R}^2$ is a bounded connected open set whose boundary is of class $W^{2,\infty}$, $\mathbf{f}\in L^2(Q)^2$ and $\mathbf{u}_0\in V$. Then the unique weak solution to* (2.1.1) *satisfies*

$$\mathbf{u}\in L^2(0,T;H^2(\Omega)^2)\cap C^0([0,T];V)\quad\textit{and}\quad\mathbf{u}_t\in L^2(0,T;H).\tag{2.5.1}$$

Proof. Let us consider the Galerkin approximations (2.2.4)–(2.2.5), where we use the special base of V. It suffices to get uniform estimates of the $\mathbf{u}_m$ in $L^\infty(0,T;V)$, the $A\mathbf{u}_m$ in $L^2(Q)^2$ and the $\mathbf{u}_m'$ in $L^2(0,T;H)$.

The estimates of the $\mathbf{u}_m$ and $A\mathbf{u}_m$ can be obtained by taking in (2.2.4) $\mathbf{v}=A\mathbf{u}_m(t)$ for each $t\in(0,T)$. Note that this is admissible, in view of the particular choice of the $\mathbf{w}_j$.

We easily deduce that

$$\begin{aligned}\frac{1}{2}\|\nabla \mathbf{u}_m(t)\|^2 + \nu\int_0^t \|A\mathbf{u}_m(s)\|^2\,ds &= \frac{1}{2}\|\nabla \mathbf{u}_{0m}\|^2 + \int_0^t (\overline{\mathbf{f}}_m(s), A\mathbf{u}_m(s))\,ds\\ &- \int_0^t b(\mathbf{u}_m(s), \mathbf{u}_m(s), A\mathbf{u}_m(s))\,ds\\ &\le \frac{1}{2}\|\nabla \mathbf{u}_0\|^2 + C\int_0^t \|\overline{\mathbf{f}}(s)\|^2\,ds + \frac{\nu}{4}\int_0^t \|A\mathbf{u}_m(s)\|^2\,ds\\ &+ C\int_0^t \|\mathbf{u}_m(s)\|^2\|\nabla \mathbf{u}_m(s)\|^4\,ds.\end{aligned}$$

Here, we have used that $\mathbf{u}_0 \in V$ and

$$\begin{aligned}|b(\mathbf{u}_m(s), \mathbf{u}_m(s), A\mathbf{u}_m(s))| &\le C\|\mathbf{u}_m(s)\|_{L^4}\|\nabla \mathbf{u}_m(s)\|_{L^4}\|A\mathbf{u}_m(s)\|\\ &\le C\|\mathbf{u}_m(s)\|^{1/2}\|\nabla \mathbf{u}_m(s)\|\|A\mathbf{u}_m(s)\|^{3/2}\\ &\le \frac{\nu}{4}\|A\mathbf{u}_m(s)\|^2 + C\|\mathbf{u}_m(s)\|^2\|\nabla \mathbf{u}_m(s)\|^4.\end{aligned}$$

These inequalities, together with (2.2.12), Gronwall's Lemma and the regularity of Ω, yield the desired bounds of the $\mathbf{u}_m$ and the $A\mathbf{u}_m$.

For the estimates of $\mathbf{u}'_m$, we take $\mathbf{v} = \mathbf{u}'_m(t)$ in (2.2.4). Now, we find that

$$\begin{aligned}\|\mathbf{u}'_m(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{u}_m(t)\|^2 &= (\overline{\mathbf{f}}_m(t), \mathbf{u}'_m(t)) - b(\mathbf{u}_m(t), \mathbf{u}_m(t), \mathbf{u}'_m(t))\\ &\le \|\overline{\mathbf{f}}(t)\|^2 + \frac{1}{2}\|\mathbf{u}'_m(t)\|^2 + C\|\mathbf{u}_m(t)\|^2_{H^2}\|\nabla \mathbf{u}_m(t)\|^2\end{aligned}$$

and, integrating in time, we find that $\mathbf{u}'_m$ is uniformly bounded in $L^2(0,T;H)$.

For more details, see for example [Constantin and Foias(1988)]. □

When $N = 3$, the situation is much more complicated. We can obtain results of this kind only when the data are small (in appropriate norms).

We have the following result:

Theorem 2.5.2. *Assume that $N = 3$, the bounded connected open set Ω is of class $W^{2,\infty}$, $\mathbf{f} \in L^2(Q)^3$ and $\mathbf{u}_0 \in V$. There exists $\alpha_0 > 0$ (that only depends on Ω and T) such that, whenever*

$$\|\nabla \mathbf{u}_0\| + \|\mathbf{f}\|_{L^2(Q)^3} \le \alpha_0, \tag{2.5.2}$$

any weak solution to (2.1.1) *furnished by Theorem 2.1.2 satisfies*

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)^3) \cap C^0([0,T];V) \quad \text{and} \quad \mathbf{u}_t \in L^2(0,T;H) \tag{2.5.3}$$

and is unique.

For the proof, we can start by trying to find adequate (new) estimates of the Galerkin approximations $\mathbf{u}_m$ and their time derivatives $\mathbf{u}'_m$. The following is found:

$$\frac{1}{2}\|\nabla \mathbf{u}_m(t)\|^2 + \nu \int_0^t \|A\mathbf{u}_m(s)\|^2\, ds \leq \frac{1}{2}\|\nabla \mathbf{u}_0\|^2 + C\int_0^t \|\bar{\mathbf{f}}(s)\|^2\, ds$$
$$+\frac{\nu}{2}\int_0^t \|A\mathbf{u}_m(s)\|^2\, ds + C\int_0^t \|\nabla \mathbf{u}_m(s)\|^6\, ds.$$

Therefore, we see that in order to conclude the assumption (2.5.2) is needed with a sufficiently small α_0.

