Controllability of discrete parabolic PDEs

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Doc Course - PDEs: Theoretical and Numerical Analysis and Control
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- M. Boulakia
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- K. Le Balc'h

Outline

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1. Introduction

2. Some facts about the Hilbert Uniqueness Method and its penalized version

3. The HUM approach in the discrete framework
3.1 The semi-discrete in space setting
3.2 Practical considerations

4. Discrete Carleman inequalities and applications
4.1 Introduction to the Lebeau-Robbiano approach
4.2 Discrete Lebeau-Robbiano inequality. Discrete elliptic Carleman estimate
4.3 Discrete parabolic Carleman estimate
4.4 The time discrete setting
4.5 The fully discrete setting

5. Numerical illustrations
5.1 1D scalar problems
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6. Discrete spectral analysis and applications
6.1 Introduction to the moments method
6.2 Moments method for 1D semi-discrete scalar problem
6.3 Boundary control for 1D semi-discrete cascade system
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6.5 Multidimensional boundary control systems

7. Time discrete boundary control problems
7.1 Boundary control for 1D time-discrete scalar problems
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Controllability of a dynamical system

General dynamical system

$$(S)\begin{cases} y'(t) = F(t, y, v), \\ y(0) = y_0, \end{cases}$$

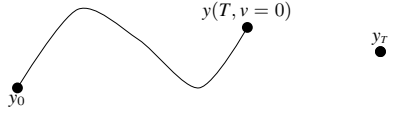
$t \mapsto y(t)$ is the state (possibly infinite dimensional) or the output.
$t \mapsto v(t)$ is the control = the input = the command = the user action on the system.

Goal : **Determine v to impose a prescribed behavior to the state y**

Many examples

- Driving a car, a plane, a satellite
- Optimizing chemical reactions
- Population dynamics, Biology
- Economics

Controllability questions : For a given $T > 0$, and y_0



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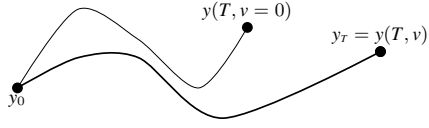
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Controllability questions : For a given $T > 0$, and y_0

Exact controllability

Let y_T be a given target.

Can I find a control v such that $y(T, v) = y_T$?



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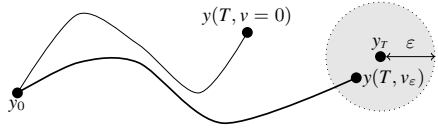
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Approximate controllability

Let y_T be a given target.

Can I find a control v such that $y(T, v)$ is close from y_T ?



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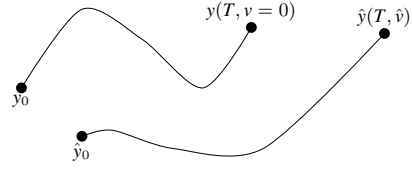
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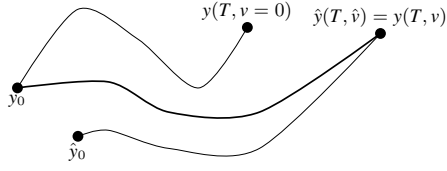
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Controllability questions : For a given $T > 0$, and y_0

Cont. to trajectories

Given an another trajectory $(\hat{y}, \hat{v})$.

Can I find a control v such that $y(T, v) = \hat{y}(T, \hat{v})$?



Linear systems: **null-controllability**

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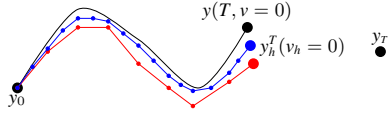
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Controllability questions : For a given $T > 0$, and y_0

Discrete exact controls

Given a discretisation (S_h) of (S), can I find a **bounded family** of discrete controls $(v_h)_h$ such that

$$y_h^T(v_h) \xrightarrow[h\to 0]{} y_T \quad ?$$



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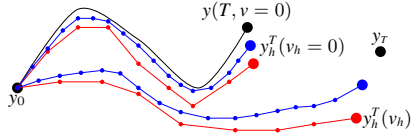
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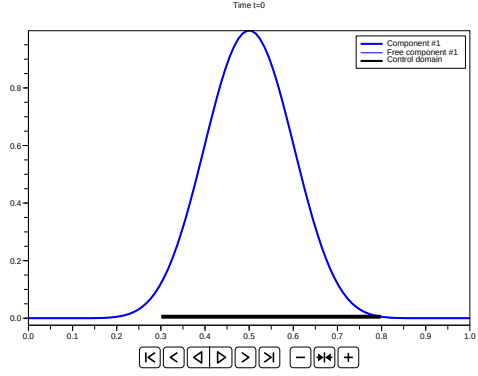
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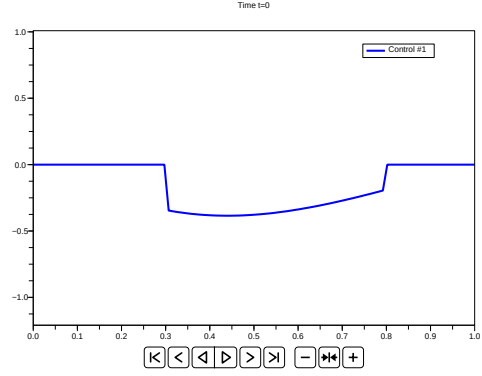
$$y_h^T(v_h) \xrightarrow[h\to 0]{} y_T \quad ?$$



Let's start with an example

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$





Abstract (linear time invariant) parabolic control problem

- Two Hilbert spaces : the state space $(E, \langle .,. \rangle)$ and the control space $(U, [.,.])$.
- $\mathcal{A} : D(\mathcal{A}) \subset E \mapsto E$ is some *elliptic* operator such that $-\mathcal{A}$ generates an analytic semigroup in E.
- $\mathcal{B} : U \mapsto D(\mathcal{A}^\star)'$ the control (bounded) operator, $\mathcal{B}^\star$ its adjoint.
- COMPATIBILITY/REGULARITY ASSUMPTION : we assume that

$$\left(t \mapsto \mathcal{B}^\star e^{-t\mathcal{A}^\star}\psi\right) \in L^2(0,T;U), \text{ and } \left[\!\left[\mathcal{B}^\star e^{-\cdot\mathcal{A}^\star}\psi\right]\!\right]_{L^2(0,T;U)} \leq C\,\|\psi\|\,, \ \ \forall \psi \in D(\mathcal{A}^\star).$$

Our controlled parabolic problem is

$$(S) \quad \begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[, \\ y(0) = y_0, \end{cases}$$

Here, $y_0 \in E$ is the initial data, $v \in L^2(]0,T[, U)$ is the control we are looking for.

THEOREM (WELL-POSEDNESS OF (S) IN A DUAL SENSE)

For any $y_0 \in E$ and $v \in L^2(0,T;U)$, there exists a unique $y = y_{v,y_0} \in C^0([0,T],E)$ such that

$$\langle y(t), \psi\rangle - \left\langle y_0, e^{-t\mathcal{A}^\star}\psi\right\rangle = \int_0^t \left[v(s), \mathcal{B}^\star e^{-(t-s)\mathcal{A}^\star}\psi\right] ds, \ \ \forall t \in [0,T], \forall \psi \in E.$$

NOTATION : $\mathcal{L}_T(v|y_0) \stackrel{\text{def}}{=} y_{v,y_0}(T)$.

Controllability questions

$$(S)\begin{cases}\partial_t y + \mathcal{A}y = \mathcal{B}v & \text{in }]0,T[,\\ y(0) = y_0.\end{cases}$$

For a given (fixed) control time $T > 0$ and any $\delta \geq 0$, we set

$$\mathrm{Adm}(y_0,\delta) \stackrel{\text{def}}{=} \left\{ v \in L^2(0,T;U),\ \text{s.t. } \left\| \mathcal{L}_T\left(v|y_0\right) \right\| \leq \delta \right\}.$$

Approximate control problem from the initial data y_0

Do we have

$$\mathrm{Adm}(y_0,\delta) \neq \emptyset, \quad \forall \delta > 0\ ?$$

Null-control problem from the initial data y_0

Do we have

$$\mathrm{Adm}(y_0,0) \neq \emptyset\ ?$$

Remarks concerning linear parabolic PDEs

- Exact controllability is hopeless in general
- Null-controllability is equivalent to controllability to the trajectories

Main academic examples

DISTRIBUTED CONTROL FOR SCALAR EQUATION $\mathcal{B}^\star = \mathbf{1}_\omega$

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_\omega v, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

with $\omega \subset \Omega$ strict subset (the case $\omega = \Omega$ is straightforward).

DIRICHLET BOUNDARY CONTROL FOR SCALAR EQUATION $\mathcal{B}^\star = \mathbf{1}_{\Gamma_0}\partial_n$

$$\begin{cases} \partial_t y - \Delta y = 0, & \text{in } \Omega \\ y = \mathbf{1}_{\Gamma_0} v, & \text{on } \partial\Omega. \end{cases}$$

where $\Gamma_0 \subset \partial\Omega$ is a subset of the boundary (the case $\Gamma_0 = \partial\Omega$ is straightforward).

(Fattorini-Russel, '71) (Lebeau-Robbiano, '95), (Fursikov-Imanuvilov, '96) (Alessandrini-Escauriaza, '08)

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COUPLED PARABOLIC SYSTEM WITH FEW DISTRIBUTED CONTROLS $\mathcal{B}^\star = \mathbf{1}_\omega B^*$

$y(t,x) \in \mathbb{R}^n$, $C(t,x) \in M_n(\mathbb{R})$, $B \in M_{n,m}(\mathbb{R})$, with $m < n$

$$\begin{cases} \partial_t y - \Delta y + C(t,x)y = \mathbf{1}_\omega B v, & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

COUPLED PARABOLIC SYSTEM WITH FEW BOUNDARY CONTROLS $\mathcal{B}^\star = \mathbf{1}_{\Gamma_0} B^* \partial_n$

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(Fernández-Cara, González-Burgos, de Teresa, '10) (Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, '11)
(B., Morancey, '25)

Numerical analysis questions

Case 1: we know that (S) is null- or approximate controllable

- Is the same property inherited by a given semi-discretisation (S_h) and/or by a full-discretisation $(S_{h,\delta t})$?
  - Do the approximate controls are uniformly bounded w.r.t. h, δt ?
  - If not, can we relax the controllability requirements at the discrete level to ensure a good approximation ?
- Is it possible to distinguish approximate and null-controllability at the discrete level ?
- What algorithm can we use to compute a suitably bounded family of controls with a reasonnable amount of computational work ?
- How do we relate algorithm parameters to discretisation parameters ?

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Case 2 : we don't know whether or not (S) is controllable

- Can we use numerical computations to get some intuition ?
- Can we use numerical computations to determine non-controllable modes in the system ?

Remarks

- From a numerical point of view : we often analyze the system **for a given initial data**.
- In most of the available results in the literature, controllability issues are stated and analyzed **for all possible initial data**.

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The Hilbert Uniqueness Method (HUM)

(Lions, '88) (Glowinski–Lions, '90s)

Ideas

- Formulate controllability problems as constrained optimisation problems.
- Write the associated **unconstrained** dual optimisation problem.
- Find conditions for the solvability of the dual problem and prove that there are satisfied.

Cost of the control

$$F(v) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^T [\![v(t)]\!]^2 \, dt, \quad \forall v \in L^2(0,T;U),$$

and for any $\delta \geq 0$, we define (it it exists !), v^δ to be the unique minimiser

$$F(v^\delta) = \inf_{v \in \text{Adm}(y_0,\delta)} F(v). \qquad (P^\delta)$$

Dual problems

- The dual pb of (P^0) is not coercive in the natural space E. We need to introduce a **big** abstract space obtained as the completion of E with respect to a suitable norm.
- The dual pb of (P^δ), $\delta > 0$ is coercive in E but is not smooth.

The penalized HUM

Primal problem

$$F_\varepsilon(v) \overset{\text{def}}{=} \frac{1}{2}\int_0^T [\![v(t)]\!]^2\, dt + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T\left(v|y_0\right)\right\|^2, \quad \forall v \in L^2(0,T;U),$$

we consider the following problem : to find $v_\varepsilon \in L^2(0,T;U)$ such that

$$F_\varepsilon(v_\varepsilon) = \inf_{v\in L^2(0,T;U)} F_\varepsilon(v). \qquad (P_\varepsilon)$$

Proposition

For any $\varepsilon > 0$, the functional F_ε is strictly convex, continuous and coercive.

Therefore, it admits a unique minimiser $v_\varepsilon \in L^2(0,T;U)$.

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Dual problem (via Fenchel-Rockafellar duality theorem)

(Ekeland-Temam, '74)

$$J_\varepsilon(q^F) \overset{\text{def}}{=} \frac{1}{2}\int_0^T \left[\!\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F\right]\!\!\right]^2 dt + \frac{\varepsilon}{2}\left\|q^F\right\|^2 + \left\langle y_0, e^{-T\mathcal{A}^\star} q^F\right\rangle, \quad \forall q^F \in E.$$

Proposition

For any $\varepsilon > 0$, the functional J_ε is strictly convex, continuous and coercive.

Therefore, it admits a unique minimiser $q^F_\varepsilon \in E$.

The penalized HUM

Remark

We do not require any particular assumption on the operators $\mathcal{A}$ and $\mathcal{B}$.

In particular **we do not assume** that the PDE (S) is (or is not) controllable.

Proposition (Duality properties precised)

For any $\varepsilon > 0$, the minimisers v_ε and q_ε^F of the functionals F_ε and J_ε respectively, are related through the formulas

$$v_\varepsilon(t) = \mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q_\varepsilon^F, \text{ for a.e. } t \in]0,T[,$$

and

$$\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) = y_{v_\varepsilon, y_0}(T) = -\varepsilon q_\varepsilon^F.$$

As a consequence, we have

$$\inf_{L^2(0,T;U)} F_\varepsilon = F_\varepsilon(v_\varepsilon) = -J_\varepsilon(q_\varepsilon^F) = -\inf_E J_\varepsilon.$$

Reading controllability properties on the penalised HUM

(B., '13)

Theorem

1. *Problem (S) is approximately controllable from the initial data y_0 if and only if*

$$\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) = y_{v_\varepsilon, y_0}(T) \xrightarrow[\varepsilon \to 0]{} 0.$$

2. *Problem (S) is null-controllable from the initial data y_0 if and only if*

$$M_{y_0}^2 \overset{\text{def}}{=} 2 \sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} F_\varepsilon \right) = 2 \sup_{\varepsilon > 0} F_\varepsilon(v_\varepsilon) < +\infty.$$

In the null-controllable case

$$[\![v_\varepsilon]\!]_{L^2(0,T;U)} \leq M_{y_0}, \quad \text{and} \quad \left\| \mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) \right\| \leq M_{y_0} \sqrt{\varepsilon}.$$

Moreover we have $[\![v^0]\!]_{L^2(0,T;U)} = M_{y_0}$ and

$$v_\varepsilon \xrightarrow[\varepsilon \to 0]{} v^0, \quad \text{strongly in } L^2(0,T;U), \quad \text{and} \quad \frac{\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right)}{\sqrt{\varepsilon}} \xrightarrow[\varepsilon \to 0]{} 0,$$

where v^0 is the unique HUM null-control (that is the one of minimal L^2-norm).

Non observable adjoint states and HUM

Non observable adjoint states : $Q^F \stackrel{\text{def}}{=} \left\{ q^F \in E, \text{ s.t. } \mathcal{B}^\star e^{-t\mathcal{A}^\star} q^F = 0, \forall t \geq 0 \right\}.$

Theorem (Convergence of the penalised HUM final state)

For any $y_0 \in E$, the penalised-HUM sequence of controls $(v_\varepsilon)_\varepsilon$ satisfies

$$\mathcal{L}_T\left(v_\varepsilon \middle| y_0\right) \xrightarrow[\varepsilon \to 0]{} \mathbb{P}_{Q^F}\left(e^{-T\mathcal{A}} y_0\right).$$

Proposition (Selfadjoint case)

Assume that $\mathcal{A}$ is selfadjoint, and set $Y_0 \stackrel{\text{def}}{=} e^{-T\mathcal{A}^\star} Q^F$ then

$$\mathbb{P}_{Q^F}\left(e^{-T\mathcal{A}} y_0\right) = e^{-T\mathcal{A}}\left(\mathbb{P}_{\overline{Y_T}} y_0\right).$$

Therefore, the system is approximately controllable from y_0 if and only if $y_0 \perp Y_0$.

- The set of (approximately) controllable initial data is $Y_0^\perp$.
- For any $y_0 \in Y_0$ we have

$$v_\varepsilon = 0, \quad \forall \varepsilon > 0,$$
$$\mathrm{Adm}(y_0, \delta) \neq \emptyset \quad \Leftrightarrow \quad \delta \geq \left\| e^{-T\mathcal{A}} y_0 \right\|.$$

Non observable adjoint states and HUM

Non observable adjoint states : $\mathcal{Q}^F \stackrel{\text{def}}{=} \left\{ q^F \in E, \text{ s.t. } \mathcal{B}^\star e^{-t\mathcal{A}^\star} q^F = 0, \forall t \geq 0 \right\}.$

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For any $y_0 \in E$, the penalised-HUM sequence of controls $(v_\varepsilon)_\varepsilon$ satisfies

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Corollary (App. controllability and Unique continuation)

The system (S) is approximately controllable from the initial data y_0 if and only if

$$\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F = 0, \quad \forall t \in [0,T]\right] \Longrightarrow \left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle = 0. \tag{UC}$$

Proposition (App. controllability and Weak observability)

The property (UC) *is equivalent to the following* **weak observability inequality**

$$\left| \left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle \right|^2 \leq C^2_{\varepsilon, y_0} \int_0^T \left[\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\right]^2 dt + \varepsilon \left\| q^F \right\|^2, \quad \forall q^F \in E, \ \forall \varepsilon > 0.$$

Reading controllability properties on the penalised HUM

Theorem (Null-controllability and Observability)

Problem (S) is null-controllable from y_0 if and only if, there exists $\widetilde{M}_{y_0} \geq 0$ such that

$$\left|\left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle\right|^2 \leq \widetilde{M}_{y_0}^2 \int_0^T \left[\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\right]^2 dt, \quad \forall q^F \in E.$$

Moreover, the best constant $\widetilde{M}_{y_0}$ is equal to the cost of the HUM control $\left[\!\left[v^0 \right]\!\right]_{L^2(0,T;U)}$.

Theorem

Problem (S) is null-controllable from any initial data if and only if, there exists $C_{obs} > 0$ such that

$$\left\| e^{-T\mathcal{A}^\star} q^F \right\|^2 \leq C_{obs}^2 \int_0^T \left[\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\right]^2 dt, \quad \forall q^F \in E.$$

In that case, for any y_0, the associated HUM control v^0 satisfies

$$\left[\!\left[v^0 \right]\!\right]_{L^2(0,T;U)} \leq C_{obs} \left\| y_0 \right\|.$$

One consequence : In finite dimension, null-controllability is equivalent to approximate controllability.

The penalized HUM approach ...

