

Mathematical Analysis of Parabolic PDEs, (Doc-Course, US).

F.Guillén-González

Depto EDAN and IMUS, Universidad de Sevilla, España. guillen@us.es.

October 4, 2026

Contents

1	Linear parabolic PDEs.	3
1.1	Weak solutions.	3
1.2	Strong solutions of second order Parabolic equations	5
1.3	Linear gradient flows	6
1.4	Maximum principle for weak solutions. Comparison principle	7
1.5	Strong Maximum principle for classical solutions. Infinite propaga- tion speed	8
1.6	Classical regularity vs compatibility constraints. Smoothing property	8
1.7	L^p regularity of the heat problem	9
2	Semilinear second order parabolic PDEs.	9
2.1	Sublinear and Lipschutz system. Banach fixed-point theorem	9
2.2	Sublinear case. Leray-Schauder fixed-point	10
2.3	Blow-up vs global in time solution	10
2.4	Gradient flows	11

3	Some chemotaxis models	12
3.1	A generic model	12
3.2	Keller-Segel model	13
3.3	Keller-Segel logistic model	13

1 Linear parabolic PDEs.

Given a function $u = u(t, x)$, we denote $u(t) = u(t, \cdot) : \Omega \rightarrow \mathbb{R}$ (separation of time and space variables).

Given X a Banach space, we consider the norms in the “Bochner” spaces $C([0, T]; X)$ and $L^p(0, T; X)$:

$$\|u\|_{C(X)} = \max_{t \in [0, T]} \|u(t)\|_X \quad \text{and} \quad \|u\|_{L^p(X)}^p = \int_0^T \|u(t)\|_X^p \quad 1 \leq p < \infty$$

Gelfand triplet: Let V, H be Hilbert spaces st. $V \subset H \equiv H' \subset V'$ (cont. and dense embeddings).

Weak solution space:

$$W(0, T) = \{u \in L^2(0, T; V), \quad u_t \in L^2(0, T; V')\} \subset C(H)$$

Theorem 1 (Lions-magenes) *For any $u, v \in W(0, T)$:*

1. *the mapping $t \in [0, T] \rightarrow u(t) \in H$ is absolutely continuous.*
2. *the scalar product mapping $t \in [0, T] \rightarrow (u(t), v(t))_H \in \mathbb{R}$ is absolutely continuous and its classical derivative exists almost everywhere in time and satisfies:*

$$\frac{d}{dt}(u(t), v(t))_H = \langle u_t(t), v(t) \rangle + \langle v_t(t), u(t) \rangle$$

1.1 Weak solutions.

Abstract variational problem: Given H, V Hilbert spaces with $V \subset H \equiv H' \subset V'$, $u_0 \in H$ and $f \in L^2(0, T; V')$, find $u \in W(0, T)$ for any $T > 0$ a weak solution satisfying the variational formulation

$$\langle u_t(t), v \rangle_{V'} + l(t; u(t), v) = \langle f(t), v \rangle_{V'}, \quad \text{a.e. } t \in (0, +\infty), \quad \forall v \in V, \quad (1)$$

and the initial condition $u(0, \cdot) = u_0$ in H .

Here, $l(t; \cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ is a V -continuous bilinear form and weakly coercive (a.e. $t > 0$); there exists $\alpha_1 > 0$:

$$l(t; u, u) \geq \alpha_1 |u|_V^2 - \alpha_2(t) \|u\|_H^2 \quad (2)$$

where $\alpha_2 \in L^1(0, T)$ and $|u|_V$ is a seminorm in V such that $\|u\|_V^2 = |u|_V^2 + \|u\|_H^2$. With respect to time is a Caratheodory function, that is

$$t \in (0, \infty) \rightarrow l(t; u, v) \text{ is (Lebesgue) measurable } (\forall u, v \in V)$$

RK: Defining the operator $L : [0, +\infty) \times V \rightarrow V'$ as $\langle L(t; u), \bar{u} \rangle = l(t; u, \bar{u})$ then problem (1) can be rewritten as

$$u_t(t, \cdot) + L(t; u(t, \cdot)) = f(t, \cdot) \text{ in } V', \text{ a.e. } t \in (0, +\infty).$$

Example: Second order linear diffusion-convection-reaction.