Remark 2.5.3. Using appropriately Lemma 2.2.5, we can deduce from (2.5.1) and (2.5.3) suitable regularity properties for p. In particular, under the assumptions in Theorems 2.5.1 or 2.5.2, we have that $p \in L^2(0,T;H^1(\Omega))$. As a consequence, the PDEs in (2.1.1) are satisfied a.e. in Q and we say that the solution is *strong.* □

For large smooth $\mathbf{f}$ and/or large $\mathbf{u}_0$, the regularity of $\mathbf{u}$ may fail.

This has been unknown since the seminal works of Leray (see [Leray(1934)]) until september 2026. The question was included in the famous list of *One-Million Dollars Millenium Open Problems* proposed by the Clay Institute in 2000, see

`http:www.claymath.org/millennium-problems/navier?stokes-equation`

Very recently, the following result has been established:

Theorem 2.5.4. *For every $\nu > 0$ there exist a smooth source $\mathbf{f}$ in $\mathbf{R}^3 \times [0,+\infty)$, a compact set $K \subset \mathbf{R}^3$ and a smooth C^∞ solution to the Navier-Stokes PDEs in $\mathbf{R}^3 \times [0,1)$ satisfying*

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0} \quad in \quad \mathbf{R}^3 \tag{2.5.4}$$

such that

$$\begin{cases} \operatorname{Supp}\mathbf{u}(\cdot\,,t) \cup \operatorname{Supp} p(\cdot\,,t) \subset K \quad \forall t \in [0,1), \\ \sup\limits_{[0,1)} \|\mathbf{u}(\cdot\,,t)\|_{L^2(\mathbf{R}^3)} < +\infty \quad and \quad \sup\limits_{[0,1)} \|\mathbf{u}(\cdot\,,t)\|_{L^\infty(\mathbf{R}^3)} = +\infty. \end{cases}$$

Therefore, no smooth solution can exist in $\mathbf{R}^3 \times [0,+\infty)$ for this $\mathbf{f}$ with $\mathbf{u}_0 = \mathbf{0}$ and this provides a (negative) answer to the previous Millennium Problem holds.

The proof of Theorem 2.5.4 is described in detail in [OpenAI-a(2026)]; more explanations can be found in [OpenAI-b(2026)]. It was produced by a group of around 10, 000 coordinating AI agents, using an OpenAI next-generation model "significantly more capable than GPT-6 Astra" in accordance with OpenAI's words. The Navier-Stokes solution was found after 88 hours and a complete formalization by Lean 4 was exhibited.

The seminal ideas come mainly from the work of D. Córdoba and L. Martínez-Zoroa in [Córdoba and Martínez-Zoroa(2023), Córdoba and Martínez-Zoroa(2025)], where the authors had previously constructed solutions that blow up in finite time for other similar systems; other relevant references are [Córdoba and others(2025), Córdoba and others(2026)].

Let us mention that a controversy has arisen following the publication of the document `https://cims.nyu.edu/ tristanb/statement.pdf` by T. Buckmaster, where he has claimed to obtain similar results in collaboration with L. Alpöge, from `Anthropic,` a short time before; see [Alpöge and Buckmaster(2026), Alpöge and others(2026)].

As the reader can easily perceive, these evenements have been too recent to allow us to understand in depth their consequences at the present moment.[3] However, it seems clear that, with adequate means, IA actions lead to the resolution of hard mathematical problems and this has far-reaching implications in the field. To this respect, together with [OpenAI-a(2026)] we can refer to [Claude (2026)], where an important improvement has been obtained towards the solution to the *Riemann's Conjecture* problem.[4]

On the other hand, after Theorem 2.5.4, at least the following apparently interesting question remains open:

> *Are weak solutions corresponding to smooth initial data* $\mathbf{u}_0$ *and vanishing right-hand side* $\mathbf{f}$ *smooth?*

2.6 The Ladyzhenskaja and Smagorinsky models

In view of the last result in Section 2.5, it seems interesting to investigate whether appropriately modified Navier-Stokes equations also lead to finite-time blowup or, contrarily, the corresponding Cauchy or Cauchy-Dirichlet problems possess smooth global in time solutions.

First, let us recall that, for a well chosen "averaged" Navier-Stokes system, there exist smooth initial data that produce solutions that blow up. More precisely, in [Tao(2016)] the author was able to find a continuous bilinear mapping $\hat{B}(\cdot\,,\cdot)$ and an initial $\mathbf{u}_0$ satisfying

$$|\partial^\alpha \mathbf{u}_0(\mathbf{x})| \leq C_{\alpha,K}(1+|\mathbf{x}|)^{-K} \quad \text{in} \quad \mathbf{R}^3 \quad \forall \alpha,\ \forall K \geq 0. \tag{2.6.1}$$

such that $\langle \hat{B}(\mathbf{v},\mathbf{v}),\mathbf{v}\rangle = 0$ for all smooth free-divergence $\mathbf{v}$ and there exists no global in time smooth solution to

$$\mathbf{u}_t + \hat{B}(\mathbf{u},\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = 0, \quad \nabla\cdot\mathbf{u} = 0 \ \text{ in } \ \mathbf{R}^3\times[0,+\infty),$$

with initial data $\mathbf{u}_0$.