... with an approximate initial data

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $Y_0 = (y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}} y_{0,\varepsilon} \xrightarrow[\varepsilon \to 0]{} e^{-T\mathcal{A}} y_0.$$

Associated HUM functionals

$$\tilde{F}_\varepsilon(v) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T [\![v(t)]\!]^2 \, dt + \frac{1}{2\varepsilon} \left\| \mathcal{L}_T\left(v \middle| y_{0,\varepsilon}\right) \right\|^2, \quad \forall v \in L^2(0,T;U),$$

$$\tilde{J}_\varepsilon(q^F) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T \left[\!\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\!\right]^2 dt + \frac{\varepsilon}{2} \left\| q^F \right\|^2 + \left\langle y_{0,\varepsilon}, e^{-T\mathcal{A}^\star} q^F \right\rangle, \quad \forall q^F \in E.$$

We denote by $\tilde{v}_\varepsilon$ the unique minimiser of $\tilde{F}_\varepsilon$.

Controllability conditions

(S) is app. cont. from $y_0 \iff \mathcal{L}_T\left(\tilde{v}_\varepsilon \middle| y_{0,\varepsilon}\right) \xrightarrow[\varepsilon \to 0]{} 0.$

$$\sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon \right) < +\infty \Longrightarrow \text{(S) is null-controllable from } y_0.$$

$$\left.\begin{array}{c} \text{(S) is null-controllable from } y_0 \\ \sup\limits_{\varepsilon > 0} \dfrac{1}{\varepsilon} \left\| e^{-T\mathcal{A}} (y_0 - y_{0,\varepsilon}) \right\|^2 < +\infty \end{array}\right\} \Longrightarrow \sup_{\varepsilon > 0} \left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon \right) < +\infty.$$

The penalized HUM approach ...

... with an approximate initial data

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $Y_0 = (y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-T\mathcal{A}} y_{0,\varepsilon} \xrightarrow[\varepsilon \to 0]{} e^{-T\mathcal{A}} y_0.$$

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Controllability conditions

$$\left.\begin{array}{c} \text{(S) is null-controllable from } y_0 \\ \sup\limits_{\varepsilon>0} \dfrac{1}{\varepsilon} \left\| e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon}) \right\|^2 < +\infty \end{array}\right\} \Longrightarrow \sup_{\varepsilon>0} \left(\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon \right) < +\infty.$$

Discussion : Assume $\mathcal{A} = \mathcal{A}^\star$ and $Q^F \neq \{0\}$, then take $y_{0,\varepsilon} = \varepsilon^\alpha z$, $z \in e^{-T\mathcal{A}^\star} Q^F$

$$\inf_{L^2(0,T;U)} \tilde{F}_\varepsilon = \frac{\varepsilon^{2\alpha-1}}{2} \left\| e^{-T\mathcal{A}} z \right\|^2 \xrightarrow[\varepsilon \to 0]{} +\infty, \quad \text{as soon as } \alpha < 1/2.$$

$y_{0,\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0$ $\Longleftarrow$ this initial data is indeed null-controllable !!.

The penalized HUM approach ...

... with an approximate initial data

For each $\varepsilon > 0$, let $y_{0,\varepsilon} \in E$ such that $Y_0 = (y_{0,\varepsilon})_\varepsilon$ is bounded in E and

$$e^{-TA}y_{0,\varepsilon} \xrightarrow[\varepsilon\to 0]{} e^{-TA}y_0.$$

Associated HUM functionals

$$\tilde{F}_\varepsilon(v) \overset{\text{def}}{=} \frac{1}{2}\int_0^T [\![v(t)]\!]^2\,dt + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T\left(v|y_{0,\varepsilon}\right)\right\|^2, \quad \forall v \in L^2(0,T;U),$$

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We denote by $\tilde{v}_\varepsilon$ the unique minimiser of $\tilde{F}_\varepsilon$.

Proposition (Relaxed observability inequality)

Let $Y_0 = (y_{0,\varepsilon})_\varepsilon$ and assume that

$$\sup_{\varepsilon>0}\frac{1}{\varepsilon}\left\|e^{-T\mathcal{A}}(y_0 - y_{0,\varepsilon})\right\|^2 < +\infty.$$

The system (S) is null-controllable from the initial data y_0 if and only if

$$\left|\left\langle y_{0,\varepsilon}, e^{-T\mathcal{A}^\star}q^F\right\rangle\right|^2 \le M_{Y_0}^2\left(\int_0^T \left[\!\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star}q^F\right]\!\!\right]^2 dt + \varepsilon\left\|q^F\right\|^2\right), \quad \forall q^F \in E.$$

We do not require the system to be null-controllable from any of the $(y_{0,\varepsilon})_\varepsilon$.

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Semi-discrete (uniform) control problems

Framework

For any $h > 0$, we are given

- A discrete state space $(E_h, \langle \cdot, \cdot \rangle_h)$.
- An approximate operator $\mathcal{A}_h$ on E_h.
- A discrete control space $(U_h, [\cdot, \cdot]_h)$.
- A linear operator $\mathcal{B}_h : U_h \to E_h$, $\mathcal{B}_h^\star$ being its adjoint $\qquad \langle \mathcal{B}_h u, x \rangle_h = [\mathcal{B}_h^\star x, u]_h$.

The semi-discrete control problem is (S_h) $\begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \\ y_h(0) = y_{0,h}. \end{cases}$

Its solution is refered to as $t \mapsto y_{v_h, y_{0,h}}(t) \in E_h$ and we set

$$\mathcal{L}_T^h\left(v_h \middle| y_{0,h}\right) \stackrel{\text{def}}{=} y_{v_h, y_{0,h}}(T).$$

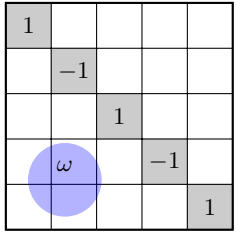
Questions

Assume that $(y_{0,h})_h$ are, in some sense, approximations of a $y_0 \in E$.

1. Can we relate the controllability properties of (S) starting from y_0 to the ones of (S_h) starting from $y_{0,h}$?
2. Can we obtain uniform bounds (w.r.t. h) for the associated controls v_h ?

Main issues related to discretisation

1. It may happen that (S_h) is not controllable even if (S) is.
   EXAMPLE : the 2D 5-point discrete Laplace operator $\mathcal{A}_h$. **(Kavian, Zuazua '06)**



There exists a non trivial $\psi_h \in E_h$ such that

$$\mathcal{A}_h^\star \psi_h = \mu_h \psi_h, \text{ and } \mathcal{B}_h^\star \psi_h = 0.$$

$$\Longrightarrow \psi_h \in Q_h^F \stackrel{\text{def}}{=} \{q_h^F \in E_h, \text{ s.t. } \mathcal{B}_h^\star e^{-t\mathcal{A}_h^\star} q_h^F = 0, \ \forall t \geq 0\},$$

For any control $v_h \in L^2(0,T;U_h)$, $\dfrac{d}{dt}\langle y_h(t), \psi_h\rangle_h + \mu_h \langle y_h(t), \psi_h\rangle = 0,$

and thus

$$\left\langle \mathcal{L}_T^h\left(v_h \middle| y_{0,h}\right), \psi_h \right\rangle_h = \langle y_h(T), \psi_h\rangle_h = e^{-\mu_h T} \langle y_{0,h}, \psi_h\rangle_h .$$

REMARK : The eigenvalue μ_h is very large $\sim \frac{C}{h^2}$ thus $\left\langle \mathcal{L}_T^h\left(v_h \middle| y_{0,h}\right), \psi_h \right\rangle_h$ is exponentially small.

2. Even if (S) and (S_h) are both controllable, it is not necessarily desirable to compute a null-control v_h of (S_h) to obtain a suitable approximation of a null-control of (S).

Penalised HUM applied to semi-discrete problems

$$F_{\varepsilon,h}(v_h) \overset{\text{def}}{=} \frac{1}{2}\int_0^T [\![v_h(t)]\!]_h^2\, dt + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T^h\left(v_h|y_{0,h}\right)\right\|_h^2, \quad \forall v_h \in L^2(0,T;U_h),$$

$$J_{\varepsilon,h}(q_h^F) \overset{\text{def}}{=} \frac{1}{2}\int_0^T \left[\!\!\left[\mathcal{B}_h^\star e^{-(T-t)\mathcal{A}_h^\star} q_h^F\right]\!\!\right]_h^2 dt + \frac{\varepsilon}{2}\left\|q_h^F\right\|_h^2 + \left\langle y_{0,h}, e^{-T\mathcal{A}_h^\star} q_h^F\right\rangle_h, \quad \forall q_h^F \in E_h.$$

- For each value of $h > 0$, all the previous results apply.
- We denote by $v_{\varepsilon,h}$ the unique minimiser of $F_{\varepsilon,h}$.

Goal

One would like to let $(\varepsilon, h) \to (0,0)$ but this should be done with some care.

Comments

1. Even if (S) is controllable from y_0, in the cases where $Q_h^F \neq \{0\}$ we may have
$$\lim_{\varepsilon\to 0}\left\|\mathcal{L}_T^h\left(v_{\varepsilon,h}|y_{0,h}\right)\right\|_h \neq 0, \quad \forall h > 0$$
2. One can prove that for any $h > 0$
$$\sup_{\varepsilon>0} [\![v_{\varepsilon,h}]\!]_{L^2(0,T;U_h)} < +\infty.$$

$\Longrightarrow$ ε and h need to be somehow related.

Relaxed null-controllability / observability

Main idea : choose $\varepsilon = \varphi(h)$

Let $h \in]0, +\infty[\mapsto \varphi(h) \in (0, +\infty)$ be a given function such that $\lim_{h\to 0} \varphi(h) = 0$.

DEFINITION

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\varphi(h)$-null controllable from Y_0, if there exists a $h_0 > 0$ such that

$$M^2_{Y_0} \stackrel{\text{def}}{=} 2 \sup_{0<h<h_0} \left(\inf_{L^2(0,T;U_h)} F_{\varphi(h),h} \right) < +\infty.$$

Here, $F_{\varphi(h),h}$ is built upon $y_{0,h}$:

$$F_{\varphi(h),h}(v_h) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^T [\![v_h(t)]\!]_h^2 \, dt + \frac{1}{2\varphi(h)} \left\| \mathcal{L}^h_T \left(v_h \middle| y_{0,h} \right) \right\|_h^2, \quad \forall v_h \in L^2(0, T; U_h),$$

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Definition

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\varphi(h)$-null controllable from Y_0, if there exists a $h_0 > 0$ such that

$$M_{Y_0}^2 \overset{\text{def}}{=} 2 \sup_{0<h<h_0} \left(\inf_{L^2(0,T;U_h)} F_{\varphi(h),h} \right) < +\infty.$$

Theorem (Relaxed observability)

For a given $Y_0 \in \prod_{h>0} E_h$, the problems (S_h) are $\varphi(h)$-null-controllable from Y_0 if and only if there exists $h_0 > 0$ and $\widetilde{M}_{Y_0} > 0$, such that, for any $0 < h < h_0$

$$\left| \left\langle y_{0,h}, e^{-T\mathcal{A}_h^\star} q_h^F \right\rangle_h \right|^2 \leq \widetilde{M}_{Y_0}^2 \left(\int_0^T \left[\!\left[\mathcal{B}_h^\star e^{-(T-t)\mathcal{A}_h^\star} q_h^F \right]\!\right]_h^2 dt + \varphi(h) \left\| q_h^F \right\|_h^2 \right), \quad \forall q_h^F \in E_h. \tag{$\star$}$$

In such case, the best constant $\widetilde{M}_{Y_0}$ is equal to M_{Y_0} and we can find controls $v_{\varphi(h),h}$ s.t.

$$\left[\!\left[v_{\varphi(h),h} \right]\!\right]_{L^2(0,T;U_h)} \leq M_{Y_0}, \text{ and } \left\| \mathcal{L}_T^h \left(v_{\varphi(h),h} \middle| y_{0,h} \right) \right\|_h \leq M_{Y_0} \sqrt{\varphi(h)}, \ \forall 0 < h < h_0. \tag{$\star\star$}$$

In fact $(\star)$ and $(\star\star)$ are "equivalent".

Relaxed null-controllability / observability

Main idea : choose $\varepsilon = \varphi(h)$

Let $h \in]0, +\infty[\mapsto \varphi(h) \in (0, +\infty)$ be a given function such that $\lim_{h\to 0} \varphi(h) = 0$.

Definition

For a given family of initial data $Y_0 = (y_{0,h})_h \in \prod_{h>0} E_h$, we say that the family of problems (S_h) is $\varphi(h)$-null controllable from Y_0, if there exists a $h_0 > 0$ such that

$$M_{Y_0}^2 \overset{\text{def}}{=} 2 \sup_{0<h<h_0} \left(\inf_{L^2(0,T;U_h)} F_{\varphi(h),h} \right) < +\infty.$$

Proposition

Assume that, for some $C_{\text{obs}} > 0$, the following relaxed observability inequality holds

$$\left\| e^{-T\mathcal{A}_h^\star} q_h^F \right\|_h^2 \leq C_{\text{obs}}^2 \left(\int_0^T \left[\!\left[\mathcal{B}_h^\star e^{-(T-t)\mathcal{A}_h^\star} q_h^F \right]\!\right]_h^2 dt + \varphi(h) \left\| q_h^F \right\|_h^2 \right), \quad \left| \begin{array}{r} \forall q_h^F \in E_h, \\ \forall 0 < h < h_0 \end{array} \right.$$

then for any **bounded** *family Y_0, the problems (S_h) are $\varphi(h)$-null-controllable from Y_0 and we have*

$$M_{Y_0} \leq C_{\text{obs}} \left(\sup_{0<h<h_0} \left\| y_{0,h} \right\|_h \right).$$

Here also $\varphi(h)$-relaxed observability and $\varphi(h)$-null-controllability are in fact equivalent.

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Solving the control problems in practice

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ (with $\varepsilon = \varphi(h)$).

PROPOSITION (GRADIENT OF $J_{\varepsilon,h}$ - GRAMIAM OPERATOR)

For any $h > 0, \varepsilon > 0$ and any $q_h^F \in E_h$, we have

$$\nabla J_{\varepsilon,h}(q_h^F) = \underbrace{\mathcal{L}_T^h\Big(\mathcal{B}_h^\star e^{-(T-\cdot)\mathcal{A}_h^\star} q_h^F \big| 0\Big)}_{\stackrel{\text{def}}{=}\Lambda^h q_h^F} + \varepsilon q_h^F + \underbrace{\mathcal{L}_T^h\big(0\big|y_{0,h}\big)}_{=\textit{Free sol. at time } T},$$

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Computation of Gramian operators

Given q_h^F, the computation of $\Lambda^h q_h^F$ amounts to

$$\boxed{t=0} \quad q_h(0) \qquad \xleftarrow{\text{Solve } -\partial_t q_h + \mathcal{A}_h^\star q_h = 0} \qquad \boxed{t=T} \quad q_h(T) = q_h^F$$

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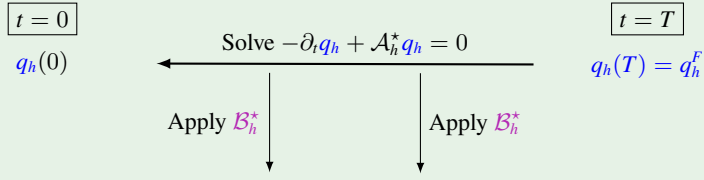
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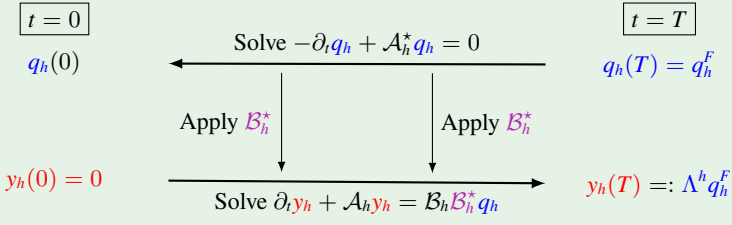
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EQUATIONS TO SOLVE

The semi/fully-discrete controls are computed by solving the equations

$$(\Lambda^h + \varepsilon \mathrm{Id}) q_h^F = -\mathcal{L}_T^h\big(0\big|y_{0,h}\big) ,$$

In practice, we can use a conjugate gradient algorithm (Λ^h is a positive symmetric operator).

Solving the control problems in practice

GENERAL PRINCIPLE : Minimise dual functionals $J_{\varepsilon,h}$ (with $\varepsilon = \varphi(h)$).

PROPOSITION (GRADIENT OF $J_{\varepsilon,h}$ - GRAMIAM OPERATOR)

For any $h > 0, \varepsilon > 0$ and any $q_h^F \in E_h$, we have

$$\nabla J_{\varepsilon,h}(q_h^F) = \underbrace{\mathcal{L}_T^h\Big(\mathcal{B}_h^\star e^{-(T-\cdot)\mathcal{A}_h^\star} q_h^F \big| 0\Big)}_{\stackrel{\text{def}}{=}\Lambda^h q_h^F} + \varepsilon q_h^F + \underbrace{\mathcal{L}_T^h\big(0\big|y_{0,h}\big)}_{=\textit{Free sol. at time } T},$$

CONDITION NUMBER

Basic estimate : $\varepsilon \left\|q_h^F\right\|_h \leq \left\|(\Lambda^h + \varepsilon \mathrm{Id}) q_h^F\right\|_h \leq (C+\varepsilon)\left\|q_h^F\right\|_h.$

$$\boxed{\mathrm{Cond}(\Lambda^h + \varepsilon \mathrm{Id}) \sim \frac{1}{\varepsilon}.}$$

The number of conjugate gradient iterates blows up when $\varepsilon \to 0$.

How to choose $h \mapsto \varphi(h)$?

Basic user guide

Two main principles

1. $\varepsilon = \varphi(h)$ should not be too small in order to maintain a reasonable condition number (i.e. computational cost)

$$\mathrm{Cond}(\Lambda^h + \varphi(h)\mathrm{Id}) \sim \frac{1}{\varphi(h)}.$$

2. The size of the computed solution at time T is

$$\|y_h(T)\|_h \approx C_{\mathrm{obs}} \sqrt{\varphi(h)}.$$

Conclusion

It seems reasonnable to choose

$$\varphi(h) \sim_{h\to 0} h^{2p},$$

where p is the order of accuracy of the numerical method under study.

Remarks

- Computing a null-control for (S_h), i.e. taking $\varepsilon = \varphi(h) = 0$, is not possible in general.
- Choosing $\varphi(h)$ much smaller than h^{2p} (like e^{-C/h^2}) is a useless computational effort.

How to choose $h \mapsto \varphi(h)$?

The simplest example in the world

We set $E = E_h = \mathbb{R}$, $\mathcal{A} = \lambda > 0$, $\mathcal{A}_h = (\lambda + \delta_h) \in \mathbb{R}$ with $\delta_h \xrightarrow[h\to 0]{} 0$, $\mathcal{B} = \mathcal{B}_h = 1$.

$$(S)\begin{cases} y' + \lambda y = v, \\ y(0) = 1, \end{cases} \text{ and } (S_h)\begin{cases} y_h' + (\lambda + \delta_h) y_h = v_h, \\ y_h(0) = 1. \end{cases}$$

Uncontrolled solution $e^{-T\mathcal{A}_h} y_{0,h} = e^{-(\lambda+\delta_h)T}$.