Let $Q_T = \Omega \times (0, T)$ ($\forall T > 0$) be the time-space cylinder domain, where $\Omega \subset \mathbb{R}^d$ is a bounded domain with regular enough boundary $\partial\Omega$.

We consider the time-dependent heat equation modeled by the linear parabolic initial and boundary-value problem:

$$\begin{cases} u_t - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + c u = f, & \text{in } (0, +\infty) \times \Omega, \\ u|_{\partial\Omega_D} = 0, \quad -A \nabla u \cdot \mathbf{n}|_{\partial\Omega_N} = 0 & \text{in } (0, +\infty) \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \quad (3)$$

Unknown is $u : (0, +\infty) \times \Omega \rightarrow \mathbb{R}$ the temperature.

Data are:

- $f : (0, +\infty) \times \Omega \rightarrow \mathbb{R}$ the external heat source,
- $u_0 : \Omega \rightarrow \mathbb{R}$ the initial temperature,
- $A(x) \in \mathbb{R}^{d \times d}$ symmetric and uniform definite positive $\xi^t A(x) \xi \geq \alpha |\xi|^2 > 0$ for any $\xi \in \mathbb{R}^d$ (uniform elliptic property) is the heat conductivity,

- $\mathbf{b} : (0, +\infty) \times \Omega \rightarrow \mathbb{R}^d$ is a given convective velocity field
- $c : (0, +\infty) \times \Omega \rightarrow \mathbb{R}$ is the reaction coefficient.

Indeed, if we take $V = H_D^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega_D} = 0\}$ and $H = L^2(\Omega)$, then given $A \in L^\infty(\Omega)^{d \times d}$, $\mathbf{b} \in L^\infty(Q_T)^d$, $c \in L^\infty(Q_T)$, $f \in L^2(0, T; V')$ ($\forall T > 0$), the variational formulation of problem (3) is given by (1) where

$$l(t; u, v) = \int_{\Omega} (A \nabla u, \nabla v) + (\mathbf{b}(t) \cdot \nabla u) v + c(t) u v \quad (4)$$

The following theorem gives the existence and uniqueness of weak solutions of (1) (in particular weak solutions of (3)).

Theorem 2 ($\exists!$ global in time weak solution) *There exists a unique weak solution u of (1) in $(0, +\infty)$, with $u \in W(0, T)$ for any $T > 0$. Moreover, the following energy inequality holds:*

$$\frac{d}{dt} \|u(t)\|_H^2 + \alpha_1 \|u\|_V^2 \leq \frac{1}{\alpha_1} \|f(t)\|_{V'}^2 + 2(\alpha_2(t) + \alpha_1) \|u\|_H^2 \quad a.e. \ t > 0. \quad (5)$$

Hint: Taking in (1) as test function $v = u(t, \cdot)$, using Cauchy-Schwarz and Young inequalities and the chain rule

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_H^2 = \langle u_t(t), u(t) \rangle_{V'}$$

Then, Gronwall's Lemma gives the weak a priori estimates. Uniqueness of weak solutions is straitforward comparing two possible solutions.

1.2 Strong solutions of second order Parabolic equations

We consider the second order problem (3). When data are more regular, the weak solutions are in fact the strong solutions satisfying the corresponding PDE of (3) point-wisely a.e. $(t, x) \in (0, +\infty) \times \Omega$.

Theorem 3 (Strong solution) *If $f \in L^2(0, T; H)$ for any $T > 0$ ($H = L^2(\Omega)$) and $u_0 \in V = H_D^1(\Omega)$, then the weak solution u of problem (3) satisfies the regularity*

$$u \in L^\infty(0, T; H^1), \quad u_t \in L^2(0, T; L^2), \quad \forall T > 0.$$

Moreover the following strong energy inequality holds:

$$\frac{d}{dt} \|u(t)\|_a^2 + \|u_t(t)\|_{L^2}^2 \leq (\|f(t)\|_{L^2} + \|c(t)\|_{L^\infty} \|u\|_{L^2} + \|\mathbf{b}(t)\|_{L^\infty} \|\nabla u\|_{L^2})^2 \quad \text{a.e. } t \in [0, +\infty), \quad (6)$$

where $\|u\|_a^2 := a(u, u) = \int_\Omega (A \nabla u, \nabla u)$. Note that $\|u\|_a$ is a seminorm equivalent to the standard H^1 -seminorm $\|\nabla u\|_{L^2}$.