Then, let us come back to the models mentioned in Section 2.1 with stress tensors given by (2.1.3)–(2.1.4) and (2.1.5) and let us see that, in these cases the opposite phenomenon appears.

Thus, let us consider the initial-boundary value problem

$$\begin{cases} \mathbf{u}_t - \nabla\cdot(\nu_r(|D\mathbf{u}|^2)D\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \quad \nabla\cdot\mathbf{u} = 0, & (\mathbf{x},t)\in Q, \\ \mathbf{u}(\mathbf{x},t) = 0, & (\mathbf{x},t)\in\Sigma, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\Omega, \end{cases} \tag{2.6.2}$$

where $\Omega\subset\mathbf{R}^3$ is (again) a bounded connected open set with $W^{1,\infty}$ boundary.

[3] Strictly speaking, the result is still subject to verification. The complexity of the proof makes it reasonable to wait until a sufficiently large number of experts have had time to review the proof.

[4] The mathematical argument in this paper was discovered and written by `Claude,` an AI developped by `Anthropic.`

In what concerns weak solutions, depending on the value of r, several different existence and/or uniqueness results can be established: for very small r, it can only be proved that a local regular solution exists for regular initial data; for moderate r, global in time weak solutions exist; for larger r, the existence and uniqueness of a solution holds, etc.

In order to illustrate the situation, we will recall now two of these results; for a collection of results and a list of open questions, see [Ladyzhenskaja(1969), J.-L. Lions(1969), Malek and others(1996)] and the more recent work [Beirao da Vega(2004)].

To fix ideas, let us assume that both $\nu_0, \nu_1 > 0$ and let W_r be the adherence of $\mathcal{V}$ in the Sobolev space $W_0^{1,r}(\Omega)^N$. Obviously, W_r is a separable reflexive Banach space for the norm

$$\|\mathbf{v}\|_{W_r} := \left(\int_\Omega |\nabla \mathbf{v}|^r \, d\mathbf{x} \right)^{1/r}.$$

It will be said that $\mathbf{u}$ is a weak solution to (2.6.2) if $\mathbf{u} \in L^r(0,T;W_r) \cap L^\infty(0,T;H)$ and

$$\begin{cases} \displaystyle\iint_Q \Big(\rho(\mathbf{u} \cdot \boldsymbol{\varphi}_t + (\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi}) - \nu_r(|D\mathbf{u}|^2) \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \Big) \, d\mathbf{x}\, dt \\ \qquad \displaystyle + \int_\Omega \mathbf{u}_0(\mathbf{x}) \cdot \varphi(\mathbf{x}, 0) \, d\mathbf{x} = 0 \end{cases}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times [0,T))^N$ satisfying $\nabla \cdot \boldsymbol{\varphi} = 0$ in Q.

The following results hold:

Theorem 2.6.1. *Assume that $N = 3$, ν_r is given by* (2.1.4) *with $\nu_0, \nu_1 > 0$ and $11/5 < r < 3$ and $\mathbf{u}_0 \in H$. Then* (2.6.2) *possesses exactly one weak solution.*

Proof. For the existence, it suffices to argue as in the proof of Theorem 2.1.2. The main points are to prove a uniform (energy) estimate for the solutions to associated Galerkin approximations and a uniform estimate of the corresponding time derivatives.

Let us see how this can be achieved. Thus, let the Galerkin approximations be defined by

$$\begin{cases} \mathbf{u}_m : [0,T] \mapsto V_m, \\ (\mathbf{u}'_m, \mathbf{v}) + k_r(\mathbf{u}_m; \mathbf{v}) + b(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}) = (\overline{\mathbf{f}}_m(t), \mathbf{v}) \quad \forall \mathbf{v} \in V_m, \quad t \in (0,T), \\ \mathbf{u}_m(0) = \mathbf{u}_{0m}, \end{cases} \tag{2.6.3}$$

where $\{\mathbf{w}_1, \mathbf{w}_2, \dots\}$ is the special basis of V introduced in Chapter 1, the $\overline{\mathbf{f}}_m$ and the $\mathbf{u}_{0m}$ are as in the proof of Theorem 2.1.2 and we have used the notation

$$k_r(\mathbf{u}; \mathbf{v}) := \int_\Omega \nu_r(|D\mathbf{u}|^2) D\mathbf{u} : D\mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v} \in W_r.$$

It is immediate that

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_m(t)\|^2 + k_r(\mathbf{u}_m(t); \mathbf{u}_m(t)) = (\overline{\mathbf{f}}_m(t), \mathbf{u}_m(t))$$

and this clearly gives

$$\|\mathbf{u}_m\|_{L^\infty(0,T;H)} + \|\mathbf{u}_m\|_{L^2(0,T;V)} + \|\mathbf{u}_m\|^{r/2}_{L^r(0,T;W_r)} \leq C(\|\mathbf{u}_0\| + \|\mathbf{f}\|_{L^2(Q)}). \tag{2.6.4}$$

Moreover, one has

$$\mathbf{u}_m'(t) = \tilde{P}_m^r(\overline{\mathbf{f}}_m(t) - K_r(\mathbf{u}_m(t)) - B(\mathbf{u}_m(t), \mathbf{u}_m(t))) \quad \forall t \in (0,T),$$

where $\tilde{P}_m^r : W_r' \mapsto V_m$ is the usual projector, that is,

$$\tilde{P}_m^r \mathbf{g} = \sum_{j=1}^{m} \langle \mathbf{g}, \mathbf{w}_j \rangle_r \, \mathbf{w}_j \quad \forall \mathbf{g} \in W_r',$$

we have introduced $K_r : W_r \mapsto W_r'$ with

$$\langle K_r(\mathbf{u}), \mathbf{v} \rangle_r = \int_\Omega \nu_r(|D\mathbf{u}|^2) D\mathbf{u} : D\mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in W_r$$

and $\langle \cdot\,, \cdot \rangle_r$ stands for the duality pairing for $W_r' \times W_r$.