Gramiam "operators"

$$\Lambda_h q^F = \frac{1 - e^{-2(\lambda+\delta_h)T}}{2(\lambda + \delta_h)} q^F, \text{ and } \Lambda q^F = \frac{1 - e^{-2\lambda T}}{2\lambda} q^F, \ \forall q^F \in \mathbb{R},$$

Proposition

The corresponding semi-discrete penalised and exact HUM controls are

$$v_{\varepsilon,h}(t) = -e^{-(T-t)(\lambda+\delta_h)} \frac{2(\lambda + \delta_h) e^{-(\lambda+\delta_h)T}}{1 - e^{-2(\lambda+\delta_h)T} + \boxed{2\varepsilon(\lambda + \delta_h)}},$$

$$v(t) = -e^{-(T-t)\lambda} \frac{2\lambda e^{-\lambda T}}{1 - e^{-2\lambda T}}.$$

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Error estimates

$$[\![v - v_{\varepsilon,h}]\!]_{L^2(0,T;U)} \leq C(\lambda, T)(|\delta_h| + \varepsilon), \quad \text{for } \delta_h \text{ and } \varepsilon \text{ small},$$

$$\mathcal{L}_T\left(v_{\varepsilon,h} \middle| 1\right) = C_1(\lambda, T)\delta_h + C_2(\lambda, T)\varepsilon + O(\varepsilon^2 + \delta_h^2),$$

with $C_i(\lambda, T) > 0$.

Conclusion : The *optimal* choice is to take $\varepsilon = \varphi(h) \sim \delta_h$.

How to choose $h \mapsto \varphi(h)$?

The simplest example in the world is not so stupid

$$\partial_t y - \partial_x^2 y = 1_\Omega v, \quad \text{in } \Omega =]0,1[,$$

in the particular case where $\omega = \Omega$.

STANDARD FINITE DIFFERENCE APPROXIMATION ON A UNIFORM GRID

$$\partial_t y_i - \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = v_i, \quad \forall i \in \{1, ..., N\}, \quad y_0 = y_{N+1} = 0.$$

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EIGENFUNCTIONS OF $\mathcal{A}$

$$\phi_k(x) = \sin(k\pi x), \quad \lambda_k = k^2\pi^2, \quad \forall k \geq 1.$$

EIGENFUNCTIONS OF $\mathcal{A}_h$

$$\phi_{k,h} = (\sin(k\pi x_i))_i, \quad \lambda_{k,h} = \frac{4\sin^2\left(\frac{k\pi h}{2}\right)}{h^2}, \quad \forall 1 \leq k \leq 1/h.$$

EQUATIONS FOR THE k-TH EIGENMODE

$$\frac{d}{dt}(y, \phi_k) + \lambda_k (y, \phi_k) = (v, \phi_k), \quad \frac{d}{dt}(y_h, \phi_{k,h}) + \lambda_{k,h}(y_h, \phi_{k,h}) = (v_h, \phi_{k,h}).$$

Here

$$\delta_{k,h} = \lambda_{k,h} - \lambda_k \underset{h \to 0}{\sim} -\frac{k^4\pi^4}{12} h^2.$$

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Here

$$\delta_{k,h} = \lambda_{k,h} - \lambda_k \underset{h \to 0}{\sim} -\frac{k^4\pi^4}{12} h^2.$$

How to prove such relaxed controllability/observability properties ?

Direct numerical analysis (Labbé-Trélat, '06)

- Galerkin framework
- Based on the properties of the continuous problem
- Sub-optimal choices for $h \mapsto \varphi(h)$.

Discrete Carleman estimates (B.-Hubert-Le Rousseau, '09-'14)

- Cartesian geometries
- Direct analysis of the semi-discrete problem
- "Optimal" penalisation $h \mapsto \varphi(h)$ up to e^{-C/h^2} or $e^{-C/h}$.

Moments method (Lopez-Zuazua, '98), (Allonsius-B.-Morancey, '18), (B. - Hernández–Santamaria '25-'26)

- 1D geometries + Boundary control of coupled systems
- Based on discrete spectral properties of the underlying discrete elliptic operator
- "Optimal" penalisation $h \mapsto \varphi(h)$ up to e^{-C/h^2}.

Moments method + Lebeau-Robbiano (Allonsius-B., '18)

- 2D Cartesian geometries and operators
- Boundary control
- Optimal penalisation ...

Crucial point

The method used to perform the analysis has no impact on the algorithm !

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Partial controllability / partial observability

Let G be a closed (e.g. final dimensional) subspace of $D(\mathcal{A}^\star)$ such that $\mathcal{A}^\star G \subset G$.

$$(S) \quad \begin{cases} \partial_t y + \mathcal{A} y = \mathcal{B} v & \text{in }]0,T[, \\ y(0) = y_0, \end{cases} \qquad (S^*) \quad \begin{cases} -\partial_t q + \mathcal{A}^\star q = 0 & \text{in }]0,T[, \\ q(T) = q^F, \end{cases}$$

Proposition

Let $M \geq 0$. The following two properties are equivalent.

1. *Partial observability*

$$\left\| e^{-T\mathcal{A}^\star} q^F \right\|^2 \leq M^2 \int_0^T \left[\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\right]^2 dt, \quad \forall q^F \in G.$$

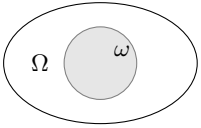
2. *Partial controllability*
   For any initial data $y_0 \in E$, there exists a $v \in L^2(0,T;U)$ such that the associated solution of (S) satisfies

$$\begin{cases} \mathbb{P}_G y(T) = 0, \\ [\![v]\!]_{L^2(0,T;U)} \leq M \left\| y_0 \right\|. \end{cases}$$

Proof : By minimizing on G the functional

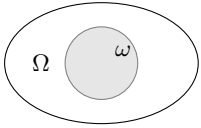
$$J(q^F) := \frac{1}{2} \int_0^T \left[\!\left[\mathcal{B}^\star e^{-(T-t)\mathcal{A}^\star} q^F \right]\!\right]^2 dt + \left\langle y_0, e^{-T\mathcal{A}^\star} q^F \right\rangle.$$

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_\omega v, \ \text{ in } \Omega \\ y = 0, \ \text{ on } \partial\Omega, \\ y(0,.) = y_0, \end{cases}$$



Let $(\phi_k, \lambda_k)_k$ the eigenelements of $\mathcal{A} = -\Delta$.
Let $E_\mu = \text{Span}\{\phi_k, \ \lambda_k \leq \mu\}$.

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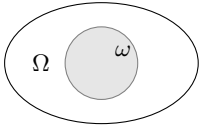


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(1) Spectral inequality (a.k.a. Lebeau-Robbiano inequality)

$$\|\psi\|_{L^2(\Omega)} \leq Ce^{C\sqrt{\mu}}\|\psi\|_{L^2(\omega)}, \ \ \forall \psi \in E_\mu.$$

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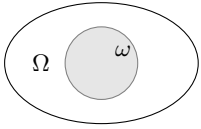
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$$\|\psi\|_{L^2(\Omega)} \leq C e^{C\sqrt{\mu}} \|\psi\|_{L^2(\omega)}, \ \ \forall \psi \in E_\mu.$$

(2) Partial (low-frequencies) observability inequality

$$\|e^{-\tau \mathcal{A}^\star} q^F\|^2_{L^2(\Omega)} \leq C \frac{e^{C\sqrt{\mu}}}{\tau} \int_0^\tau \|e^{-s\mathcal{A}^\star} q^F\|^2_{L^2(\omega)} \, ds, \ \ \forall q^F \in E_\mu.$$

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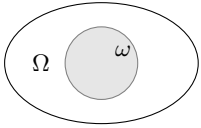
$$\|e^{-\tau \mathcal{A}^\star} q^F\|^2_{L^2(\Omega)} \leq C \frac{e^{C\sqrt{\mu}}}{\tau} \int_0^\tau \|e^{-s\mathcal{A}^\star} q^F\|^2_{L^2(\omega)} \, ds, \ \ \forall q^F \in E_\mu.$$

(3) Partial (low-frequencies) controllability:
For any $y_0 \in E$, $\tau > 0$ there exists $v \in L^2(]0,\tau[\times\omega)$ such that the solution of

$$\begin{cases} \partial_t y - \Delta y = \mathbf{1}_\omega v, \ \text{ in } \Omega \\ \quad y(0,.) = y_0 \in E \end{cases}$$

satisfies $\mathbb{P}_{E_\mu} y(\tau) = 0$ and moreover $\|v\|_{L^2(]0,\tau[\times\omega)} \leq C\tau^{-1} e^{C\sqrt{\mu}} \|y_0\|_{L^2(\Omega)}$.

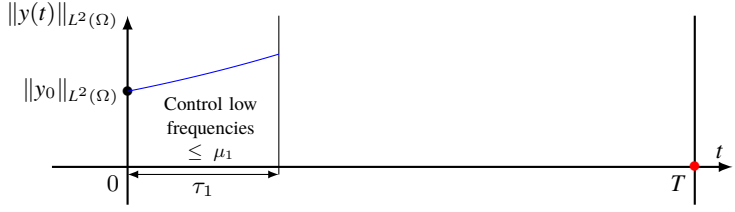
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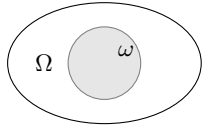
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(4) CONSTRUCTION OF THE CONTROL : Time slicing procedure.



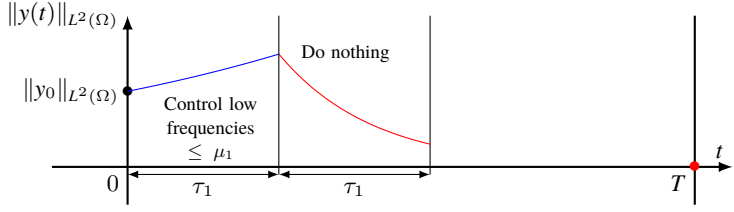
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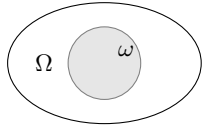
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(4) Construction of the control : Time slicing procedure.



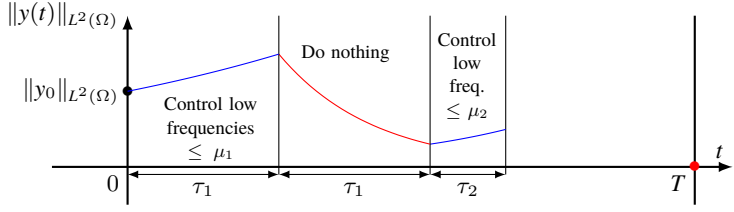
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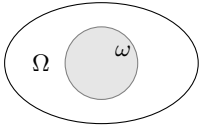
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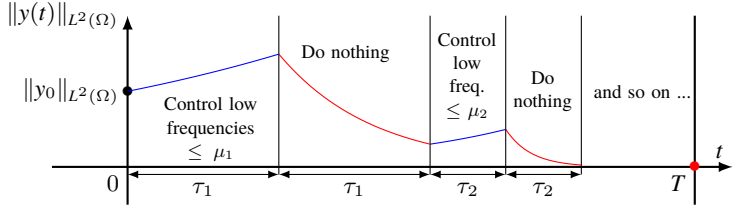


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(4) CONSTRUCTION OF THE CONTROL : Time slicing procedure.



At the end, the control v is shown to satisfy

$$\|v\|_{L^2(]0,T[\times\omega)} \leq C\|y_0\|_{L^2(\Omega)}.$$

CONSEQUENCE

By duality, we obtain the uniform observability inequality for the adjoint system.

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Discrete Lebeau-Robbiano inequality

(B.-Hubert-Le Rousseau, '09-'11)

We assume that $\mathcal{A}_h$ is SPD and let $(\psi_{j,h}, \mu_{j,h})_j$ its eigenelements.

Assumption : Discrete Lebeau-Robbiano spectral inequality

There exists $h_0 > 0$, $\alpha \in [0,1)$, $\beta > 0$, and $\kappa, \ell > 0$ such that, for any $h < h_0$ and for any $(a_j)_j \in \mathbb{R}^{\mathbb{N}}$, we have

$$\Big\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star \Big(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big) \Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \tag{$\mathcal{H}_{\alpha,\beta}$}$$

Theorem

Assume that assumption $(\mathcal{H}_{\alpha,\beta})$ holds, then there exists $h_0 > 0$, $C > 0$ such that, the relaxed observability inequality holds as soon as the function φ satisfies

$$\liminf_{h \to 0} \frac{\varphi(h)}{e^{-C/h^\beta}} > 0.$$

Thus, for any bounded family of initial data $Y_0 \in E_{\text{init}}$, and for any $0 < h < h_0$ we have

$$\big[\!\big[v_{\varphi(h),h} \big]\!\big]_{L^2(0,T;U_h)} \leq C_{\text{obs}} \|y_{0,h}\|_h \,, \text{ and } \Big\| \mathcal{L}_T^h \big(v_{\varphi(h),h} \big| y_{0,h} \big) \Big\|_h \leq C_{\text{obs}} \|y_{0,h}\|_h \sqrt{\varphi(h)}.$$

Discrete Lebeau-Robbiano inequality

Construction of the discrete control

Let $E_{\mu,h} = \mathrm{Span}\{\psi_{j,h},\ \ \mu_{j,h} \leq \mu\}$. Assumption $(\mathcal{H}_{\alpha,\beta})$ says that

$$\left\|q_h\right\|_h^2 \leq \kappa e^{\kappa\mu^\alpha} \left[\!\left[\mathcal{B}_h^\star q_h\right]\!\right]_h^2, \quad \forall q_h \in E_{\mu,h}, \forall \mu < \frac{\ell}{h^\beta}.$$

1. Non-uniform partial observability

$$\left\|e^{-\tau\mathcal{A}_h^\star} q_h^F\right\|_h^2 \leq C\frac{Ce^{C\mu^\alpha}}{\tau} \int_0^\tau \left[\!\left[\mathcal{B}_h^\star e^{-s\mathcal{A}_h^\star} q_h^F\right]\!\right]_h^2 ds, \ \ \forall q_h^F \in E_{\mu,h}, \forall \tau > 0.$$

Discrete Lebeau-Robbiano inequality

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2. Partial controllability

For any $y_{0,h} \in E_{\mu,h}$, and any $\tau > 0$ there exists a $v_h \in L^2(0,\tau;U_h)$ such that

$$\begin{cases} [\![v_h]\!]_{L^2(0,\tau;U_h)} \leq C\dfrac{e^{C\mu^\alpha}}{\tau^{1/2}} \left\|y_{0,h}\right\|_h, \\ \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \ \ y_h(0) = y_{0,h}, \quad \boxed{\mathbb{P}_{E_{\mu,h}} y_h(\tau) = 0.} \end{cases}$$

Discrete Lebeau-Robbiano inequality

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1. Non-uniform partial observability

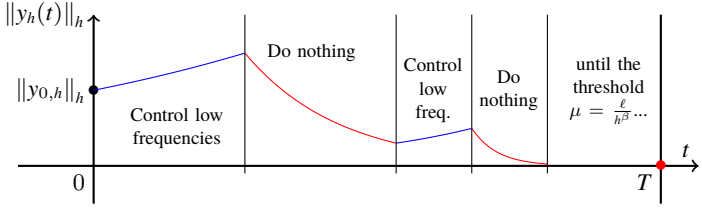
$$\left\|e^{-\tau\mathcal{A}_h^\star} q_h^F\right\|_h^2 \leq C\frac{Ce^{C\mu^\alpha}}{\tau} \int_0^\tau \left[\!\left[\mathcal{B}_h^\star e^{-s\mathcal{A}_h^\star} q_h^F\right]\!\right]_h^2 ds, \ \ \forall q_h^F \in E_{\mu,h}, \forall \tau > 0.$$

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3. Construction of the control : Time slicing procedure.



$$\Big\| \sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big\|_h^2 \leq \kappa e^{\kappa \mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star \Big(\sum_{\mu_{j,h} \leq \mu} a_j \psi_{j,h} \Big) \Big]\!\Big]_h^2, \quad \boxed{\forall \mu < \frac{\ell}{h^\beta}}. \qquad (\mathcal{H}_{\alpha,\beta})$$

Important observation

Excepted in very particular cases, the assumption $(\mathcal{H}_{\alpha,\beta})$ has no chance to hold true without restriction on μ, for dimension reasons.

See also the counter-example of Kavian.

Finite difference framework

We assume now that $\mathcal{A}_h$ is the finite-difference discretisation of $-\nabla \cdot (\gamma \nabla .)$, γ being a Lipschitz continuous coefficient and $\mathcal{B}_h = 1_\omega$ with $\omega =\subset \Omega$.

$$\Big\| \sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big\|_h^2 \leq \kappa e^{\kappa\mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star\Big(\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big)\Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \tag{$\mathcal{H}_{\alpha,\beta}$}$$

The Carleman estimate

For a suitable weight function $(t,x)\mapsto \eta(t,x)$, $\varepsilon_0>0$, $s_0>0$ and $C>0$.

For any $s\geq s_0$ such that $sh\leq\varepsilon_0$ and any $u\in\mathcal{C}^2([0,\tau],E_h)$, with $u(0)=0$, we have

$$s^3\int_0^\tau \|e^{s\eta}u\|_h^2\,dt + s\left\|e^{s\eta(0,.)}\partial_t u(0)\right\|_h^2 + se^{s\eta(\tau)}\|\partial_t u(\tau)\|_h^2 + s^3e^{2s\eta(\tau)}\|u(\tau)\|_h^2$$
$$\leq C\int_0^\tau \left\|e^{s\eta}(-\partial_t^2+\mathcal{A}_h)u\right\|_h^2\,dt + Cse^{2s\eta(\tau)}\langle\mathcal{A}_h u(\tau),u(\tau)\rangle_h + Cs\left\|\mathcal{B}_h^\star e^{s\eta(0,.)}\partial_t u(0)\right\|_h^2.$$

$$\Big\| \sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big\|_h^2 \leq \kappa e^{\kappa\mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star\Big(\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big)\Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \tag{$\mathcal{H}_{\alpha,\beta}$}$$

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$$s^3 e^{2s\eta(\tau)} \|u(\tau)\|_h^2 \leq Cse^{2s\eta(\tau)} \langle \mathcal{A}_h u(\tau), u(\tau)\rangle_h + Cse^{Cs} \|\mathcal{B}_h^\star \partial_t u(0)\|_h^2,$$

as soon as $(-\partial_t^2+\mathcal{A}_h)u=0$.

$$\Big\| \sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big\|_h^2 \leq \kappa e^{\kappa\mu^\alpha} \Big[\!\Big[\mathcal{B}_h^\star\Big(\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big)\Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \tag{$\mathcal{H}_{\alpha,\beta}$}$$

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as soon as $(-\partial_t^2+\mathcal{A}_h)u=0$.

We take: $u(t)=\displaystyle\sum_{\mu_{j,h}\leq\mu} a_j \frac{\sinh(\sqrt{\mu_{j,h}}t)}{\sqrt{\mu_{j,h}}}\psi_{j,h}$, sol. of the elliptic Cauchy problem:

$$\begin{cases} (-\partial_t^2+\mathcal{A}_h)u=0,\\ u(0)=0,\\ \partial_t u(0)=\displaystyle\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}. \end{cases}$$

Discrete Lebeau-robbiano inequality

Sketch of proof

$$\Big\| \sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big\|_h^2 \leq \kappa e^{\kappa\mu^\alpha}\Big[\!\Big[\mathcal{B}_h^\star\Big(\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big)\Big]\!\Big]_h^2, \quad \forall \mu < \frac{\ell}{h^\beta}. \tag{$\mathcal{H}_{\alpha,\beta}$}$$

The Carleman estimate

For a suitable weight function $(t,x)\mapsto \eta(t,x)$, $\varepsilon_0>0$, $s_0>0$ and $C>0$.