Finally, assuming the $H^2(\Omega)$ -regularity of the elliptic boundary-value problem

$$-\nabla \cdot (A \nabla u) = g \quad \text{in } \Omega, \quad u|_{\partial\Omega_D} = 0, \quad -A \nabla u \cdot \mathbf{n}|_{\partial\Omega_N} = 0, \quad (7)$$

then

$$u \in L^2(0, T; H^2) \text{ for any } T > 0 \text{ (and } u \in C^0([0, +\infty); H^1))$$

and the PDE the Neumann boundary condition of (3) are satisfied point-wisely a.e.

Essentially, if $f \in L^2$ then $u, u_t, \nabla u, D^2 u \in L^2$.

Hint: Take in (1) as test function $v = u_t$ and using the equality

$$a(u, u_t) = \frac{1}{2} \frac{d}{dt} a(u, u)$$

the equality (6) can be deduced. Then, regularity $u \in L^2(0, T; H^2)$ holds by a bootstrap argument via elliptic problem (7)

1.3 Linear gradient flows

In the case without convection ($\mathbf{b} = 0$) and nonnegative reaction ($c \geq 0$), and time-independent data $c \neq c(t)$ and $f \neq f(t)$, the problem has a gradient flow structure

$$u_t + \frac{\delta E(u)}{\delta u} = 0$$

where

$$E(u) = \frac{1}{2} \int_\Omega ((A \nabla u) \cdot \nabla u + c u^2) dx - \langle f, u \rangle.$$

In fact $\frac{\delta E(u)}{\delta u} = -\nabla \cdot (A \nabla u) + c u - f$

Theorem 4 (Large-time convergence via Lyapunov function) *The following free-energy equality holds:*

$$\frac{d}{dt}E(u(t)) + \|u_t(t)\|_{L^2}^2 = 0 \quad \text{a.e. } t \in [0, +\infty). \quad (8)$$

In particular, the free-energy $E(u(t))$ is strictly decreasing excepting in critical points: $u_ \in V$ solution of the corresponding steady solution*

$$(A\nabla u_*, \nabla v) + (c u_*, v) = \langle f, v \rangle \quad \forall v \in V$$

Moreover, one has the following asymptotic in time convergence

$$u(t) \rightarrow u_* \text{ in } H^1, \text{ as } t \rightarrow +\infty. \quad (9)$$

1.4 Maximum principle for weak solutions. Comparison principle

[?] Stampacchia's method

Assuming nonnegative reaction $c(t, x) \geq 0$.

Let $Q_T = \Omega \times (0, T)$ be the time-space cylinder domain, and let the parabolic boundary ∂Q_T consist of the initial time slice and the lateral boundary:

$$\partial Q_T = (\{t = 0\} \times \Omega) \cup ([0, T) \times \partial\Omega)$$

Theorem 5 *Let u be the weak solution of (3) with $u_0 \in L^\infty(\Omega)$. It holds*

- *If $f(t, x) \leq 0$ then $u(t, x) \leq M := \max\{0, \sup_{\partial Q_T} u\}$ a.e. $(t, x) \in Q_T$.*
- *If $f(t, x) \geq 0$ then $u(t, x) \geq m := \min\{0, \inf_{\partial Q_T} u\}$ a.e. $(t, x) \in Q_T$.*

Moreover, in the case $c \equiv 0$, then $u \leq \sup_{\partial Q_T} u$ and $u \geq \inf_{\partial Q_T} u$ respectively.

Hint: Use Stampacchia's (truncation) method: If $f \leq 0$, test (1) by $v = (u - M)_+ \in V$ (or $v = (u + m)_-$ if $f \geq 0$), using that $\partial_t v_+ = \partial_t v$ and $\nabla v_+ = \nabla v$ when $v(t, x) \geq 0$;

otherwise $\partial_t v_+ = 0$ and $\nabla v_+ = 0$ when $v(t, x) < 0$. Take special attention to reaction term.

Key Implication: Comparison Principle.