Note that K_r is continuous, uniformly monotone and coercive. In the sequel, we will use the mapping $\mathcal{K}_r : L^r(0,T;W_r) \mapsto L^{r'}(0,T;W_r')$, with $\mathcal{K}_r(\mathbf{u})(t) := K_r(\mathbf{u}(\cdot\,,t))$ a.e.

Taking into account that $\|\tilde{P}_m^r \mathbf{g}\|_{W_r'} \leq C \|\mathbf{g}\|_{W_r'}$ for any $\mathbf{g} \in W_r'$ and some C independent of m, it is not difficult to prove that

$$\|\mathbf{u}_m'\|_{L^{r'}(0,T;W_r')} \leq C(\|\mathbf{f}\|_{L^2(Q)} + \|\mathbf{u}_m\|_{L^r(0,T;W_r)}^{r-1} + \|B(\mathbf{u}_m, \mathbf{u}_m)\|_{L^{r'}(0,T;W_r')}) \tag{2.6.5}$$

for all m.

In order to estimate the nonlinear term, we note the following:

- Since $r > 11/5$, one has $L^r(0,T;W_r) \cap L^\infty(0,T;H) \hookrightarrow L^s(Q)^N$ for some $s > 2r'$.
- For any $\mathbf{u} \in L^r(0,T;W_r) \cap L^\infty(0,T;H)$ and any $\mathbf{v} \in L^r(0,T;W_r)$, one has

$$\iint_Q u_i \partial_i u_j v_j \, d\mathbf{x}\, dt = - \iint_Q u_i u_j \partial_i v_j \, d\mathbf{x}\, dt \leq \|\mathbf{u}\|_{L^{2r'}(Q)}^2 \|\mathbf{v}\|_{L^r(0,T;W_r)}.$$

Therefore, $\|B(\mathbf{u}_m, \mathbf{u}_m)\|_{L^{r'}(0,T;W_r')} \leq C \left(\|\mathbf{u}_m\|_{L^\infty(0,T;H)} + \|\mathbf{u}_m\|_{L^r(0,T;W_r)} \right)^2$. Using (2.6.4) and (2.6.5) we see that $\|\mathbf{u}_m'\|_{L^{r'}(0,T;W_r')}$ is uniformly bounded.

Thus, at least for a subsequence $\{\mathbf{u}_{m'}\}$, one has

$$\begin{array}{l} \mathbf{u}_{m'} \to \mathbf{u} \text{ weakly-}* \text{ in } L^\infty(0,T;H) \text{ and weakly in } L^r(0,T;W_r), \\ \mathbf{u}_{m'}' \to \mathbf{u}_t \text{ weakly in } L^{r'}(0,T;W_r') \text{ and} \\ \mathbf{u}_{m'} \to \mathbf{u} \text{ strongly in } L^2(0,T;H) \text{ and a.e.} \end{array} \tag{2.6.6}$$

As in the proof of Theorem 2.1.2, we can take limits in the nonlinear term and deduce that $B(\mathbf{u}_{m'}, \mathbf{u}_{m'}) \to B(\mathbf{u}, \mathbf{u})$ weakly in $L^{r'}(0,T;W_r')$. On the other hand, since $\mathcal{K}_r(\mathbf{u}_m)$ is uniformly bounded in $L^{r'}(0,T;W_r')$, it can be assumed that

$$\mathcal{K}_r(\mathbf{u}_{m'}) \to \boldsymbol{\chi} \text{ weakly in } L^{r'}(0,T;W_r').$$

But $\mathcal{K}_r : L^r(0,T;W_r) \mapsto L^{r'}(0,T;W_r')$ is (uniformly) monotone and hemicontinuous and this implies $\boldsymbol{\chi} = \mathcal{K}_r(\mathbf{u})$.

The rest of the proof of existence is left to the reader.

The proof of uniqueness is as follows. Let $\mathbf{u}^1$ and $\mathbf{u}^2$ be two weak solutions to (2.6.2) and let us set $\mathbf{w} := \mathbf{u}^1 - \mathbf{u}^2$. Then $\mathbf{w} \in L^r(0,T;W_r) \cap L^\infty(0,T;H)$, $\mathbf{w}_t \in L^{r'}(0,T;W_r')$,

$$\mathbf{w}_t + K_r(\mathbf{u}^1) - K_r(\mathbf{u}^2) = -B(\mathbf{u}^1, \mathbf{w}) - B(\mathbf{w}, \mathbf{u}^2)$$

and $\mathbf{w}|_{t=0} = 0$. Consequently,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\mathbf{w}(\cdot\,,t)\|^2 &+ \langle K_r(\mathbf{u}^1(\cdot\,,t)) - K_r(\mathbf{u}^2(\cdot\,,t)), \mathbf{w}(\cdot\,,t)\rangle \\
&= -\langle B(\mathbf{w}(\cdot\,,t), \mathbf{u}^2(\cdot\,,t)), \mathbf{w}(\cdot\,,t))\rangle \\
&\le C\,\|\mathbf{w}(\cdot\,,t)\|^2_{L^{2r'}}\,\|\nabla \mathbf{u}^2(\cdot\,,t)\|_{L^r} \\
&\le C\,\|\mathbf{w}(\cdot\,,t)\|^{2-2a}\,\|\nabla \mathbf{w}(\cdot\,,t)\|^{2a}_{L^r}\,\|\nabla \mathbf{u}^2(\cdot\,,t)\|_{L^r},
\end{aligned}$$

where we have used that $2r'$ is smaller than the Sobolev embedding exponent r^* of W_r (true because $r > 11/5$).