For any $s\geq s_0$ such that $\boxed{sh\leq\varepsilon_0}$ and any $u\in\mathcal{C}^2([0,\tau],E_h)$, with $u(0)=0$, we have

$$s^3e^{2s\eta(\tau)}\|u(\tau)\|_h^2 \leq Cse^{2s\eta(\tau)}\langle\mathcal{A}_hu(\tau),u(\tau)\rangle_h + Cse^{Cs}\|\mathcal{B}_h^\star\partial_tu(0)\|_h^2,$$

as soon as $(-\partial_t^2+\mathcal{A}_h)u=0$.

$$\partial_tu(0)=\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}, \quad u(\tau)=\sum_{\mu_{j,h}\leq\mu} a_j\frac{\sinh(\sqrt{\mu_{j,h}}t)}{\sqrt{\mu_{j,h}}}\psi_{j,h}, \quad \mathcal{A}_hu(\tau)=\sum_{\mu_{j,h}\leq\mu} a_j\sqrt{\mu_{j,h}}\sinh(\sqrt{\mu_{j,h}}t)\psi_{j,h}.$$

It follows

$$\|u(\tau)\|_h^2 \geq \frac{1}{\mu}\sum_{\mu_{j,h}\leq\mu}|a_j|^2\sinh^2(\sqrt{\mu_{j,h}}\tau) = \frac{1}{\mu}\langle\mathcal{A}_hu(\tau),u(\tau)\rangle_h, \quad \text{and} \quad \|\partial_tu(0)\|_h^2=\Big[\!\Big[\mathcal{B}_h^\star\Big(\sum_{\mu_{j,h}\leq\mu} a_j\psi_{j,h}\Big)\Big]\!\Big]_h^2.$$

Finally, we need $s\sim\sqrt{\mu}$ which gives the condition $\sqrt{\mu}h\leq\varepsilon_0$.

A discrete Carleman inequality

Sketch of proof

The continuous case

We consider a weight function ψ in $\mathcal{C}^2([0,\tau]\times\overline{\Omega},\mathbb{R})$, sich that

$$|\nabla\psi|\geq c \text{ and } \psi>0 \text{ in }]0,\tau[\times\Omega, \quad \partial_{n_x}\psi(t,x)<0 \text{ in } (0,\tau)\times\partial\Omega,$$
$$\partial_t\psi\geq c \text{ on } \{0\}\times(\Omega\setminus\omega), \quad \nabla_x\psi(T,.)=0 \text{ and } \partial_t\psi(T,.)\leq -c \text{ on } \Omega.$$

We set $\eta=e^{\lambda\psi}$ **for** λ **large enough** and $\rho=e^{s\eta}$, and

$$P=-\partial_t^2-\nabla\cdot(\gamma\nabla\cdot).$$

We compute the conjugate operator $\rho^{-1}P\rho$

Assume $d=1$ and $\gamma=1$

$$f=Pu=-\partial_t^2u-\partial_x^2u,$$

becomes (with $r=\rho^{-1}$, $u=\rho v$)

$$\underbrace{\partial_t^2v+\partial_x^2v+r(\partial_t^2\rho)v+r(\partial_x^2\rho)v}_{=Av}+\underbrace{2r(\partial_t\rho)(\partial_tv)+r(\partial_x\rho)(\partial_xv)}_{=Bv}=-rf.$$

Then

$$2(Av,Bv)_{L^2_{t,x}}\leq\|rf\|_{L^2_{t,x}}.$$

We conclude by integrating by parts in the term (Av,Bv) and using the properties of the weights (and by choosing λ large enough).

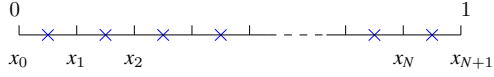
Conclusion : The inequality holds for any s large enough.

A discrete Carleman inequality

Sketch of proof

The 1D uniform discretisation case

✔ for smooth Cartesian multi-D geometry



$\mathfrak{M} = (x_i)_{1\le i\le N}$

$\overline{\mathfrak{M}} = (x_{i+\frac{1}{2}})_{0\le i\le N}$

- Primal/Dual discrete functions

$$u = (u_i)_{1\le i\le N} \in \mathbb{R}^{\mathfrak{M}}, \qquad v = (v_{i+\frac{1}{2}})_{0\le i\le N} \in \mathbb{R}^{\overline{\mathfrak{M}}}.$$

- Discrete derivatives (Dirichlet BC : $\Leftrightarrow u_0 = u_{N+1} = 0$)

$$\left((Du)_{i+\frac{1}{2}} = \frac{u_{i+1}-u_i}{h}\right)_{0\le i\le N} \in \mathbb{R}^{\overline{\mathfrak{M}}}, \quad \left((\bar{D}v)_i = \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h}\right)_{1\le i\le N} \in \mathbb{R}^{\mathfrak{M}},$$

- Discrete integrals and norms

$$\|u\|_h^2 = \int_\Omega u^2 = \sum_{i=1}^N h u_i^2, \quad \int_\Omega v = \sum_{i=0}^N h v_{i+\frac{1}{2}}, \ldots$$

The discrete operator reads $\mathcal{A}_h = -\bar{D}(\gamma D\cdot)$

Products derivatives

$f_i \in \mathbb{R}^{\mathfrak{M}}$, $g_i \in \mathbb{R}^{\overline{\mathfrak{M}}}$

$$D(f_1 f_2) = (Df_1)\overline{f_2} + (Df_2)\overline{f_1},$$

$$\bar{D}(g_1 g_2) = (\bar{D}g_1)\overline{g_2} + (\bar{D}g_2)\overline{g_1}.$$

Products averages

$$\overline{f_1 f_2} = \overline{f_1}\,\overline{f_2} + \frac{h^2}{4}(Df_1)(Df_2).$$

Repeated averages

$$\bar{\bar{f}} = f + \frac{h^2}{4}\bar{D}Df,$$

Discrete integration by parts

$f \in \mathbb{R}^{\overline{\mathfrak{M}}}$ and $g \in \mathbb{R}^{\mathfrak{M}}$

$$\int_\Omega f(Dg) = -\int_\Omega (\bar{D}f)g + f_{N+1}g_{N+\frac{1}{2}} - f_0 g_{\frac{1}{2}}.$$

A discrete Carleman inequality

Sketch of proof

- We consider **the same weight** as in the continuous case which is sampled on the meshes.
- Discrete integration by parts and discrete derivation of products lead to new terms

$$\underbrace{\partial_t^2 v + r\overline{\overline{\rho}}\,\bar{D}Dv + r(\partial_t^2\rho)\,v + r(\bar{D}D\rho)\,\overline{\overline{v}}}_{=Av} + \underbrace{2r(\partial_t\rho)(\partial_t v) + 2r\overline{D\rho}\,\overline{Dv}}_{=Bv}$$
$$= -rf + \cdots$$

- All the additional terms are **small** with respect to h but contain **high powers of the parameters** s and λ. Typical term :

$$r\overline{\overline{\rho}} = 1 + \frac{h^2}{4} r(\bar{D}D\rho) \approx 1 + \frac{h^2}{4} r\partial_x^2\rho \approx 1 + \frac{h^2 s^2}{4}(\partial_x\eta)^2 \underbrace{r\rho}_{=1} + \cdots$$

- We shall need a condition relating s and h of the following form

$$sh \leq \varepsilon_0,$$

where ε_0 only depends on the data.

- In the proof of the discrete Lebeau-Robbiano, we take $s \sim \sqrt{\mu}$, which gives the threshold $\mu \leq \frac{\varepsilon_1}{h^2}$.

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3. The HUM approach in the discrete framework
3.1 The semi-discrete in space setting
3.2 Practical considerations

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4.1 Introduction to the Lebeau-Robbiano approach
4.2 Discrete Lebeau-Robbiano inequality. Discrete elliptic Carleman estimate
4.3 Discrete parabolic Carleman estimate
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7.1 Boundary control for 1D time-discrete scalar problems
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Another (parabolic) discrete Carleman estimate

(B. - Le Rousseau, '14)

GOALS

- To obtain relaxed observability estimates for some time depend coefficients parabolic equations.
- To study some semi-linear case.
- To derive discrete parabolic Carleman estimates (*a la Fursikov-Imanuvilov*).

CARLEMAN WEIGHTS

- Standard weights in space $\eta = e^{\lambda\psi} - e^{\lambda\|\psi\|_\infty} < 0$.
- Singular weights in time $\theta(t) = (t+\alpha h)^{-1}(T-t+\alpha h)^{-1}$.

THEOREM

For any $\tau > 0$ large enough, there exists $\alpha > 0$ and $h_0 > 0$ such that for any function $q_h \in \mathcal{C}^1([0,T], E_h)$ and any $h < h_0$ we have

$$\begin{aligned}\|(\tau\theta)^{\frac{1}{2}} e^{\tau\theta\eta} Dq_h\|^2_{L^2(0,T;E_h)} + \|(\tau\theta)^{\frac{3}{2}} e^{\tau\theta\eta} q_h\|^2_{L^2(0,T;E_h)} \\ \leq C\big(\|e^{\tau\theta\eta}(-\partial_t + \mathcal{A}_h^*)q_h\|^2_{L^2(0,T;E_h)} + \|(\tau\theta)^{\frac{3}{2}} e^{\tau\theta\eta}\mathcal{B}_h^\star q_h\|^2_{L^2(0,T;E_h)}\big) \\ + Ch^{-2}\left(\left\|e^{\tau\theta\eta} q_h(0)\right\|_h^2 + \left\|e^{\tau\theta\eta} q_h(T)\right\|_h^2\right)\end{aligned}$$

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- Singular weights in time $\theta(t) = (t+\alpha h)^{-1}(T-t+\alpha h)^{-1}$.

THEOREM (RELAXED OBSERVABILITY INEQUALITY)

There exists $C > 0$ s.t. for any function $a_h \in L^\infty(]0,T[, E_h)$, and any $h \leq \min(h_0, h_1)$ with $h_1 \sim \|a_h\|_\infty^{-\frac{2}{3}}$, any solution of $-\partial_t q_h + \mathcal{A}_h^\star q_h + a_h q_h = 0$ satisfies

$$\|q_h(0)\|_h^2 \leq C_{\text{obs}} \|\mathcal{B}_h^\star q_h\|^2_{L^2(0,T;E_h)} + e^{-\frac{C^{-1}}{h} + T\|a_h\|_\infty} \|q_h(T)\|_h^2.$$

with $C_{\text{obs}} = e^{C(1+\frac{1}{T}+T\|a_h\|_\infty + \|a_h\|_\infty^{\frac{2}{3}})}$.

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The time discrete setting

Problem formulation

Assume we are given $y_0 \in L^2(\Omega)$. Does there exist a sequence $(v^{n+1})_{0\leq n\leq M-1} \subset L^2(\Omega)$ such that the solution in $H_0^1(\Omega)$ to

$$(S) \quad \begin{cases} \dfrac{y^{n+1}-y^n}{\delta t} - \Delta y^{n+1} = \mathbf{1}_\omega v^{n+1}, & \forall n \in \{0,\dots,M-1\}, \\ y^0 = y_0. \end{cases}$$

satisfies the controllability condition $\boxed{y^M = 0}$?

In case of a positive answer can we bound $\displaystyle\sum_{n=1}^{M} \delta t \|v^n\|^2_{L^2(\Omega)}$ uniformly with respect to δt ?

First answers

- If $\omega \neq \Omega$, this problem does not have any solution **(Zheng, '08)**
- However, it was proved that it is positive if we relax the controllability condition by

$$P_{E_{C/\delta t}} y^M = 0.$$

Problem formulation

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In case of a positive answer can we bound $\displaystyle\sum_{n=1}^{M} \delta t \|v^n\|^2_{L^2(\Omega)}$ uniformly with respect to δt ?

First answers

- Associated relaxed observability inequalities where proved in **(Ervedoza-Valein, '10)**

$$\left\|q^1\right\|^2 \leq C_{obs}^2 \left(\sum_{n=1}^{M} \delta t \|q^n\|^2_{L^2(\omega)} \right) + C(\delta t)^\beta \left\|q^F\right\|^2,$$

with

$$(S^*) \quad \begin{cases} \dfrac{q^n - q^{n+1}}{\delta t} - \Delta q^n = 0, & \forall n \in \{1,\dots,M\}, \\ q^{M+1} = q^F. \end{cases}$$

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satisfies the controllability condition $\boxed{y^M = 0}$?

In case of a positive answer can we bound $\displaystyle\sum_{n=1}^{M} \delta t \|v^n\|^2_{L^2(\Omega)}$ uniformly with respect to δt ?

First answers

- In both cases, spectral techniques are used

$$\Longrightarrow \text{ Not adapted to time-dependent or nonlinear problems.}$$

- The value of β is suboptimal.

THEOREM ((B.-Hernández–Santamaría, '20))

Let $T > 0$, $M \in \mathbb{N}^*$ and $\delta t = T/(M+1)$. Let $(a^n)_n \subset L^\infty(\Omega)$ with $\|a^n\|_{L^\infty} \leq A$ for some A.

There exists C_{obs}, C_1, that depend only on A such that any solution (in $H_0^1(\Omega)$) of

$$(S^*) \quad \begin{cases} \dfrac{q^n - q^{n+1}}{\delta t} - \Delta q^n + a^n q^n = 0, \quad \forall n \in \{1, \dots, M\}, \\ q^{M+1} = q^F, \end{cases}$$

satisfies

$$\left\|q^1\right\|^2 \leq C_{obs}^2 \left(\sum_{n=1}^{M} \delta t \|q^n\|_{L^2(\omega)}^2 \right) + C_1 e^{-\frac{C_1}{\delta t^{1/4}}} \|q^F\|_{H_0^1(\Omega)}^2.$$

TOOL : A suitable time-discrete Carleman estimate

Theorem (B.-Hernández–Santamaría, '20)

Let $T > 0$, $M \in \mathbb{N}^*$ and $\delta t = T/(M+1)$. Let $(a^n)_n \subset L^\infty(\Omega)$ with $\|a^n\|_{L^\infty} \leq A$ for some A.

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satisfies

$$\left\|q^1\right\|^2 \leq C_{obs}^2 \left(\sum_{n=1}^{M} \delta t \|q^n\|_{L^2(\omega)}^2 \right) + C_1 e^{-\frac{C_1}{\delta t^{1/4}}} \|q^F\|_{H_0^1(\Omega)}^2.$$

Proposition (Relaxed controllability in H^{-1})

For any $y_0 \in L^2(\Omega)$ there exists a $(v^n)_n \subset L^2(\Omega)$ such that the unique solution in $H_0^1(\Omega)$ of

$$(S) \quad \begin{cases} \dfrac{y^{n+1} - y^n}{\delta t} - \Delta y^{n+1} + a^{n+1} y^{n+1} = \mathbf{1}_\omega v^{n+1}, & \forall n \in \{0, \dots, M-1\}, \\ y^0 = y_0, \end{cases}$$

satisfies $\|y^M\|_{H^{-1}} \leq C e^{-\frac{C}{\delta t^{1/4}}} \|y_0\|_{L^2}$ and $\sum_{}^{M} \delta t \|v^n\|_{L^2}^2 \leq C_{obs}^2 \|y_0\|_{L^2}^2$.

Main result

Theorem

For any $y_0 \in L^2(\Omega)$, and any function $\varphi : (0,+\infty) \to (0,+\infty)$ such that $\varphi(\delta t) \gtrapprox e^{-1/\delta t^{1/4}}$, the unique minimizer of the penalized HUM functional

$$F_{\varphi(\delta t),\delta t}(v^{\delta t}) = \frac{1}{2}\left(\sum_{n=1}^{M} \delta t \|v^n\|^2_{L^2(\omega)}\right) + \frac{1}{2\varphi(\delta t)}\|y^M\|^2_{L^2},$$

is such that the associated time discrete trajectory

$$(S) \quad \begin{cases} \dfrac{y^{n+1}-y^n}{\delta t} - \Delta y^{n+1} + a^{n+1}y^{n+1} = \mathbf{1}_\omega v^{n+1}, & \forall n \in \{0,\dots,M-1\}, \\ y^0 = y_0, \end{cases}$$

satisfies

$$\|y^M\|_{L^2} \leq C_{obs}\sqrt{\varphi(\delta t)}\,\|y_0\|_{L^2}$$

$$\sum_{n=1}^{M} \delta t \|v^n\|^2_{L^2} \leq C^2_{obs}\,\|y_0\|^2_{L^2}\,.$$

Other applications : semi-linear equations

$$\frac{y^{n+1}-y^n}{\delta t} - \Delta y^{n+1} + f(y^{n+1}) = \mathbf{1}_\omega v^{n+1}, \quad \forall n \in \{0,\dots,M-1\}.$$

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The fully discrete setting

We have introduced and analyzed the $\varphi(h)$-null-controllability hold for

$$(S_h)\quad \begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h, \\ y_h(0) = y_{0,h}. \end{cases}$$

What about time discretization of such a system ?