$u_t + Lu \leq v_t + Lv$ in the weak sense in Q_T and $u \leq v$ on ∂Q_T in the trace sense, then $u \leq v$ in Q_T

1.5 Strong Maximum principle for classical solutions. Infinite propagation speed

[Evans] with $c = 0$, using Harnack's inequality (Th 11), and with $c \geq 0$ (Th 12), (E. Hopf's boundary/interior argument)

Theorem 6 *Let $u \in C^{1,2}(Q_T) \cap C^0(\overline{Q_T})$ be classical solution of (3) with $u_0 \in C(\overline{\Omega})$. It holds*

- If $u_t + Lu \leq 0$ in Q_T and u attains a nonnegative maximum at a point $(t_0, x_0) \in Q_T$, then u is identically constant on Q_{t_0} .
- If $u_t + Lu \geq 0$ in Q_T and u attains a nonpositive minimum at a point $(t_0, x_0) \in Q_T$, then u is identically constant on Q_{t_0} .

Key Implications:

- No Interior Peaks: Unless the physical system is in a steady, uniform equilibrium (constant function), a concentration or temperature spike cannot spontaneously form inside the domain.
- Infinite propagation speed

1.6 Classical regularity vs compatibility constraints. Smoothing property

[Ladyzhenskaya-Solonnikov-Uraltseva, Evans]

Theorem 7 Assume $u_0 \in L^2(\Omega)$, $f, a_{ij}, b_i, c \in C^\infty([0, +\infty) \times \overline{\Omega})$, then the solution of (3) satisfies $u \in C^\infty([\varepsilon, +\infty) \times \overline{\Omega})$ for any $\varepsilon > 0$.
Moreover, if $u_0 \in C^\infty(\overline{\Omega})$ and data (f, u_0) satisfies the (infinite) compatibility conditions

$$g_0 := u_0 \in H_0^1(\Omega), \quad g_1 := Lg_0 - f(0, \cdot) \in H_0^1(\Omega), \dots \quad g_{j+1} := (Lg_j) - \frac{\partial^j}{\partial t^j} f(0, \cdot) \in H_0^1(\Omega), \dots$$

then

$$u \in C^\infty([0, +\infty) \times \overline{\Omega})$$

1.7 L^p regularity of the heat problem

[Feireisl-Novotny]

If $f \in L^p(Q)$ with $1 < p < \infty$ and for simplicity $u_0 = 0$ (if not, fractional Sobolev spaces appear), then there exists a unique solution u with the L^p -regularity:

$$u, u_t, \nabla u, D^2 u \in L^p(Q)$$

2 Semilinear second order parabolic PDEs.

$$u_t - \Delta u = f(u), \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \tag{10}$$

”Informal principle”: If we can prove appropriate estimates for solutions of nonlinear PDEs, assuming such solutions exist, then in fact these solutions do exist.

2.1 Sublinear and Lipschutz system. Banach fixed-point theorem

[Evans]

Hypotheses:

1. $f : z \in \mathbb{R}^M \rightarrow f(z) \in \mathbb{R}^M$ global Lipschitz in $\mathbb{R}^M$
2. $|f(z)| \leq C(1 + |z|)$ (f sublinear in $\mathbb{R}^M$)

Then, there exists a unique global in time weak solution of the system (10).

Hint: Let $X = C([0, T]; L^2(\Omega)^M)$ and the mapping $R : \bar{u} \in X \rightarrow u \in X$ the solution of the heat system with RHS $f(\bar{u}) \in \mathbb{R}^M$, is contractive if $T \ll$ such that $C L T^{1/2} < 1$. After repeat in $[T, 2T]$, etc.

2.2 Sublinear case. Leray-Schauder fixed-point

Theorem 8 Assume $R : X \rightarrow X$ cont, compact and the set

$$\{u \in X : u = \alpha R u \text{ for some } \alpha \in [0, 1]\}$$

is bounded in X , The R has at least a fixed-point.

Hypothesis: $|f(z)| \leq C(1 + |z|)$ (f sublinear), then there exists at least a global in time weak solution of (10).

Hint: By testing by u , to bound possible weak solutions in weak space $W(0, T)$.

2.3 Blow-up vs global in time solution

[Evans] Case $f(u) = u^2$. By maximum principle, the possible solutions are nonnegative if u_0 is nonnegative.

Let (λ_1, ϕ_1) be first eigenvalue-eigenfunction of Laplace-Dirichlet problem.