From the uniform monotonicity of K_r and Korn's inequality in W_r, we see that, for some $\alpha > 0$, one has

$$\begin{aligned}
\frac{d}{dt}\|\mathbf{w}(\cdot\,,t)\|^2 &+ +\alpha\|\nabla \mathbf{w}(\cdot\,,t)\|^2 + \alpha\|\nabla \mathbf{w}(\cdot\,,t)\|^r_{L^r} \\
&\le \frac{\alpha}{2}\|\nabla \mathbf{w}(\cdot\,,t)\|^r_{L^r} + C\,\|\nabla \mathbf{u}^2(\cdot\,,t)\|^b_{L^r}\|\mathbf{w}(\cdot\,,t)\|^{2ab}_{L^{2r'}},
\end{aligned}$$

where $b := r/(r-2a)$. Since $r > 11/5$, one has $b < r$. Therefore, from Gronwall's Lemma we deduce that $\|\mathbf{w}(\cdot\,,t)\| \equiv 0$, that is, $\mathbf{u}^1 = \mathbf{u}^2$.

This ends the proof. □

Theorem 2.6.2. *Under the assumptions of Theorem 2.6.1, if $\partial\Omega$ is of class $W^{2,\infty}$, $r \ge 5/2$ and $\mathbf{u}_0 \in V$, the solution to* (2.6.2) *satisfies:*

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V),\ \mathbf{u}_t \in L^2(0,T;H) \text{ and } p \in L^2(0,T;H^1(\Omega)).$$

Sketch of the proof. As before, it suffices to establish appropriate estimates of the Galerkin approximations. For the proof, the following steps can be followed:

• $L^2(0,T;H^2(\Omega))$ regularity: Take $v = A\mathbf{u}_m(\cdot\,,t)$ in (2.6.3) for $t \in [0,T]$. Then, we see after some computations that, if $r \ge 5/2$, one has

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}_m\|^2 &+ \nu_0\|A\mathbf{u}_m\|^2 + \nu_1\int_\Omega |D\mathbf{u}_m|^{r-2}|A\mathbf{u}_m|^2\,d\mathbf{x} \\
&= -\int_\Omega (\mathbf{u}_m\cdot\nabla\mathbf{u}_m)\cdot A\mathbf{u}_m\,d\mathbf{x} + (\overline{\mathbf{f}}(t), A\mathbf{u}_m) \\
&\le \frac{\nu_0}{2}\|A\mathbf{u}_m\|^2 + \frac{\nu_1}{2}\int_\Omega |D\mathbf{u}_m|^{r-2}|A\mathbf{u}_m|^2\,d\mathbf{x} + \psi(t)\|\nabla\mathbf{u}_m\|^2
\end{aligned}$$

for some $\psi \in L^1(0,T)$; see [Malek and others(1996), Du and Gunzburger(1991)] for the details.

Consequently, Gronwall's Lemma yields uniform bounds for $\mathbf{u}_m$ in the spaces $L^\infty(0,T;V)$ and $L^2(0,T;H^2(\Omega)^3)$ and we have $\mathbf{u} \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V)$.

• L^2 regularity of the time derivative $\mathbf{u}_t$: If we take $v = \mathbf{u}'_m(\cdot\,,t)$ in (2.6.3) for any t, after some estimates similar to those above, one has:

$$\|\mathbf{u}'_m\|^2 + \frac{d}{dt}\left(\int_\Omega \mu_r(|D\mathbf{u}_m|^2)\,d\mathbf{x}\right) \le \frac{1}{2}\|\mathbf{u}'_m\|^2 + C(\|A\mathbf{u}_m\|^2 + \|\mathbf{f}(\cdot\,,t)\|^2),$$

where the real function μ_r is given by

$$\mu_r(s) := \frac{1}{2}\int_0^s \nu_r(\sigma)\,d\sigma \quad \forall \sigma \in \mathbf{R}_+.$$

Clearly, this provides a uniform estimate of $\|\mathbf{u}'_m\|_{L^2(Q)}$. □

It is also true that, for $r > 5/2$, stronger regularity assumptions on $\partial\Omega$, $\mathbf{f}$ and $\mathbf{u}_0$ lead to more regular properties of $\mathbf{u}$ and p. In particular, C^∞ data produce C^∞ solutions.

Let us now consider the Ladyzhenskaja-Smagorinsky model

$$\begin{cases} \mathbf{u}_t - \nu_r(\|D\mathbf{u}\|^2)\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0, \quad \nabla\cdot\mathbf{u} = 0, & (\mathbf{x},t) \in Q, \\ \mathbf{u}(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \Sigma, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{2.6.7}$$

where ν_r is again given by (2.1.4) and $r = 4$.

We see that, in this system, the viscosity coefficient depends on the velocity field through the L^2 norm $\|\nabla\mathbf{u}(\cdot\,,t)\|$. These equations have been used to describe the behaviour of *turbulent flows*. Thus, we must assume that $\mathbf{u}$ and p represent *averaged quantities*. As an effect of nonlinearity, the viscous term is modified and becomes as before.