We study **unconditionally stable schemes** : the θ-scheme with $\theta \in [1/2, 1]$

$$(S_{h,\delta t})\quad \begin{cases} \dfrac{y_h^{n+1} - y_h^n}{\delta t} + \mathcal{A}_h(\theta y_h^{n+1} + (1-\theta) y_h^n) = \mathcal{B}_h v_h^{n+1},\ \forall n \in [\![0, M-1]\!], \\ y_h^0 = y_{0,h} \in E_h, \end{cases}$$

where, $\delta t = T/M$, $v_{h,\delta t} = (v_h^n)_{1\le n\le M} \in (U_h)^M$ is a fully-discrete control function whose cost is defined by

$$[\![v_{h,\delta t}]\!]_{L^2_{\delta t}(0,T;U_h)} \overset{\text{def}}{=} \left(\sum_{n=1}^{M} \delta t\, [\![v_h^n]\!]_h^2 \right)^{\frac{1}{2}}.$$

The value at the final time iteration of the controlled solution of $(S_{h,\delta t})$ is denoted by

$$\mathcal{L}_T^{h,\delta t}\left(v_{h,\delta t} \middle| y_{0,h}\right) \overset{\text{def}}{=} y_h^M.$$

The fully discrete setting

The penalised HUM primal functional

$$F_{\varepsilon,h,\delta t}(v_{h,\delta t}) \overset{\text{def}}{=} \frac{1}{2}\left[\!\left[v_{h,\delta t}\right]\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T^{h,\delta t}\left(v_{h,\delta t}\middle| y_{0,h}\right)\right\|_h^2.$$

Definition (Dual functional)

We define the functional

$$J_{\varepsilon,h,\delta t}(q_h^F) \overset{\text{def}}{=} \frac{1}{2}\left[\!\left[\mathcal{B}_h^\star \mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right)\right]\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{\varepsilon}{2}\left\|q_h^F\right\|_h^2 - \left\langle y_{0,h}, q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1\right\rangle_h, \quad \forall q_h^F \in E_h,$$

where $\mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right) = (q_h^n)_{1\le n\le M}$ *is the solution of the following adjoint problem*

$$\left\{\begin{array}{c} q_h^{M+1} = q_h^F, \\ \dfrac{q_h^M - q_h^{M+1}}{\delta t} + \theta \mathcal{A}_h q_h^M = 0, \\ \dfrac{q_h^n - q_h^{n+1}}{\delta t} + \mathcal{A}_h(\theta q_h^n + (1-\theta)q_h^{n+1}) = 0, \ \ \forall n \in [\![1, M-1]\!]. \end{array}\right. \tag{$S^*_{h,\delta t}$}$$

The fully discrete setting

The penalised HUM primal functional

$$F_{\varepsilon,h,\delta t}(v_{h,\delta t}) \stackrel{\text{def}}{=} \frac{1}{2}\left[\!\left[v_{h,\delta t}\right]\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{1}{2\varepsilon}\left\|\mathcal{L}_T^{h,\delta t}\left(v_{h,\delta t}\middle|y_{0,h}\right)\right\|_h^2.$$

$$J_{\varepsilon,h,\delta t}(q_h^F) \stackrel{\text{def}}{=} \frac{1}{2}\left[\!\left[\mathcal{B}_h^\star\mathcal{L}_T^{*,h,\delta t}\left(q_h^F\right)\right]\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \frac{\varepsilon}{2}\left\|q_h^F\right\|_h^2 - \left\langle y_{0,h}, q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1\right\rangle_h, \quad \forall q_h^F \in E_h,$$

Theorem (Duality)

The functionals $F_{\varepsilon,h,\delta t}$ and $J_{\varepsilon,h,\delta t}$ are in duality, in the sense that their respective minimisers $v_{\varepsilon,h,\delta t} \in L^2(0,T;U_h)$ and $q^F_{\varepsilon,h,\delta t} \in E_h$ satisfy

$$\inf_{L^2_{\delta t}(0,T;U_h)} F_{\varepsilon,h,\delta t} = F_{\varepsilon,h,\delta t}(v_{\varepsilon,h,\delta t}) = -J_{\varepsilon,h,\delta t}(q^F_{\varepsilon,h,\delta t}) = -\inf_{E_h} J_{\varepsilon,h,\delta t},$$

and moreover

$$v_{\varepsilon,h,\delta t} = \mathcal{B}_h^\star\mathcal{L}_T^{*,h,\delta t}\left(q^F_{\varepsilon,h,\delta t}\right).$$

Theorem (Case $\theta \in]1/2, 1]$)

Assume that the discrete Lebeau-Robbiano inequality $(\mathcal{H}_{\alpha,\beta})$ holds and let ϕ be such that

$$\liminf_{h\to 0} \frac{\varphi(h)}{e^{-C/h^\beta}} > 0.$$

Then, there exists $h_0 > 0, C_T > 0, C_{\text{obs}} > 0$ such that for any $0 < h < h_0$ and any $\delta t \leq C_T |\log \varphi(h)|^{-1}$, the following relaxed observability inequality holds

$$\left\|q_h^1 - \delta t(1-\theta)\mathcal{A}_h q_h^1\right\|_h^2 \leq C_{\text{obs}}^2 \left(\left[\!\left[\mathcal{B}_h^\star q_h^n\right]\!\right]^2_{L^2_{\delta t}(0,T;U_h)} + \varphi(h) \left\|q_h^F\right\|_h^2 \right), \quad \forall q_h^F \in E_h.$$

Thus, for any such δt and h and any initial data $y_{0,h} \in E_h$, the full-discrete control $v_{\varphi(h),h,\delta t}$, obtained by minimising $F_{\varphi(h),h,\delta t}$ (or equivalently $J_{\varphi(h),h,\delta t}$) satisfies

$$\left[\!\left[v_{\varphi(h),h,\delta t}\right]\!\right]_{L^2_{\delta t}(0,T;U_h)} \leq C_{\text{obs}} \left\|y_{0,h}\right\|_h,$$

$$\left\| \mathcal{L}_T^{h,\delta t}\left(v_{\varphi(h),h,\delta t} \middle| y_{0,h}\right) \right\|_h \leq C_{\text{obs}} \sqrt{\varphi(h)} \left\|y_{0,h}\right\|_h.$$

Case $\theta = 1/2$: We need the (much stronger) condition $\delta t \rho(\mathcal{A}_h) \leq \delta$ for some δ.

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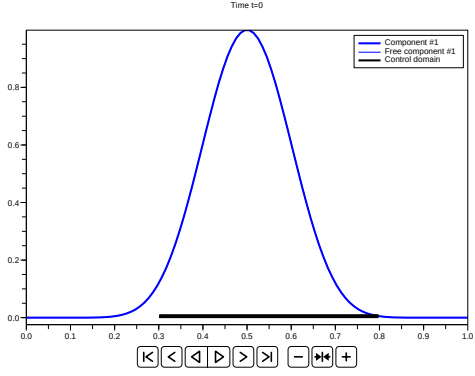
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The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$



The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Conjugate gradient iterates

N	M				
	20	80	320	1280	$+\infty$
20	14	16	16	16	16
50	22	26	29	29	31
100	30	38	44	49	48
200	45	58	69	77	82

(A) Case $\varphi(h) = h^2$

N	M				
	20	80	320	1280	$+\infty$
20	24	30	28	27	32
50	83	87	87	93	106
100	235	240	233	262	265
200	778	850	1098	1230	1374

(B) Case $\varphi(h) = h^4$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Optimal energy

$$\inf_{L^2_{\delta t}(0,T;U_h)} F_{\varphi(h),h,\delta t}$$

N	M				
	20	80	320	1280	$+\infty$
20	$7.17 \cdot 10^{-2}$	$6.54 \cdot 10^{-2}$	$6.38 \cdot 10^{-2}$	$6.34 \cdot 10^{-2}$	$6.33 \cdot 10^{-2}$
50	$7.98 \cdot 10^{-2}$	$7.08 \cdot 10^{-2}$	$6.85 \cdot 10^{-2}$	$6.79 \cdot 10^{-2}$	$6.78 \cdot 10^{-2}$
100	$8.5 \cdot 10^{-2}$	$7.44 \cdot 10^{-2}$	$7.15 \cdot 10^{-2}$	$7.07 \cdot 10^{-2}$	$7.05 \cdot 10^{-2}$
200	$9.1 \cdot 10^{-2}$	$7.75 \cdot 10^{-2}$	$7.39 \cdot 10^{-2}$	$7.3 \cdot 10^{-2}$	$7.27 \cdot 10^{-2}$

Case $\varphi(h) = h^2$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Optimal energy

$$\inf_{L^2_{\delta t}(0,T;U_h)} F_{\varphi(h),h,\delta t}$$

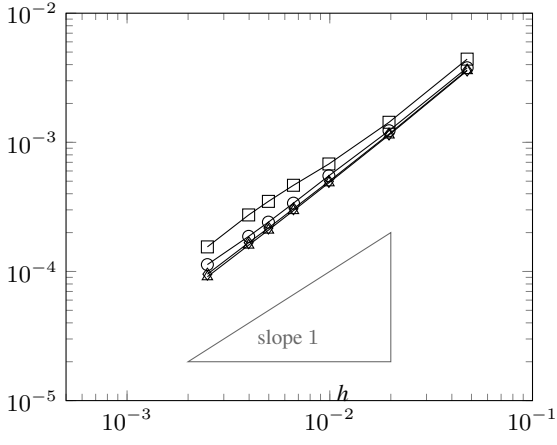
N	M				
	20	80	320	1280	$+\infty$
20	0.11	$8.92 \cdot 10^{-2}$	$8.43 \cdot 10^{-2}$	$8.3 \cdot 10^{-2}$	$8.26 \cdot 10^{-2}$
50	0.12	$8.94 \cdot 10^{-2}$	$8.29 \cdot 10^{-2}$	$8.12 \cdot 10^{-2}$	$8.07 \cdot 10^{-2}$
100	0.12	$9.1 \cdot 10^{-2}$	$8.33 \cdot 10^{-2}$	$8.13 \cdot 10^{-2}$	$8.06 \cdot 10^{-2}$
200	0.13	$9.33 \cdot 10^{-2}$	$8.41 \cdot 10^{-2}$	$8.17 \cdot 10^{-2}$	$8.09 \cdot 10^{-2}$

Case $\varphi(h) = h^4$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}\nu,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Convergence analysis



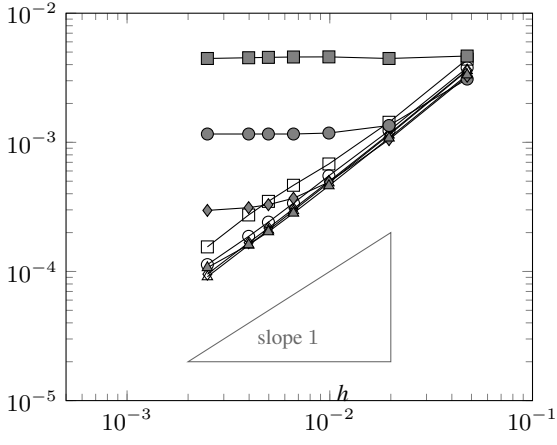
Size of the target $M = 20$
$M = 80$
$M = 320$
$M = 1280$

Case $\varphi(h) = h^2$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}\nu,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Convergence analysis



Size of the target $M = 20$
$M = 80$
$M = 320$
$M = 1280$

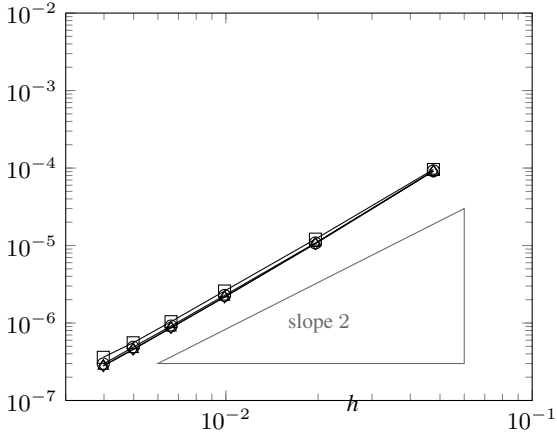
Exact solution $M = 20$
$M = 80$
$M = 320$
$M = 1280$

Case $\varphi(h) = h^2$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}\nu,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Convergence analysis



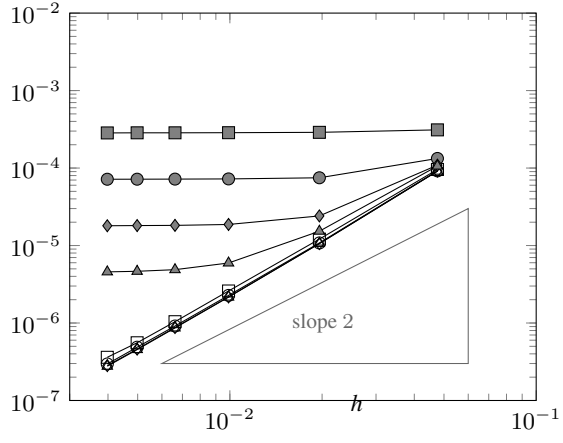
Size of the target $M = 320$
$M = 1280$
$M = 5120$
$M = 20480$

Case $\varphi(h) = h^4$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}\nu,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Convergence analysis



Size of the target $M = 320$
$M = 1280$
$M = 5120$
$M = 20480$

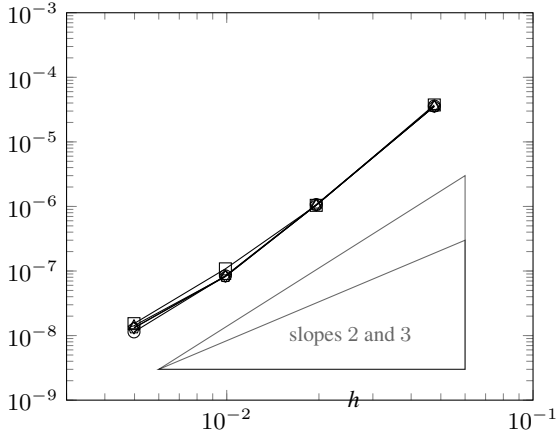
Exact solution $M = 320$
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Case $\varphi(h) = h^4$

The 1D heat equation

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Convergence analysis



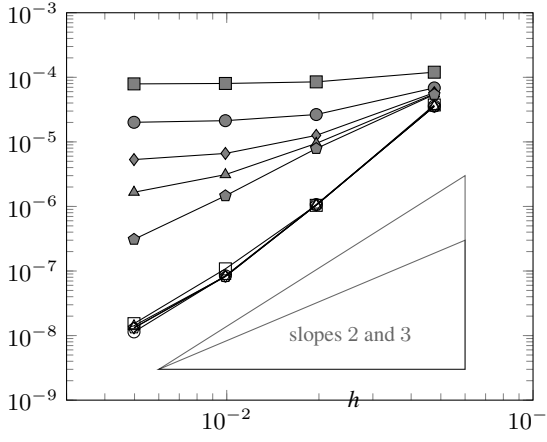
Size of the target $M = 1280$
$M = 5120$
$M = 20480$
$M = 81920$
$M = +\infty$

Case $\varphi(h) = 1000h^6$

The 1D heat equation

$$\partial_t y - 0.1\partial_x^2 y = 1_{]0.3,0.8[}\nu,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$

Convergence analysis



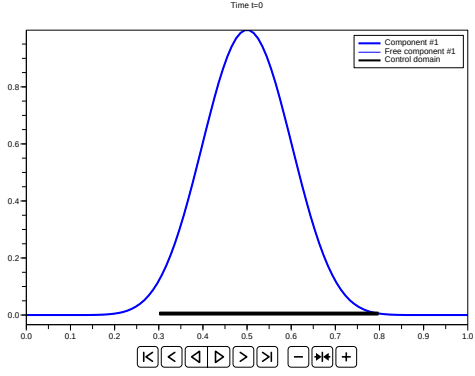
Size of the target $M = 1280$
$M = 5120$
$M = 20480$
$M = 81920$
$M = +\infty$

Exact solution $M = 1280$
$M = 5120$
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Case $\varphi(h) = 1000h^6$

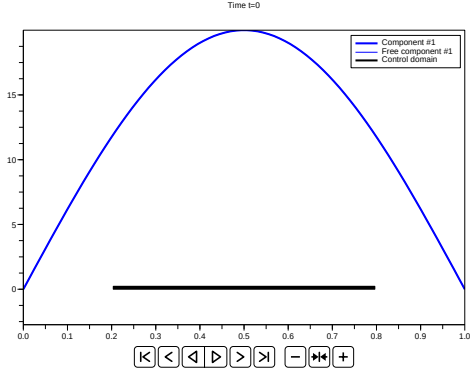
A 1D parabolic equation with unstable modes

$$\partial_t y - 0.1\partial_x^2 y - 1.5y = 1_{]0.3,0.8[}v,$$
$$T = 1, y_0(x) = \sin(\pi x)^{10}.$$



A nonlinear case

$$\partial_t y - 0.1\partial_x^2 y - 5y\log^{1.4}(1+|y|) = 1_{]0.2,0.8[}v,$$
$$T = 0.5, y_0(x) = 20\sin(\pi x).$$



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A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

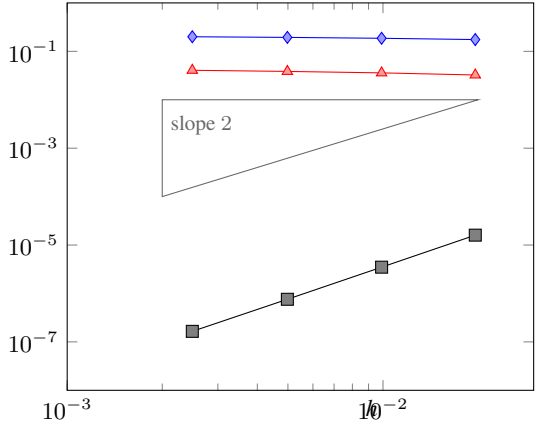
SHORT REVIEW OF KNOWN RESULTS

- In the case $a_{21} = \text{cte}$ the system is null-controllable if and only if $a_{21} \neq 0$ (Kalman-like condition)
  (Ammar-Khodja–Benabdallah–Dupaix–González-Burgos, '09)
- In the case where $\text{Supp}(a_{21}) \cap \omega \neq \emptyset$, the system is null-controllable
  (González-Burgos–de Teresa, '10)
- In the case where $\text{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} has a constant sign, the system is null-controllable
  (Rosier–de Teresa, '11)
- In the case where $\text{Supp}(a_{21}) \cap \omega = \emptyset$ and a_{21} changes it sign:
  - There are structural conditions for the system to be even approximatively controllable.
    (B.– Olive, '13)
  - A minimal time condition for the null-controllability can occur
    (Ammar-Khodja–Benabdallah–González-Burgos–de Teresa, '14) (B.-Morancey, '25)

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 1 : $a_{21}(x) = \mathbf{1}_{]0.2,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.

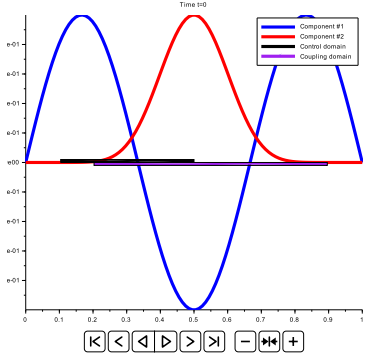


App. Cont. ✔
Null Cont. ✔

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

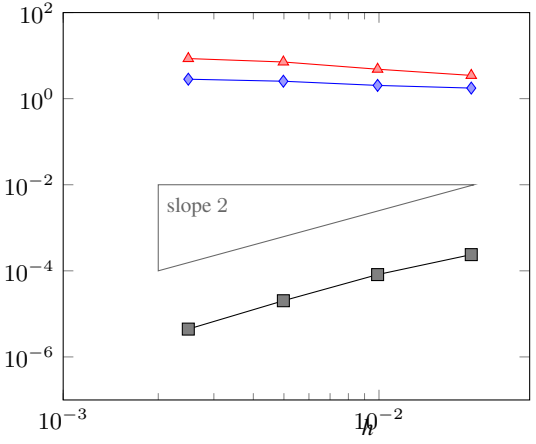
CASE 1 : $a_{21}(x) = \mathbf{1}_{]0.2,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.



A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 2 : $a_{21}(x) = \mathbf{1}_{]0.7,0.9[}(x)$, $\omega =]0.1, 0.5[$, $y_0(x) = \left(\sin(3\pi x), \sin(\pi x)^{10}\right)^t$.



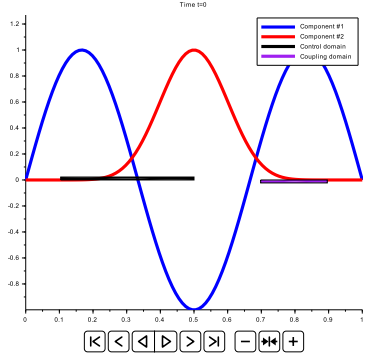
Cost of the control
Size of the target
Optimal energy

App. Cont. ✔
Null Cont. ✔

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

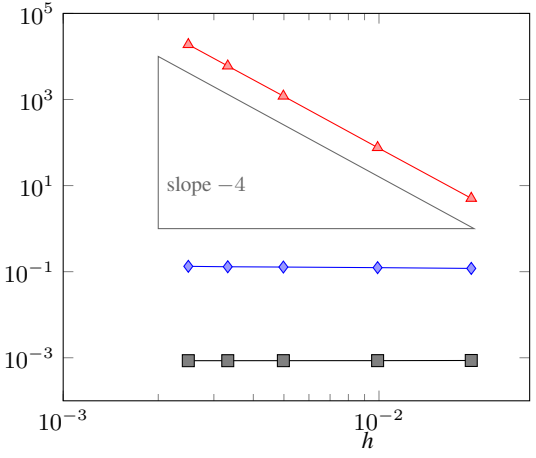
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A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 3 : $a_{21}(x) = (x-\alpha)\mathbf{1}_{]0,0.5[}(x)$, $\omega =]0.5, 1[$, $y_0(x) = (\sin(2\pi x), 3\sin(2\pi x))^t$.