One has $\phi_1 > 0$ in Ω and $\|\nabla v\|_{L^2}^2 \geq \lambda_1 \|v\|_{L^2}^2$ for all $v \in H_0^1(\Omega)$,

- Global in time weak estimates for small enough initial data:

If $\|u_0\|_{L^2}^2 \leq \lambda_1/2$ then one has the weak estimate $\|u(t)\|_{L^2}^2 \leq \lambda_1$ for any $t > 0$.

Hint: Assuming $\|u(t)\|_{L^2}^2 \leq \lambda_1$ for $t \in [0, t_*]$, then testing by u , one arrives at $\|u(t_*)\|_{L^2}^2 \leq \lambda_1/2$. Then one can extend to $[t_*, 2t_*]$, etc.

- Blow-up for large enough initial data u_0 . In fact, if $\int_{\Omega} u_0 \phi_1 > \lambda_1$, then there exists $T_* > 0$ such that $\lim_{t \rightarrow T_*^-} \|u(t, \cdot)\|_{L^1} = +\infty$

Hint: $\eta(t) = \int_{\Omega} u(t, x) \phi_1(x) dx$ satisfies $\eta'(t) \geq -\lambda_1 \eta(t) + \eta(t)^2$, hence, if $\eta(0) > \lambda_1$ then $\exp(\lambda_1 t) \eta(t) \rightarrow +\infty$ as $t \rightarrow T_*$.

2.4 Gradient flows

[Zheng]

Double-well potential $F(u) = (u^2 - 1)^2/4$.

$F(u)$ has two global minima $u = \pm 1$ and one local maximum $u = 0$.

In fact, $F(u)$ is coercive and weakly convex (it is bounded from below by a convex function).

Allen-Cahn structure (second-order semilinear PDE)

$$u_t + \frac{\delta E(u)}{\delta u} = 0$$

where

$$E(u) = \frac{1}{2} \int_{\Omega} (A \nabla u, \nabla u) + \int_{\Omega} F(u) - \langle f, u \rangle.$$

In fact, the previous variational derivative is

$$\frac{\delta E(u)}{\delta u} = -\nabla \cdot (A \nabla u) + F'(u) - f.$$

Properties:

- (Maximum principle) If $-1 \leq u_0 \leq 1$ in Ω , then $-1 \leq u \leq 1$ in Q .

- (Energy equality)

$$\frac{d}{dt} E(u(t)) + \int_{\Omega} (u_t)^2 dx = 0$$

- Asymptotic behaviour. For any initial data u_0 , the whole trajectory $u(t)$ converges as $t \rightarrow +\infty$ to u_* , being u_* a critical point of $E(u)$, that is

$$-\nabla \cdot (A \nabla u_*) + F'(u_*) - f = 0.$$

Cahn-Hilliard structure (fourth-order PDE)

$$u_t - \nabla \cdot \left(M(u) \nabla \left(\frac{\delta E(u)}{\delta u} \right) \right) = 0$$

where $M(u)$ is the mobility.

Properties:

- (Conservation). Assuming isolated BC $\nabla \frac{\delta E(u)}{\delta u} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then

$$\frac{d}{dt} \int_{\Omega} u(t) = 0.$$

- (Energy equality)

$$\frac{d}{dt} E(u(t)) + \int_{\Omega} M(u(t)) \left| \nabla \left(\frac{\delta E(u)}{\delta u} \right) \right|^2 dx = 0$$

- (Maximum principle) Assuming degenerate mobility $M(-1) = 0 = M(1)$, then if $-1 \leq u_0 \leq 1$ in Ω , one has $-1 \leq u \leq 1$ in Q .

RK: Changing the free-energy, depending on higher derivatives, for instance

$$E(u) = \frac{1}{2} \int_{\Omega} (A \Delta u, \Delta u) + \int_{\Omega} F(\nabla u) - \langle f, u \rangle$$

then Allen-Cahn is fourth-order and Cahn-Hilliard is sixth-order.

3 Some chemotaxis models

The drift-diffusion equation models how particles (such as electrons, holes, ions, or chemical molecules) move through a medium due to two combined physical processes: drift (directed movement pushed by an external force or electric field) and diffusion (random thermal movement from high to low concentration)

The chemotaxis is a drift process where the movement is pushed by the gradient of a chemical signal, which can be either attractive (from high to low chemical concentration) or repulsive.