For (2.6.7), we can prove results very similar to Theorems 2.6.1 and 2.6.2. More precisely, it will be said that $\mathbf{u}$ is a weak solution to (2.6.7) if $\mathbf{u} \in L^r(0,T;V) \cap L^\infty(0,T;H)$ and

$$\begin{cases} \displaystyle\iint_Q \Big(\rho(\mathbf{u}\cdot\boldsymbol{\varphi}_t + (\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi}) - \nu_r(\|D\mathbf{u}\|^2)D\mathbf{u} : D\boldsymbol{\varphi}\Big)\,d\mathbf{x}\,dt \\ \qquad\qquad + \displaystyle\int_\Omega \mathbf{u}_0(\mathbf{x})\cdot\varphi(\mathbf{x},0)\,d\mathbf{x} = 0 \end{cases}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(\Omega\times[0,T))^N$ satisfying $\nabla\cdot\boldsymbol{\varphi} = 0$ in Q.

The following results hold:

1. Assume that $N = 3$, ν_r is given by (2.1.4) with $\nu_0, \nu_1 > 0$ and $\mathbf{u}_0 \in H$. Then (2.6.7) possesses exactly one weak solution.

2. Under the previous assumptions, if $\partial\Omega$ is of class $W^{2,\infty}$ and $\mathbf{u}_0 \in V$, the solution to (2.6.7) satisfies:

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V),\ \mathbf{u}_t \in L^2(0,T;H) \text{ and } p \in L^2(0,T;H^1(\Omega)).$$

Chapter 3

Some Results Concerning Nonlinear Elasticity

This chapter concerns some problems with origin in nonlinear Elasticity. We will only consider a particular kind of stationary systems in three dimensions.

We will mainly follow the approach in Ciarlet's books [Ciarlet(1983), Ciarlet(2022)], where the motivation and description of the underlying physical and geometrical notions are shown in detail.

3.1 Hyperelasticity and Polyconvexity

For convenience, the notation $\mathbb{M}^3_+$ will be used to denote the set of square matrices $F \in \mathbf{R}^{3\times 3}$ such that $\det(F) > 0$.

We will deal with *hyperelastic media* in bounded connected open sets $\Omega \subset \mathbf{R}^3$ with boundary of class $W^{1,\infty}$. By definition, they are those homogeneous elastic materials governed by a field of displacements $\mathbf{u} = \mathbf{u}(\mathbf{x})$ such that

- The gradient $\nabla\mathbf{u}$ satisfies $\det(\mathrm{Id.} + \nabla\mathbf{u})(\mathbf{x}) > 0$ a.e. and
- There exists a continuous a.e. differentiable function $\Phi : \mathbb{M}^3_+ \mapsto \mathbf{R}$ such that the stress tensor associated with $\nabla\mathbf{u}$ is given by $\sigma^* = \Phi'(\mathrm{Id.} + \nabla\mathbf{u})$.[1]

Here, we have denoted by Φ' the matrix derivative of Φ, that is,

$$(\Phi'(F))_{ij} = \frac{\partial\Phi}{\partial F_{ij}}(F) \quad \forall F \in \mathbb{M}^3_+, \quad \forall i,j = 1,2,3.$$

The function Φ is usually called the *stored energy.*

A particular class of hyperelastic media is formed by the so called *St. Venant-Kirchhoff materials.* They correspond to the choice

$$\Phi(F) = -\nu_1 \operatorname{trace}(F^T F) + \nu_2 \operatorname{trace}((F^T)^2) + \nu_3 \operatorname{trace}(\operatorname{adj}(F^T F))$$

[1] Among other things, this means that the resultant of the forces exerted by the material external to a ball B on the particles inside the ball is given by $\int_{\partial B} \sigma^* \cdot \mathbf{n}\, d\Gamma$, where (as usual) $\mathbf{n}$ denotes the unit exterior normal vector.

for some $\nu_i > 0$.

It will always be assumed in the sequel that

$$\Phi(F) \geq -C \quad \forall F \in \mathbb{M}^3_+.$$

Let $\{\Gamma_0, \Gamma_1\}$ be a partition of $\partial\Omega$ and let the functions $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ be given, with

$$\mathbf{f} \in L^\infty(\Omega)^3,\ \mathbf{h} \in L^\infty(\partial\Omega)^3 \text{ and } \mathbf{g} = \overline{\mathbf{u}}|_{\Gamma_0} \text{ with } \overline{\mathbf{u}} \in W^{1,\infty}(\Omega)^3.$$

We will interpret $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ as prescribed fields of external volumetric forces, boundary displacements and boundary surface forces, respectively. Then, taking into account the laws of mechanics, it is reasonable to believe that (if it exists) the field of displacements produced by these data is the solution to the extremal problem

$$\left\{ \begin{array}{l} \text{Minimize } I(\mathbf{u}) := \displaystyle\int_\Omega \Big(\Phi(\mathrm{Id.} + \nabla\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \Big)\, d\mathbf{x} - \int_{\Gamma_1} \mathbf{h} \cdot \mathbf{u}\, d\Gamma \\ \text{Subject to } \mathbf{u} \in \mathbb{U}, \end{array} \right. \tag{3.1.1}$$

where $\mathbb{U}$ is a well chosen subset of $H^1(\Omega)^3$ where $\det(\mathrm{Id.} + \nabla\mathbf{u}) > 0$ a.e. and $\mathbf{u}|_{\Gamma_0} = \mathbf{g}$. Here, I is called the *total energy* functional.