Cost of the control
Size of the target
Optimal energy

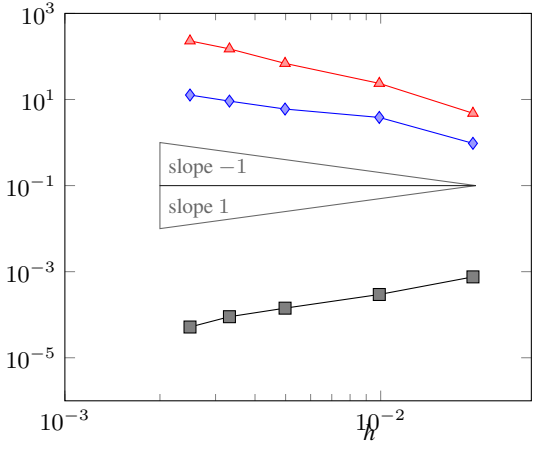
App. Cont. ✖
Null Cont. ✖

FIGURE 1: $\alpha = 1/4$

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 3 : $a_{21}(x) = (x-\alpha)\mathbf{1}_{]0,0.5[}(x)$, $\omega =]0.5, 1[$, $y_0(x) = (\sin(2\pi x), 3\sin(2\pi x))^t$.



Cost of the control
Size of the target
Optimal energy

App. Cont.	✔
Null Cont.	✔ for $T > T_{\min}$

FIGURE 1: $\alpha = 1/8$

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 4 : $a_{21}(x) = \mathbf{1}_{[\alpha-d,\alpha]} - \mathbf{1}_{[\alpha,\alpha+d]}$, $y_0(x) = \left(\sin(\pi x)^{10}, -2\sin(2\pi x)^{10}\right)^t$.

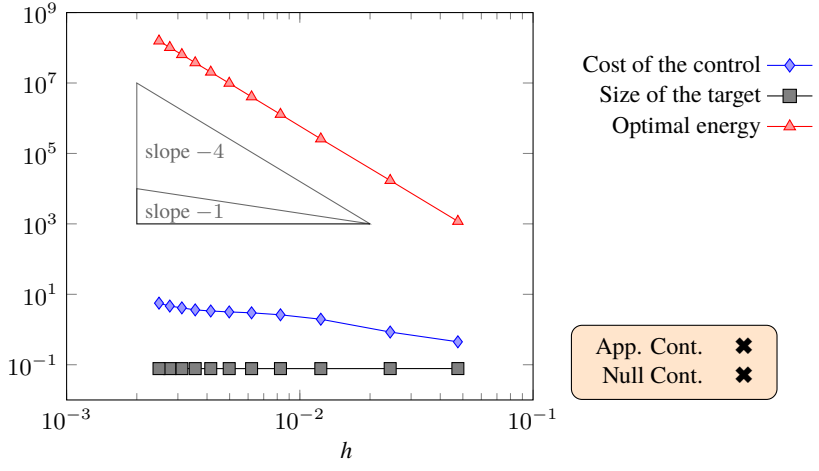


FIGURE 2: $\alpha = 1/2, d = \frac{1}{2\sqrt{3}}$, $\omega =]0.8, 1.0[$.

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 4 : $a_{21}(x) = \mathbf{1}_{[\alpha-d,\alpha]} - \mathbf{1}_{[\alpha,\alpha+d]}$, $y_0(x) = \left(\sin(\pi x)^{10}, -2\sin(2\pi x)^{10}\right)^t$.

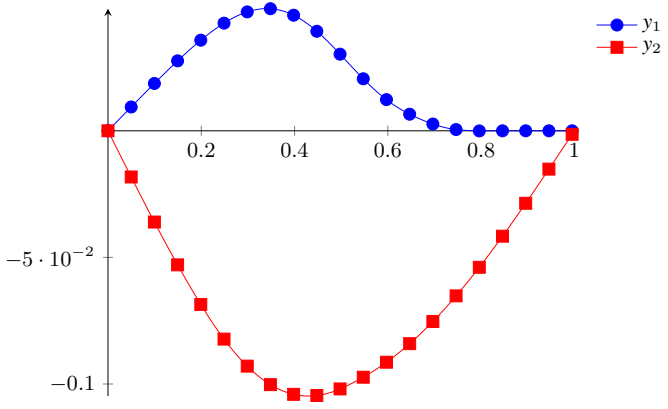


FIGURE 2: $\alpha = 1/2, d = \frac{1}{2\sqrt{3}}$, $\omega =]0.8, 1.0[$.

A two equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 \\ a_{21}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{1}_\omega v.$$

CASE 4 : $a_{21}(x) = \mathbf{1}_{[\alpha-d,\alpha]} - \mathbf{1}_{[\alpha,\alpha+d]}$, $y_0(x) = \left(\sin(\pi x)^{10}, -2\sin(2\pi x)^{10}\right)^t$.

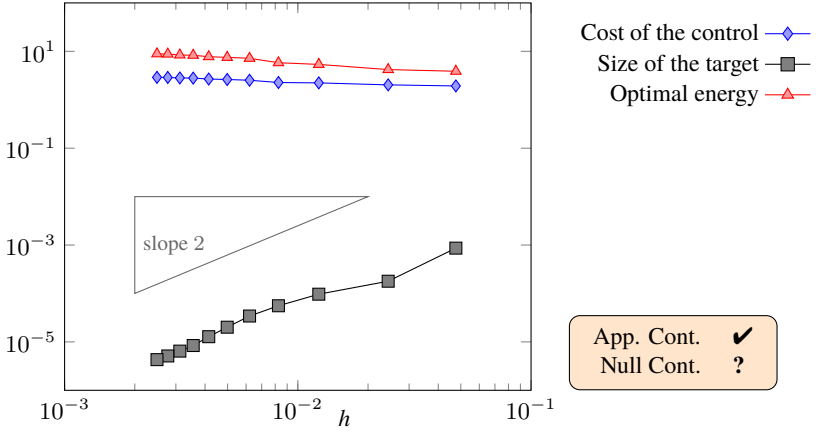


FIGURE 3: $\alpha = 1/2, d = \frac{1}{2\sqrt{3}}$, $\omega =]\mathbf{0}, \mathbf{0.2}[\cup]\mathbf{0.8}, \mathbf{1.0}[$.

Simultaneous control of two 2×2 cascade systems

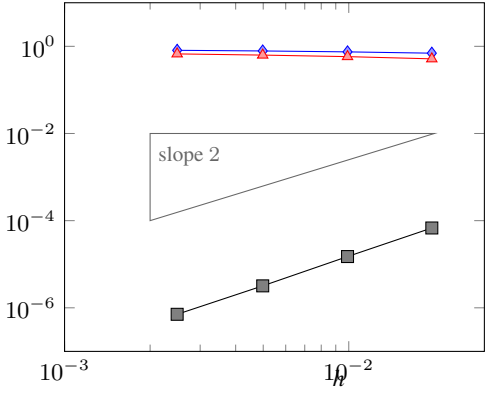
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

Here also necessary and sufficient conditions for approximate controllability are known

Simultaneous control of two 2×2 cascade systems

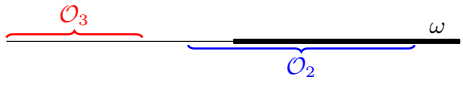
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CASE 1 : $\mathcal{O}_2 \cap \omega \neq \emptyset$



Cost of the control
Size of the target
Optimal energy

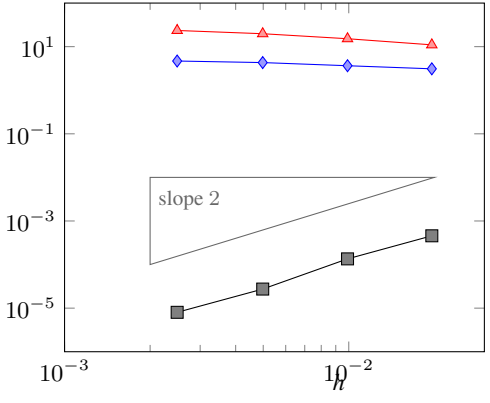
App. Cont. ✔
Null Cont. ?



Simultaneous control of two 2×2 cascade systems

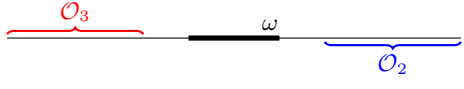
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 2 : $\mathcal{O}_2$ and $\mathcal{O}_3$ are located in different connected components of $\Omega \setminus \omega$



Cost of the control
Size of the target
Optimal energy

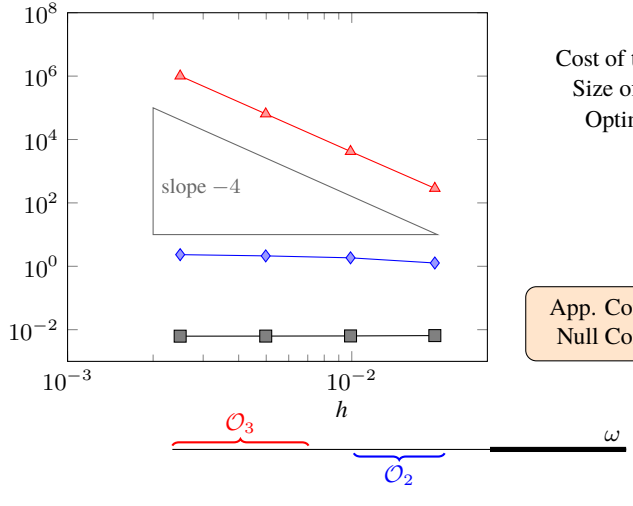
App. Cont. ✔
Null Cont. ✔



Simultaneous control of two 2×2 cascade systems

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

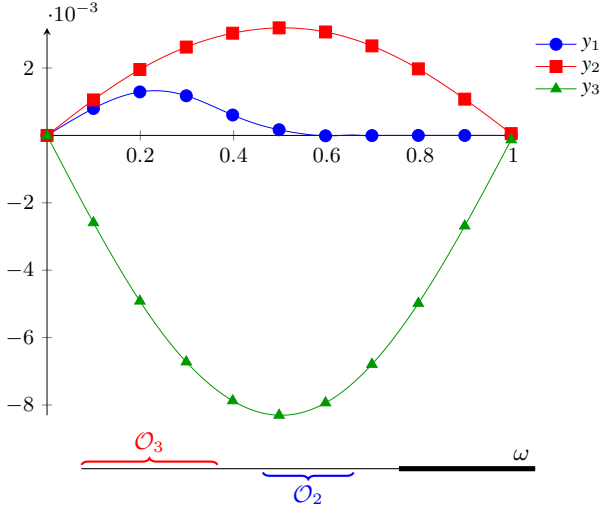
CASE 3: $\mathcal{O}_2$ and $\mathcal{O}_3$ are located in the same connected component of $\Omega \setminus \omega$



Simultaneous control of two 2×2 cascade systems

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

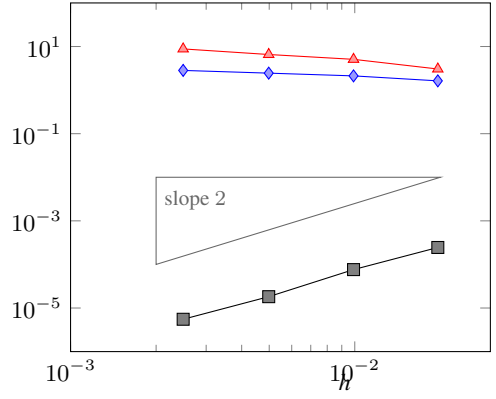
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Simultaneous control of two 2×2 cascade systems

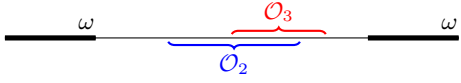
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 4.1: $\mathcal{O}_2 =]0.35, 0.65[$, $\mathcal{O}_3 =]0.5, 1/\sqrt{2}[$, $\omega =]0, 0.2[\cup]0.8, 1.0[$



Cost of the control
Size of the target
Optimal energy

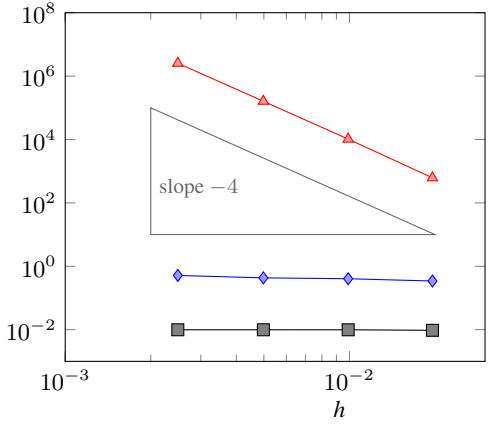
App. Cont. ✔
Null Cont. ?



Simultaneous control of two 2×2 cascade systems

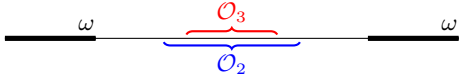
$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 4.2: $\mathcal{O}_2 =]0.35, 0.65[$, $\mathcal{O}_3 =]0.4, 0.6[$, $\omega =]0, 0.2[\cup]0.8, 1[$



Cost of the control
Size of the target
Optimal energy

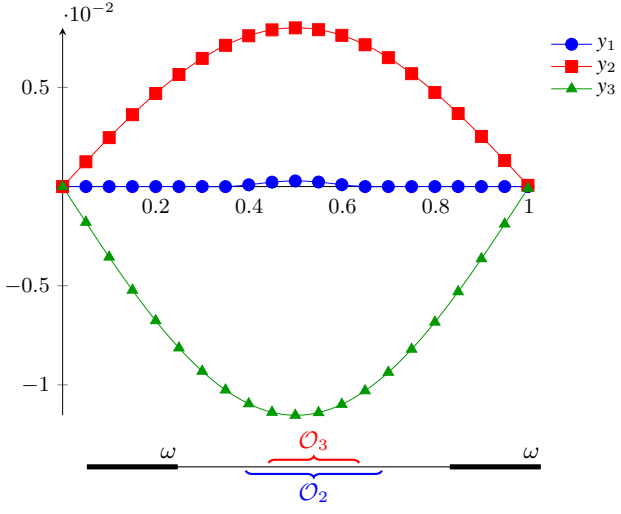
App. Cont.	✖
Null Cont.	✔for $T > T_{\min}$



Simultaneous control of two 2×2 cascade systems

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_2}(x) & 0 & 0 \\ \mathbf{1}_{\mathcal{O}_3}(x) & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

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A three equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

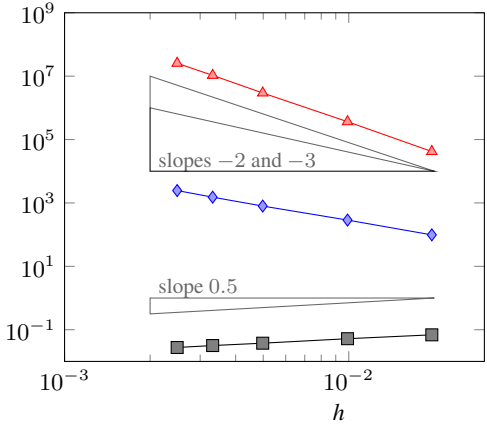
VERY SHORT REVIEW :

- If the supports of all the a_{ij} intersect the control domain ω and keeps a constant sign on a part of ω, then the system is null-controllable.
- Necessary and sufficient conditions for approximate controllability are known in the general case.

A three equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 1: $a_{21} = \mathbf{1}_{]0,0.5[}$, $a_{32} = 1$, $\omega =]0.5, 1[$.



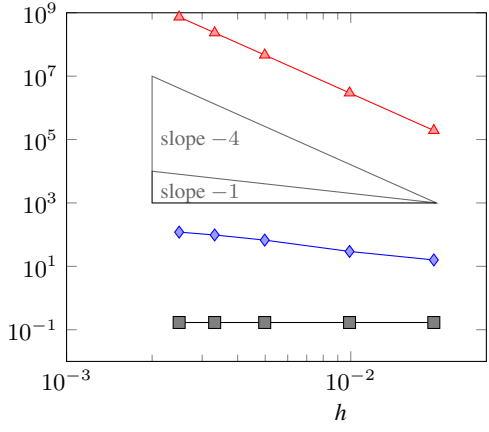
Cost of the control
Size of the target
Optimal energy

App. Cont. ✔
Null Cont. ?

A three equation cascade system

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 0 & 0 & 0 \\ a_{21}(x) & 0 & 0 \\ 0 & a_{32}(x) & 0 \end{pmatrix} y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{1}_\omega(x) v.$$

CASE 2: $a_{21} = \mathbf{1}_{]0,0.5[}$, $a_{32}(x) = x - 1/2$, $\omega =]0.5, 1[$.