3.1 A generic model

[Bellomo-Bellouquid-Tao-Winkler]

$$\begin{cases} u_t - \Delta u \pm \nabla \cdot (u \nabla v) = f(t, x, u, v), & \text{in } Q, \\ v_t - \Delta v - v = g(t, x, u, v), & \text{in } Q, \\ (\nabla u) \cdot \mathbf{n}|_{\partial\Omega} = 0 = (\nabla v) \cdot \mathbf{n}|_{\partial\Omega}, & \text{in } (0, T) \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, & \text{in } \Omega. \end{cases} \quad (11)$$

Nonnegative hypotheses:

$$f(t, x, 0, v) \geq 0 \text{ for any } v \geq 0, \text{ and } g(t, x, u, 0) \geq 0 \text{ for any } u \geq 0.$$

Global classical solutions in time with value in Sobolev spaces vs Blow-up.

Theorem 9 (Local existence. Extensibility criterium) *Let $q > d$. For any nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,p}(\Omega)$, There exists $T_{max} \in (0, \infty]$ and a unique classical solution (u, v) of (11) in $(0, T_{max})$:*

$$u, v \in C^0([0, T_{max}) \times \overline{\Omega}) \cap C^{1,2}((0, T_{max}) \times \overline{\Omega})$$

Moreover, if $T_{max} < \infty$ then $\lim_{t \rightarrow T_{max}^-} (\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,q}(\Omega)}) = +\infty$

3.2 Keller-Segel model

$$\begin{cases} u_t - \Delta u + \nabla \cdot (u \nabla v) = 0, & \text{in } Q, \\ v_t - \Delta v - v = u, & \text{in } Q, \\ (\nabla u) \cdot \mathbf{n}|_{\partial\Omega} = 0 = (\nabla v) \cdot \mathbf{n}|_{\partial\Omega}, & \text{in } (0, T) \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, & \text{in } \Omega. \end{cases} \quad (12)$$

If $d = 1$, there is not blow-up.

If $d = 2$ or $d = 3$, then for large enough initial data, there is blow-up at finite time.

3.3 Keller-Segel logistic model

$$\begin{cases} u_t - \Delta u + \chi \nabla \cdot (u \nabla v) = r u - \mu u^2, & \text{in } Q, \\ v_t - \Delta v - v = \alpha u, & \text{in } Q, \\ (\nabla u) \cdot \mathbf{n}|_{\partial\Omega} = 0 = (\nabla v) \cdot \mathbf{n}|_{\partial\Omega}, & \text{in } (0, T) \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, & \text{in } \Omega. \end{cases} \quad (13)$$

If $d = 1$ or $d = 2$, there exists a unique strong solution

If $d = 3$:

[Lankeit] Global weak solution.

[Winkler] Strong solution whenever $\mu \gg \chi, \alpha$.

References

- [Allaire] Allaire G. *Numerical analysis and optimization: an introduction to mathematical modelling and numerical simulation*. Oxford University Press, (2007).
- [Bellomo-Bellouquid-Tao-Winkler] Bellomo N., Bellouquid A., Tao Y., Winkler M. *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues*. Mathematical Models and Methods in Applied Sciences Vol. 25, No. 9 (2015) 1663-1763
- [Evans] Evans L.C. *Partial Differential Equations*. American Mathematical Society, second edition 2010.
- [Feireisl-Novotny] Feireisl E., Novotny A. *Singular Limits in Thermodynamics of Viscous Fluids. Advances in Mathematical Fluid Mechanics*. Birkhäuser Verlag, Basel (2009).
- [Ladyzhenskaya-Solonnikov-Uraltseva] Ladyzhenskaya O.A., Solonnikov V.A., Uraltseva N.N. *Linear and Quasilinear Equations of Parabolic Type*. American Mathematical Society, 1968.
- [Lankeit] Lankeit J. *Long-term behaviour in a chemotaxis-fluid system with logistic source*. Math. Models Methods Appl. Sci. 26, 2071-2109 (2016)
- [Winkler] Winkler M. *Global large-data solutions in a chemotaxis-Navier-Stokes system modeling cellular swimming in fluid drops*. Commun. Partial Differ. Equ. 37, 319-351 (2012).
- [Zheng] Zheng S. *Nonlinear Evolution Equations*. Chapman & Hall/CRC 2004.