Observe that $I(\mathbf{u})$ makes sense for all $\mathbf{u} \in H^1(\Omega)^3$ provided $\det(\mathrm{Id.} + \nabla\mathbf{u})(\mathbf{x}) > 0$ a.e. Also, note that, if $\mathbf{u}$ were a solution to (3.1.1), at least formally we would have $I'(\mathbf{u})\mathbf{v} = 0$ for all $\mathbf{v} \in H^1(\Omega)^3$ such that $\mathbf{u} + \epsilon\mathbf{v} \in \mathbb{U}$ for any small $|\epsilon|$.

Consequently, with an appropriate choice of $\mathbb{U}$, it can be expected that $\mathbf{u}$ solves the PDE problem

$$\left\{ \begin{array}{ll} -\nabla \cdot \Phi'(\mathrm{Id.} + \nabla\mathbf{u}) = \mathbf{f}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{u}(\mathbf{x}) = \mathbf{g}(\mathbf{x}), & \mathbf{x} \in \Gamma_0, \\ \Phi'(\mathrm{Id.} + \nabla\mathbf{u}) \cdot \mathbf{n} = \mathbf{h}(\mathbf{x}), & \mathbf{x} \in \Gamma_1. \end{array} \right. \tag{3.1.2}$$

The reminder of this section is mainly concerned with the existence of solution to (3.1.1) and some variants for appropriate assumptions on Φ, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$.

In accordance with well known results, if the stored energy function Φ were convex and coercive (for instance) in the sense that

$$\Phi(F) \geq C|F|^2 - C \quad \forall F \in \mathbf{R}^{3\times 3}$$

we would certainly be able to assert the existence of at least one solution to (3.1.1). To this purpose, it would suffice to apply the classical statement of the *Fundamental Principle* of the Calculus of Variations. However, the convexity of Φ is not a realistic assumption; see [Truesdell and Noll(2004)]. And this is a major difficulty in the present context.

In 1977, J. Ball introduced a generalized convexity concept that made it possible to minimize the total energy for many materials; see [Ball(1976), Ball(1982)]. It is called *polyconvexity* and is made precise as follows:

Definition 3.1.1. *Let the function $\Phi : \mathbb{M}^3_+ \mapsto \mathbf{R}$ be given. It will be said that Φ is polyconvex if there exists a convex function $G : \mathbb{M}^3_+ \times \mathbb{M}^3_+ \times \mathbf{R}_+ \mapsto \mathbf{R}$ such that*

$$\Phi(F) = G(F, \mathrm{adj}(F), \det(F)) \quad \forall F \in \mathbb{M}^3_+$$

where, for every $F \in \mathbf{R}^{3\times 3}$, $\mathrm{adj}(F)$ denotes the corresponding matrix of adjoints (i.e. the transpose of the matrix of cofactors of F).

As a first example of polyconvex hyperelastic media, we have the so called *Ogden materials*. In the simplest case, this corresponds to functions Φ of the form

$$\Phi(F) = a|F|^2 + b|\mathrm{adj}(F)|^2 + \psi(\det(F)) \quad \forall F \in \mathbb{M}^3_+, \tag{3.1.3}$$

where $\psi : \mathbf{R}_+ \mapsto \mathbf{R}$ is a convex function.

More generally, it is said that we deal with an Ogden material whenever there exist constants a_i, α_i, b_j and β_j such that the $a_i, b_j > 0$, the $\alpha_i, \beta_j \geq 1$ and

$$\begin{cases} \Phi(F) = \displaystyle\sum_{i=1}^{m} a_i \,\mathrm{trace}((F^TF)^{\alpha_i/2}) \\ \qquad\quad + \displaystyle\sum_{j=1}^{n} b_j \,\mathrm{trace}(\mathrm{adj}(F^TF)^{\beta_j/2}) + \psi(\det(F)) \quad \forall F \in \mathbb{M}^3_+, \end{cases}$$

where again ψ is convex.

E 3.1. Prove that Ogden materials are polyconvex. Also, prove that St. Venant-Kirchhoff materials are not polyconvex.

3.2 An existence result

The main result in this section (and also the main contribution by Ball in the field) is as follows:

Theorem 3.2.1. *Let Φ be as in Definition 3.1.1. Assume that the following properties hold:*

- *If $F^n \to F$ and $H^n \to H$ in $\mathbb{M}^3_+$ and $\beta^n \to 0^+$, then $G(F^n, H^n, \beta^n) \to +\infty$.*
- *There exist $p \geq 2$, $q \geq p'$ and $r > 1$ and constants $C > 0$ such that*

$$G(F, H, \beta) \geq C(\|F\|_p^p + \|H\|_q^q + |\beta|^r) - C \quad \forall (F, H, \beta) \in \mathbb{M}^3_+ \times \mathbb{M}^3_+ \times \mathbf{R}_+.$$

Let $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ satisfy

$$\mathbf{f} \in L^\rho(\Omega)^3, \ \mathbf{g} = \overline{\mathbf{u}}|_{\Gamma_0} \text{ with } \overline{\mathbf{u}} \in W^{1,p}(\Omega)^3 \ \text{ and } \ \mathbf{h} \in L^\sigma(\Gamma_1)^3$$

with $\rho, \sigma \geq 1$ large enough to have

$$L^\rho(\Omega) \subset (W^{1,p}(\Omega))' \text{ and } L^\sigma(\Gamma_1) \subset (W^{1-1/p,p}(\Gamma_1))'$$

and let us set

$$\begin{aligned} \mathbb{U} := \{\mathbf{u} \in W^{1,p}(\Omega)^3 : \mathrm{adj}(\mathrm{Id.} + \nabla\mathbf{u}) \in L^q(\Omega)^{3\times 3}, \ \det(\mathrm{Id.} + \nabla\mathbf{u}) \in L^r(\Omega), \\ \det(\mathrm{Id.} + \nabla\mathbf{u}) > 0 \ \text{ a.e. in } \Omega, \ \mathbf{u}|_{\Gamma_0} = \mathbf{g}\} \end{aligned}$$

Then there exists at least one solution to the extremal problem (3.1.1).