Cost of the control
Size of the target
Optimal energy

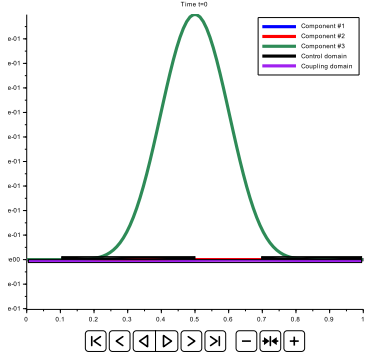
App. Cont. ✖
Null Cont. ✖

A 3 equation system with 2 disjoint controls

(Olive, '12)

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_{\omega_1}(x) & 0 \\ 0 & \mathbf{1}_{\omega_2}(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$y_0(x) = \left(0, 0, \sin(\pi x)^{10}\right)^t,\ \omega_1 =]0.7, 1.0[,\ \omega_2 =]0.1, 0.5[$$

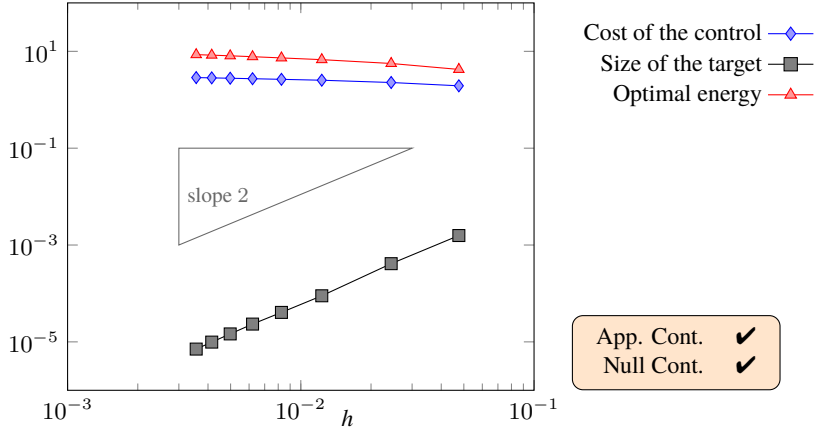


A 3 equation system with 2 disjoint controls

(Olive, '12)

$$\partial_t y - 0.1\partial_x^2 y + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_{\omega_1}(x) & 0 \\ 0 & \mathbf{1}_{\omega_2}(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

$$y_0(x) = \left(0, 0, \sin(\pi x)^{10}\right)^t,\ \omega_1 =]0.7, 1.0[,\ \omega_2 =]0.1, 0.5[$$



App. Cont. ✔
Null Cont. ✔

Outline

1. Introduction

2. Some facts about the Hilbert Uniqueness Method and its penalized version

3. The HUM approach in the discrete framework
3.1 The semi-discrete in space setting
3.2 Practical considerations

4. Discrete Carleman inequalities and applications
4.1 Introduction to the Lebeau-Robbiano approach
4.2 Discrete Lebeau-Robbiano inequality. Discrete elliptic Carleman estimate
4.3 Discrete parabolic Carleman estimate
4.4 The time discrete setting
4.5 The fully discrete setting

5. Numerical illustrations
5.1 1D scalar problems
5.2 1D Parabolic systems

6. Discrete spectral analysis and applications
6.1 Introduction to the moments method
6.2 Moments method for 1D semi-discrete scalar problem
6.3 Boundary control for 1D semi-discrete cascade system
6.4 Numerical illustrations
6.5 Multidimensional boundary control systems

7. Time discrete boundary control problems
7.1 Boundary control for 1D time-discrete scalar problems
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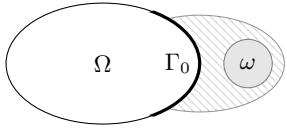
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7. Time discrete boundary control problems
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Remarks on boundary control problems

Scalar case

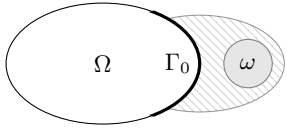
Distributed controllability $\Longrightarrow$ Boundary controllability



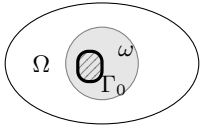
Remarks on boundary control problems

Scalar case

Distributed controllability $\Longrightarrow$ Boundary controllability



Boundary controllability $\Longrightarrow$ Distributed controllability



Remarks on boundary control problems

Scalar case

$$\text{Boundary controllability} \iff \text{Distributed controllability}$$

Remarks on boundary control problems

Scalar case

$$\text{Boundary controllability} \iff \text{Distributed controllability}$$

Non scalar case with less controls than equations

- Extension trick does not work anymore
- Lebeau-Robbiano method fails :

  No spectral inequality available for boundary observation operator

- Parabolic Carleman method fails :

  Elimination of an observation cannot be done on the boundary

Take home message

For some parabolic systems distributed controllability is no more equivalent to boundary controllability.

Remarks on boundary control problems

Scalar case

$$\text{Boundary controllability} \iff \text{Distributed controllability}$$

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Take home message

For some parabolic systems distributed controllability is no more equivalent to boundary controllability.

A "new" approach : the moments method

- Main advantages:
  - Gives sharp results in some situations where other approaches fail.
  - Handles the case where there is a minimal null-control time.
- Main drawbacks:
  - Rely on precise spectral analysis of the underlying operators
  - (For 2nd order parabolic equations) : seems to be restricted to the 1D case

The moments method

Reminder

(λ_k, ϕ_k) : eigenelements of $\mathcal{A} = \mathcal{A}^\star$ (let say a scalar elliptic operator)

Proposition

A function $v \in L^2(0,T;U)$ is a null-control for the problem

$$\partial_t y + \mathcal{A} y = \mathcal{B} v, \;\; y(0) = y_0,$$

if and only if

$$-\left\langle y_0, e^{-\lambda_k T}\phi_k \right\rangle = \int_0^T \left[v(t), e^{-\lambda_k (T-t)} \mathcal{B}^\star \phi_k \right] dt, \;\; \forall k \geq 1.$$

This is a **moment problem**.

Strategy : If we are able to build a family of functions $\overline{q_k} \in L^2(0,T;U)$ such that

$$\int_0^T \left[\overline{q_l}(t), e^{-\lambda_k (T-t)} \mathcal{B}^\star \phi_k \right] dt = \delta_{kl}, \;\; \forall k, l \geq 1,$$

then the control problem can be **formally** solved by

$$v(t) = -\sum_{l \geq 1} \left\langle y_0, e^{-\lambda_l T}\phi_l \right\rangle \overline{q_l}(t).$$

The moments method

Reminder

(Fattorini–Russell, '71-'74), (Benabdallah–B.–Gonzalez-Burgos–Olive, '14), (B., '26+)

Let $\Sigma = (\sigma_k)_k$ be an increasing sequence of distinct positive numbers. For a given ρ we consider the properties

$$\sigma_1 \geq \rho, \qquad \sigma_{k+1} - \sigma_k \geq \rho, \ \forall k \geq 1, \qquad \sigma_k \leq \frac{k^2}{\rho}, \ \forall k \geq 1. \tag{H_ρ}$$

Theorem (Biorthogonal families of exponential functions)

Assume that $\Sigma = (\sigma_k)_k$ satisfies (H_ρ) for some $\rho > 0$.

Then, for any $T > 0$, there exists a sequence of functions $(q_k)_{k\in\mathbb{N}} \subset L^2(0,T)$ such that

$$\begin{cases} \displaystyle\int_0^T e^{-(T-s)\sigma_l} q_k(s)\, ds = \delta_{kl}, \ \forall k, l \in \mathbb{N}, \\ \|q_k\|_{L^2(0,T)} \leq K e^{K\sqrt{\sigma_k} + \frac{K}{T}}, \ \forall k \geq 1, \end{cases}$$

where K depends on ρ but not on the particular sequence Σ.

Remarks

- **Good news :** Conditions (H_ρ) are satisfied in 1D for heat-like equations.
- **Bad news :** There are not satisfied in higher dimension.
- **Crucial point :** The estimates are uniform with respect to the family Σ.

The moments method

Reminder

(Fattorini–Russell, '71-'74), (Benabdallah–B.–Gonzalez-Burgos–Olive, '14), (B., '26+)

Let $\Sigma = (\sigma_k)_k$ be an increasing sequence of distinct positive numbers. For a given ρ we consider the properties

$$\sigma_1 \geq \rho, \qquad \sigma_{k+1} - \sigma_k \geq \rho, \ \forall k \geq 1, \qquad \sigma_k \leq \frac{k^2}{\rho}, \ \forall k \geq 1. \tag{H_ρ}$$

Theorem (Biorthogonal families of exponential-kind functions)

Assume that $\Sigma = (\sigma_k)_k$ satisfies (H_ρ) for some $\rho > 0$. Let $\eta \in \mathbb{N}^*$ fixed.

Then, for any $T > 0$, there exists a sequence of functions $(q_{k,j})_{\substack{k\in\mathbb{N} \\ j\in\{0,\dots,\eta\}}} \subset L^2(0,T)$ such that

$$\begin{cases} \displaystyle\int_0^T (T-s)^i e^{-(T-s)\sigma_l} q_{k,j}(s)\, ds = \delta_{kl}\delta_{ij}, \ \forall k,l \in \mathbb{N}, \forall i,j \in \{0,\dots,\eta\}, \\ \|q_{k,j}\|_{L^2(0,T)} \leq K e^{K\sqrt{\sigma_k} + \frac{K}{T}}, \ \ \forall k \geq 1, \forall j \in \{0,\dots,\eta\}, \end{cases}$$

where K depends on ρ, η but not on the particular sequence Σ.

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The moments method

Reminder

What we want

$$\int_0^T \left[\overline{q_l}(t), e^{-\lambda_k(T-t)}\mathcal{B}^\star\phi_k\right] dt = \delta_{kl}, \quad \forall k, l \geq 1,$$

$$\text{and set } \; v = -\sum_{l\geq 1} \left\langle y_0, e^{-\lambda_l T}\phi_l \right\rangle \overline{q_l}.$$

Construction

We define

$$\overline{q_k}(t) = q_k(t)\frac{\mathcal{B}^\star\phi_k}{[\![\mathcal{B}^\star\phi_k]\!]^2} \in U,$$

so that $[\![\overline{q_k}]\!]_{L^2(0,T;U)} = \|q_k\|_{L^2(0,T)}\,[\![\mathcal{B}^\star\phi_k]\!]^{-1}$.

Distributed control

$U = L^2(\Omega)$, $\qquad \mathcal{B}^\star\phi_k = 1_\omega\phi_k$

$$[\![\mathcal{B}^\star\phi_k]\!] = \|\phi_k\|_{L^2(\omega)} \geq C_\omega, \quad \forall k \geq 1.$$

Boundary control

$U = \mathbb{R}$, $\qquad \mathcal{B}^\star\phi_k = \gamma(1)\phi_k'(1)$

$$[\![\mathcal{B}^\star\phi_k]\!] = |\gamma(1)\phi_k'(1)| \geq C\sqrt{\lambda_k} \geq C, \quad \forall k \geq 1.$$

The moments method

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Conclusion in both cases:

We have

$$\left[\!\left[\left\langle y_0, e^{-\lambda_l T} \phi_l \right\rangle \overline{q_l} \right]\!\right]_{L^2(0,T;U)} \leq C e^{-\lambda_l T} e^{K\sqrt{\lambda_l} + \frac{K}{T}} \leq C e^{-\lambda_l T/2} e^{\frac{K'}{T}},$$

and thus the series defining v converges and we have

$$\|v\|_{L^2(0,T,U)} \leq K e^{\frac{K}{T}}$$

Outline

1. Introduction

2. Some facts about the Hilbert Uniqueness Method and its penalized version

3. The HUM approach in the discrete framework
3.1 The semi-discrete in space setting
3.2 Practical considerations

4. Discrete Carleman inequalities and applications
4.1 Introduction to the Lebeau-Robbiano approach
4.2 Discrete Lebeau-Robbiano inequality. Discrete elliptic Carleman estimate
4.3 Discrete parabolic Carleman estimate
4.4 The time discrete setting
4.5 The fully discrete setting

5. Numerical illustrations
5.1 1D scalar problems
5.2 1D Parabolic systems

6. Discrete spectral analysis and applications
6.1 Introduction to the moments method
6.2 Moments method for 1D semi-discrete scalar problem
6.3 Boundary control for 1D semi-discrete cascade system
6.4 Numerical illustrations
6.5 Multidimensional boundary control systems

7. Time discrete boundary control problems
7.1 Boundary control for 1D time-discrete scalar problems
7.2 Boundary control for 1D time-discrete cascade systems

The 1D scalar case : Moments method at the discrete level

Framework

$$(S_h) \quad \begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h \\ y_h(0) = y_{0,h} \in E_h, \end{cases}$$

- $\mathcal{A}_h$ is the finite difference discretization operator

$$(\mathcal{A}_h y)_i = -\frac{1}{h}\left(\gamma_{i+1/2}\frac{y_{i+1}-y_i}{h} - \gamma_{i-1/2}\frac{y_i - y_{i-1}}{h}\right), \quad \forall i \in \{1,...,N\},$$
$$y_0 = y_{N+1} = 0,$$

  with $\gamma_{i+1/2} = \gamma(x_{i+1/2})$, $x_{i+1/2} = (x_i + x_{i+1})/2$.
- $\mathcal{B}_h$ is the discrete control operator

$$\text{either } \mathcal{B}_h = \mathbf{1}_\omega = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix} \quad \text{or} \quad \mathcal{B}_h = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\gamma_{N+1/2}}{h^2} \end{pmatrix}$$

- Note that the discrete observation operator $\mathcal{B}_h^\star$ is either $\mathcal{B}_h^\star = \mathbf{1}_\omega$ or $\mathcal{B}_h^\star = -\gamma\partial_r$

  Indeed in the boundary control case : $\mathcal{B}_h^\star y_h = \gamma_{N+1/2}\frac{y_N}{h} = -\gamma_{N+1/2}\frac{0-y_N}{h} = -\gamma_{N+1/2}\frac{y_{N+1}-y_N}{h}$.

Remark : For any fixed h the system (S_h) **is** controllable !

The 1D scalar case : Moments method at the discrete level

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- $\mathcal{B}_h$ is the discrete control operator

Remark : For any fixed h the system (S_h) **is** controllable !

Theorem (Uniform $\phi(h)$-null controllability)

There exists $C > 0$ such that for any ϕ satisfying

$$\liminf_{h \to 0} \frac{\phi(h)}{e^{-C/h^2}} > 0,$$

there exists $\bar{C} > 0$, $h_0 > 0$ such that for any $h < h_0$, any $y^{0,h}$, there exists a $v_h \in L^2(0,T,U_h)$ such that

$$\|v_h\|_{L^2(0,T;U_h)} \leq \bar{C} \|y^{0,h}\|_h, \quad \text{and} \quad \|y^h(T)\|_h \leq \bar{C} \|y^{0,h}\|_h \sqrt{\phi(h)}.$$

The 1D scalar case : Moments method at the discrete level

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$$(S_h) \quad \begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h \\ y_h(0) = y_{0,h} \in E_h, \end{cases}$$

Tools : Biotho. families + discrete spectral properties for $\mathcal{A}_h$

(Allonsius-B.-Morancey, '18)

- Uniform growth rate for eigenvalues $\lambda_k^h \geq Ck^2, \ \forall k, \forall h.$
- Uniform spectral gap $\lambda_{k+1}^h - \lambda_k^h \geq \rho, \ \forall h, \ \forall k \leq \dfrac{C_1}{h}.$
- Uniform lower bounds for discrete eigenfunctions

$$\|\mathcal{B}_h^\star \phi_k^h\| = \|\phi_k^h\|_{L^2(\omega)} \geq C, \ \forall h, \forall k \leq \frac{C_1}{h} \text{ for distributed control,}$$

$$\|\mathcal{B}_h^\star \phi_k^h\| = |\partial_r \phi_k^h| \geq C, \ \forall h, \forall k \leq \frac{C_1}{h} \text{ for boundary control.}$$

Remarks

- Those properties are not straightforward for general diffusion coefficients on general grids.
- Our results are uniform for a constant portion of the spectrum.
- Optimal value of C_1 is somehow explicit.

The 1D scalar case : Moments method at the discrete level

Framework

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An explicit case : the discrete Laplace operator on uniform grid

$$\mathcal{A}_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Explicit formulas for any $1 \leq k \leq N$, $h = \frac{1}{N+1}$

$$\phi_{k,h} = (\sin(k\pi x_i))_i, \ \forall i \in \{1, ..., N\},$$

$$\lambda_{k,h} = \frac{4\sin^2\left(\frac{k\pi h}{2}\right)}{h^2}, \ \forall k \in \{1, ..., N\}.$$

$\rightsquigarrow$ The above properties can be proved by hand.

The 1D scalar case : Moments method at the discrete level

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$$(S_h) \quad \begin{cases} \partial_t y_h + \mathcal{A}_h y_h = \mathcal{B}_h v_h \\ y_h(0) = y_{0,h} \in E_h, \end{cases}$$

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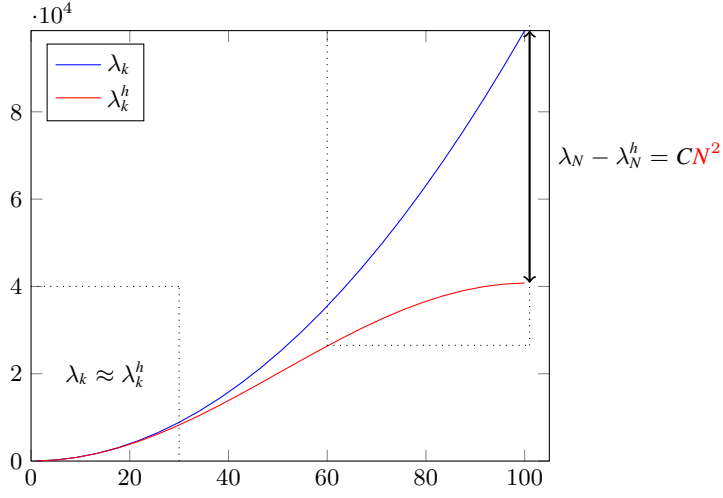
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Discrete spectral properties

Laplace spectrum vs Discrete Laplace spectrum



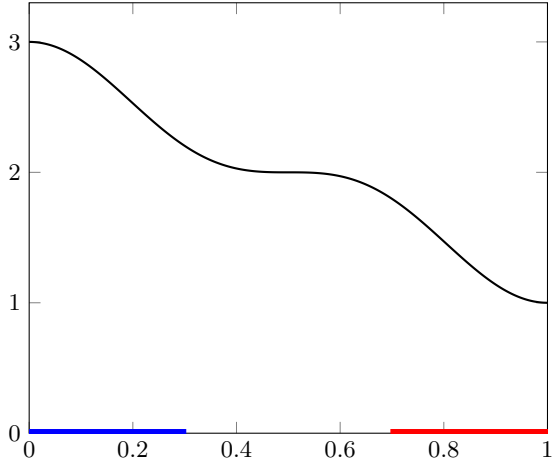
TYPICAL ERROR ESTIMATE: $|\lambda_k^h - \lambda_k| \approx Ch^2\lambda_k^2 \approx Ch^2k^4$ $\rightsquigarrow$ Useless for large k

HOWEVER : Uniform discrete gap holds $\inf_{k \leq N} |\lambda_{k+1}^h - \lambda_k^h| \approx C$

Discrete spectral properties

Example

Diffusion coefficient : $\gamma(x) = 2 + \cos(\pi x)^3$

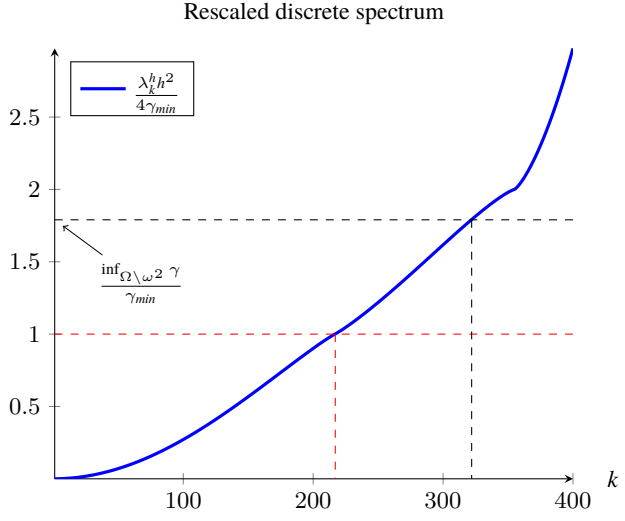


Two observation domains : $\omega^1 = (0, 0.3), \omega^2 = (0.7, 1)$.