For the proof, the main technical ingredient is the following result, that allows to take limits in a minimizing sequence:

Lemma 3.2.2. *Let us assume that* $p \geq 2$, $\mathbf{u} \in W^{1,p}(\Omega)^3$ *and* $\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}) \in L^{p'}(\Omega)^{3\times 3}$. *Then* $\det(\mathrm{Id.} + \nabla \mathbf{u}) \in L^1(\Omega)$. *On the other hand, if*

$$\begin{cases} \mathbf{u}^n \to \mathbf{u} \text{ weakly in } W^{1,p}(\Omega)^3, \\ \mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}^n) \to M \text{ weakly in } L^{p'}(\Omega)^{3\times 3} \text{ and} \\ \det(\mathrm{Id.} + \nabla \mathbf{u}^n) \to \beta \text{ weakly in } L^1(\Omega), \end{cases} \tag{3.2.1}$$

then $M = \mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u})$ *and* $\beta = \det(\mathrm{Id.} + \nabla \mathbf{u})$.

Proof. First of all, note that, if $\mathbf{u} \in W^{1,p}(\Omega)^3$, one has $\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}) \in L^{p/2}(\Omega)^{3\times 3}$, since the components of $\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u})$ are polynomial in the $\partial_i u_j$ of order 2.

On the other hand, under the assumptions of the lemma, $\det(\mathrm{Id.} + \nabla \mathbf{u}) \in L^1(\Omega)$.

Let us assume that (3.2.1) holds. With the convention that the indices 4, 5 and 6 are respectively identified to 1, 2 and 3, we have

$$\begin{aligned} (\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}))_{ij} = {} & \partial_{i+2}\left((x_{j+2} + u_{j+2})\partial_{i+1}(x_{j+1} + u_{j+1})\right) \\ & - \partial_{i+1}\left((x_{j+2} + u_{j+2})\partial_{i+2}(x_{j+1} + u_{j+1})\right) \end{aligned}$$

for all $i, j = 1, 2, 3$. Consequently, the $(\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}^n))_{ij}$ converge weakly in $W^{-1,s'}(\Omega)$ to the corresponding $(\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}))_{ij}$ for some $s > 1$.

This proves that $M = \mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u})$.

On the other hand, we also have that

$$\det(\mathrm{Id.} + \nabla \mathbf{u}) = \sum_{i=1}^{3} \partial_i \left((x_1 + u_1)(\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u})_{i1}\right),$$

whence $\det(\mathrm{Id.} + \nabla \mathbf{u}^n)$ converges weakly in $W^{-1,r'}(\Omega)$ to the corresponding $\det(\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}))$ for some $r > 1$.

Accordingly, $\beta = \det(\mathrm{Id.} + \nabla \mathbf{u})$ and this ends the proof. □

Proof of Theorem 3.2.1:

Let $\{\mathbf{u}^n\}$ be a minimizig sequence for (3.1.1). Then, in view of the assumptions on G, one has

$$\|\mathbf{u}^n\|_{W^{1,p}} + \|\mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}^n)\|_{L^p} + \|\det(\mathrm{Id.} + \nabla \mathbf{u}^n)\|_{L^r} \leq C$$

for some $p \geq 2$, $q \geq p'$ and $r > 1$. Also, it can be assumed that $\det(\mathrm{Id.} + \nabla \mathbf{u}^n) \geq \kappa > 0$.

After an appropriate extraction, one has (3.2.1) for some M and β. Consequently, in view of Lemma 3.2.2, one has

$$\begin{cases} \mathbf{u}^n \to \mathbf{u} \text{ weakly in } W^{1,p}(\Omega)^3, \\ \mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}^n) \to \mathrm{adj}(\mathrm{Id.} + \nabla \mathbf{u}) \text{ weakly in } L^q(\Omega)^{3\times 3} \text{ and} \\ \det(\mathrm{Id.} + \nabla \mathbf{u}^n) \to \det(\mathrm{Id.} + \nabla \mathbf{u}) \text{ weakly in } L^r(\Omega), \end{cases}$$

This, together with the convexity of G and the fact that $\det(\mathrm{Id.} + \nabla \mathbf{u}) \geq \kappa$, proves that $\mathbf{u}$ belongs to $\mathbb{U}$ and solves (3.1.1).

Therefore, the existence of a solution has been established. □

Unfortunately, it is unknown whether the solution to (3.1.1) is regular enough to satisfy an equilibrium PDE like the one in (3.1.2) even in a weak sense.

E 3.2. Consider a simplified Ogden material described by (3.1.3). Is it possible in this case to deduce an equilibrium PDE satisfied by the displacements field, at least for small data $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$?

Remark 3.2.3. At present, how to extend the results in this section to a time-dependent context is not completely understood. This would provide a description of the behavior of a hyperelastic material subject to the action of time depending forces. □

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