Discrete spectral properties

Example

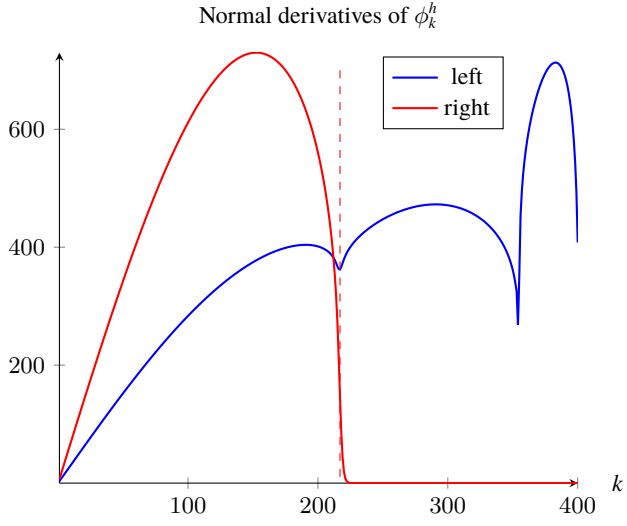
DIFFUSION COEFFICIENT : $\gamma(x) = 2 + \cos(\pi x)^3$



Discrete spectral properties

Example

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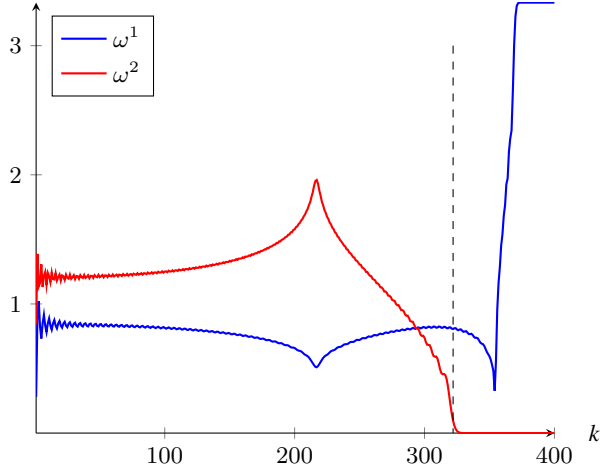


Discrete spectral properties

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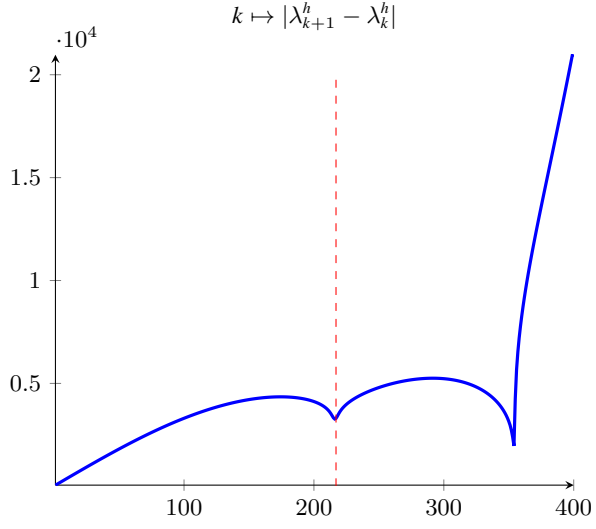
$$k \mapsto \|\phi_k^h\|^2_{L^2(\omega^i)}/|\omega^i|$$



Discrete spectral properties

Example

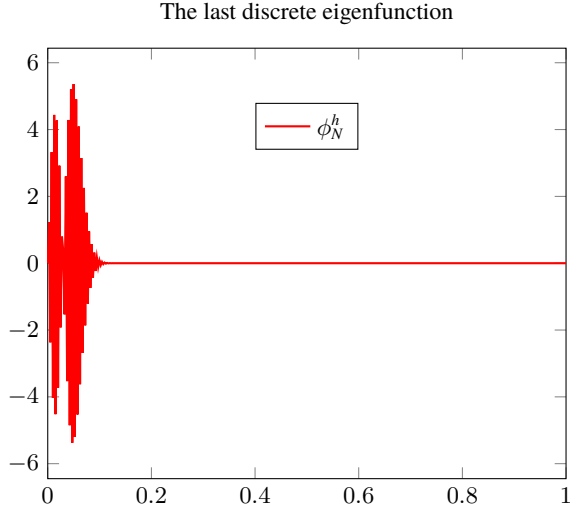
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Discrete spectral properties

Example

DIFFUSION COEFFICIENT : $\gamma(x) = 2 + \cos(\pi x)^3$



Discrete spectral properties

(Allonsius-B.-Morancey, '18)

Theorem (Uniform lower bounds for discrete eigenfunctions)

Assume some regularity of the mesh and of the diffusion coefficient. There exists $C_1, C_2 > 0$ such that

$$\|\mathcal{B}_h^\star \phi_k^h\| = \|\phi_k^h\|_{L^2(\omega)} \geq C_2, \qquad \forall h, \forall k \leq \frac{C_1}{h} \text{ in the distributed control case,}$$

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Theorem (Uniform spectral gap)

Assume some regularity of the mesh and of the diffusion coefficient. There exists $C_1, \rho > 0$ such that

$$\lambda_{k+1}^h - \lambda_k^h \geq \rho \sqrt{\lambda_k^h}, \quad \forall h, \quad \forall k \leq \frac{C_1}{h}.$$

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Lemma (Main tool)

There exists $C_1, C_2 > 0$ such that, for any $\lambda > 0$ and any solution u^h of $\mathcal{A}_h u^h = \lambda u^h$, with $\left\|u^h\right\|_h = 1$, we have

$$\left(|u_i|^2 + \frac{\gamma_{i-1/2}}{\lambda} \left| \frac{u_i - u_{i-1}}{h} \right|^2 \right) \geq C_1, \quad \forall i \in \{1, ..., N+1\}, \qquad \text{as soon as } \lambda h^2 \leq C_2.$$

- Boundary observation : apply the Lemma to $u^h = \phi_k^h$ and take $i = N + 1$.
- Distributed observation : apply the Lemma to $u^h = \phi_k^h$ and sum over i s.t. $x_i \in \omega$ ...

Discrete spectral properties

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Lemma (Main tool - continuous case)

There exists $C > 0$ such that for any $\lambda > 0$ and any u solution of the ODE $-\partial_x(\gamma(x)\partial_x u) = \lambda u$ with $\|u\|_{L^2} = 1$, we have

$$\left(|u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 \right) \geq C, \quad \forall y \in (0,1).$$

A sketch of proof of the Lemma

$$\begin{cases} -\partial_x(\gamma(x)\partial_x u) = \lambda u, \\ \|u\|_{L^2(0,1)} = 1 \end{cases}$$

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$$U'(x) = M_\lambda(x)U(x) + Q(x)U(x),$$

$$M_\lambda(x) = \begin{pmatrix} 0 & \frac{\sqrt{\lambda}}{\sqrt{\gamma(x)}} \\ -\frac{\sqrt{\lambda}}{\sqrt{\gamma(x)}} & 0 \end{pmatrix}, \quad Q \text{ ind. of } \lambda.$$

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- $M_\lambda(x)$ is anti-symmetric
- $M_\lambda(x)$ and $M_\lambda(y)$ commute.
- The resolvant associated to M_λ is

$$S_\lambda(y,x) = \exp\left(\int_x^y M_\lambda(t)\,dt\right),$$

and satisfies $\|S_\lambda(x,y)\| = 1$.

A sketch of proof of the Lemma

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and satisfies $\|S_\lambda(x,y)\| = 1$.

$\Rightarrow$ for some $C > 0$ independent of λ

$$|u(x)|^2 \le \|U(x)\|^2 \le C\|U(y)\|^2, \quad \forall x, y \in (0,1)$$

By integration with respect to x we get

$$1 \le C\left(|u(y)|^2 + \frac{\gamma(y)}{\lambda}|u'(y)|^2\right), \quad \forall y$$

A sketch of proof of the Lemma

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$$1 \le C\left(|u(y)|^2 + \frac{\gamma(y)}{\lambda}|u'(y)|^2\right), \quad \forall y$$

$$-\frac{1}{h}\left(\gamma_{i+1/2}\frac{u_{i+1}-u_i}{h} - \gamma_{i-1/2}\frac{u_i - u_{i-1}}{h}\right) = \lambda u_i$$

A sketch of proof of the Lemma

$$\begin{cases} -\partial_x(\gamma(x)\partial_x u) = \lambda u, \\ \|u\|_{L^2(0,1)} = 1 \end{cases}$$

$$U(x) = \begin{pmatrix} u(x) \\ \sqrt{\frac{\gamma(x)}{\lambda}}u'(x) \end{pmatrix},$$

$$U'(x) = M_\lambda(x)U(x) + Q(x)U(x),$$

$$M_\lambda(x) = \begin{pmatrix} 0 & \frac{\sqrt{\lambda}}{\sqrt{\gamma(x)}} \\ -\frac{\sqrt{\lambda}}{\sqrt{\gamma(x)}} & 0 \end{pmatrix}, \quad Q \text{ ind. of } \lambda.$$

- $M_\lambda(x)$ is anti-symmetric
- $M_\lambda(x)$ and $M_\lambda(y)$ commute.
- The resolvant associated to M_λ is

$$S_\lambda(y,x) = \exp\left(\int_x^y M_\lambda(t)\,dt\right),$$

and satisfies $\|S_\lambda(x,y)\| = 1$.

$\Rightarrow$ for some $C > 0$ independent of λ

$$|u(x)|^2 \le \|U(x)\|^2 \le C\|U(y)\|^2, \quad \forall x, y \in (0,1)$$

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- $M^h_{\lambda,i}$ is **not** anti-symmetric
- $M^h_{\lambda,i}$ and $M^h_{\lambda,j}$ **do not** commute.
- The discrete resolvant associated to $M^h_{\lambda,\cdot}$ is

$$S^h_\lambda(i,j) = (I + hM^h_{\lambda,i}) \cdots (I + hM^h_{\lambda,j-1}).$$

- We proved that : for some $C, C_1 > 0$, if $h^2\lambda < C_1$

$$\|S^h_\lambda(i,j)\| \le C, \quad \forall i, j.$$

A sketch of proof of the Lemma

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- We proved that : for some $C, C_1 > 0$, if $h^2\lambda < C_1$

$$\|S^h_\lambda(i,j)\| \le C, \quad \forall i,j.$$

$\Rightarrow$ for some $C > 0$ independent of λ and h (with $h^2\lambda < \alpha$)

$$|u_i|^2 \le \|U_i\|^2 \le C\|U_j\|^2, \quad \forall i,j.$$

By summation, we get

$$1 \le C\left(|u_j|^2 + \frac{\gamma_{j-1/2}}{\lambda}\left|\frac{u_j-u_{j-1}}{h}\right|^2\right), \forall j.$$

Outline

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A very basic introduction to tensor products

To deal with Cartesian structures (multi-D, systems) the tensor product formalism is quite useful.

MATRICES AS TENSOR PRODUCTS : $M_{n,m}(\mathbb{R}) \sim \mathbb{R}^n \otimes \mathbb{R}^m$.

- For any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ we denote $x \otimes y$ to be the rank one matrix $x.\,{}^t y \in M_{n,m}(\mathbb{R})$.
- Properties :

$$(\lambda x_1 + x_2) \otimes y = \lambda(x_1 \otimes y) + (x_2 \otimes y),$$
$$x \otimes (\lambda y_1 + y_2) = \lambda(x \otimes y_1) + (x \otimes y_2).$$

$M_{n,m}(\mathbb{R})$ is generated by the elementary tensor products.

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VARIOUS WAYS TO EXPRESS A MATRIX

$$M = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij} e_i \otimes e_j, \qquad \text{using all the entries,}$$

$$M = \sum_{i=1}^{n} e_i \otimes r_i, \qquad \text{using the lines=rows } r_i \in \mathbb{R}^m,$$

$$M = \sum_{j=1}^{m} c_j \otimes e_j, \qquad \text{using the columns } c_j \in \mathbb{R}^n,$$

$$M = \sum_{j=1}^{m} \hat{m}_j \otimes \phi_j, \qquad \text{using a special basis } (\phi_j)_j \text{ of } \mathbb{R}^n,$$

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OPERATORS ON MATRICES AS TENSOR PRODUCTS

- Let $A_1 \in L(\mathbb{R}^n, \mathbb{R})$ and $A_2 \in L(\mathbb{R}^m, \mathbb{R})$ be two linear forms. We define $A_1 \otimes A_2 \in L(\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R})$ by
$$(A_1 \otimes A_2)(x \otimes y) = (A_1 x)(A_2 y), \ \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m, \text{ extended by linearity.}$$

- More generally, if $A_1 \in L(\mathbb{R}^n, \mathbb{R}^p)$ and $A_2 \in L(\mathbb{R}^m, \mathbb{R}^q)$ we set $A_1 \otimes A_2 \in L(\mathbb{R}^n \otimes \mathbb{R}^m, \mathbb{R}^p \otimes \mathbb{R}^q)$
$$(A_1 \otimes A_2)(x \otimes y) = (A_1 x) \otimes (A_2 y), \ \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m, \text{ extended by linearity.}$$

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TYPICAL EXAMPLE 1 : $\mathbb{R}^2$-valued discrete unknowns on a 1D grid $\rightsquigarrow E_h \otimes \mathbb{R}^2$

Let $\mathcal{A}_h$ be the (scalar) discrete Laplace operator on E_h.

$\mathcal{A}_h \otimes \mathrm{Id}_2$ is the discrete Laplace operator applied component wise to a 1D unknown.

$\mathcal{A}_h \otimes \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is a discrete diffusion operator with different diffusions.

$\mathcal{A}_h \otimes \mathrm{Id}_2 + \mathrm{Id}_h \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a discrete Laplace operator + coupling between components.

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TYPICAL EXAMPLE 2 : Operators on a 2D Cartesian grid $\rightsquigarrow E_h \otimes E_k$

Let $\mathcal{A}_h$ be the discrete Laplace operator on E_h, $\mathcal{A}_k$ be the discrete Laplace operator on E_k

$\mathcal{A}_h \otimes \mathrm{Id}_k$ is the discrete $-\partial_x^2$ operator applied to a 2D unknown.

$\mathrm{Id}_h \otimes \mathcal{A}_k$ is the discrete $-\partial_y^2$ operator applied to a 2D unknown.

$(\mathcal{A}_h \otimes \mathrm{Id}_k) + (\mathrm{Id}_h \otimes \mathcal{A}_k)$ is the 2D discrete Laplace operator.

1D cascade system with boundary control

Question

Can we prove the uniform **boundary** $\phi(h)$-null controllability of the following **1D** semi-discrete coupled system ?

$$\begin{cases} \partial_t \alpha_i + \dfrac{-\alpha_{i-1} + 2\alpha_i - \alpha_{i+1}}{h^2} = 0, & \forall 1 \leq i \leq N, \\ \partial_t \beta_i + \dfrac{-\beta_{i-1} + 2\beta_i - \beta_{i+1}}{h^2} + \alpha_i = 0, & \forall 1 \leq i \leq N, \\ \beta_0 = \alpha_{N+1} = \beta_{N+1} = 0, \\ \alpha_0 = v(t). \end{cases}$$

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More compact form $\alpha^h = (\alpha_i)_i, \beta^h = (\beta_i)_i \in E_h$

$$\partial_t \begin{pmatrix} \alpha^h \\ \beta^h \end{pmatrix} + \begin{pmatrix} \mathcal{A}_h & 0 \\ 0 & \mathcal{A}_h \end{pmatrix} \begin{pmatrix} \alpha^h \\ \beta^h \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{E_h} & 0 \end{pmatrix} \begin{pmatrix} \alpha^h \\ \beta^h \end{pmatrix} = \begin{pmatrix} \mathcal{B}_h v \\ 0 \end{pmatrix}.$$

$$\mathcal{A}_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & & & & \\ 0 & & & & 0 \\ \vdots & & & & -1 \\ 0 & & \cdots & 0 & -1 \quad 2 \end{pmatrix} \qquad \mathcal{B}_h = \frac{1}{h^2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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Even more compact form using tensor products $y^h = \alpha^h \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta^h \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in E_h \otimes \mathbb{R}^2$

$$\partial_t y^h + \mathcal{A}_h \otimes \mathrm{Id}_2\, y^h + \mathrm{Id}_{E_h} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y^h = \mathcal{B}_h \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(t).$$

$$\mathcal{A}_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & & & & \\ 0 & & & & 0 \\ \vdots & & & & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \qquad \mathcal{B}_h = \frac{1}{h^2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

1D cascade system with boundary control

The general framework

- State space $E_h \otimes \mathbb{R}^n$, coupling matrix $C \in M_n(\mathbb{R})$.
- Underlying elliptic operators

$$\mathcal{L}_h = \mathcal{A}_h \otimes \mathrm{Id}_n + \mathrm{Id}_{E_h} \otimes C, \quad \text{and its adjoint} \quad \mathcal{L}_h^\star = \mathcal{A}_h \otimes \mathrm{Id}_n + \mathrm{Id}_{E_h} \otimes C^\star.$$

- Control space $U_h = \mathbb{R} \otimes \mathbb{R}^m \approx \mathbb{R}^m$, algebraic localization of the controls $B \in M_{n,m}(\mathbb{R})$.
- Control operator

$$\mathcal{B}_h \otimes B : \mathbb{R} \otimes \mathbb{R}^m \longrightarrow E_h \otimes \mathbb{R}^n.$$

Example of the cascade system $m = 1$

$$B = \begin{pmatrix} 1 \\ 0 \\ | \\ | \\ 0 \end{pmatrix} \in \mathbb{R}^n, \quad C = \begin{pmatrix} 0 & \text{———} & 0 \\ 1 & \diagdown & | \\ 0 & \diagdown & | \\ | & \diagdown & | \\ 0 & \text{———} & 0\ 1\ 0 \end{pmatrix} \in M_n(\mathbb{R}).$$

1D cascade system with boundary control

$$\mathcal{L}_h^\star = \mathcal{A}_h \otimes \mathrm{Id}_n + \mathrm{Id}_{E_h} \otimes C^\star$$

$$B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \iddots & \vdots \\ 0 & \iddots & 0 \\ \vdots & \iddots & 1 \\ 0 & \cdots & 0 \end{pmatrix}, \quad B^\star = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad C^\star = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \iddots & 0 \\ \vdots & \iddots & \iddots & \iddots & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

Spectral decomposition of $\mathcal{L}_h^\star$

Each eigenvalue λ_k^h is of geometric (resp. algebraic) multiplicity 1 (resp. n).

$$\begin{cases} \mathcal{L}_h^\star(\phi_k^h \otimes e_1) = \lambda_k^h(\phi_k^h \otimes e_1), \\ \mathcal{L}_h^\star(\phi_k^h \otimes e_2) = \lambda_k^h(\phi_k^h \otimes e_2) + (\phi_k^h \otimes e_1), \\ \qquad\qquad \vdots \\ \mathcal{L}_h^\star(\phi_k^h \otimes e_n) = \lambda_k^h(\phi_k^h \otimes e_n) + (\phi_k^h \otimes e_{n-1}), \end{cases}$$

Adjoint semigroup expression

$$e^{-t\mathcal{L}_h^\star}(\phi_k^h \otimes e_i) = e^{-t\lambda_k^h} e^{-t(\mathcal{L}_h^\star - \lambda_k^h)}(\phi_k^h \otimes e_i) = e^{-t\lambda_k^h} \sum_{j=1}^{i} \frac{(-t)^{i-j}}{(i-j)!} \phi_k^j \otimes e_j.$$

1D cascade system with boundary control

$$\mathcal{L}_h^\star = \mathcal{A}_h \otimes \mathrm{Id}_n + \mathrm{Id}_{E_h} \otimes C^\star$$

$$B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & & & & 0 \\ 1 & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & & 0 & 1 & 0 \end{pmatrix}, \quad B^\star = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad C^\star = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

We test the control problem against each eigenelement $\phi_k^h \otimes e_i$

$$-\langle e^{-T\mathcal{L}_h} y_0^h, \phi_k^h \otimes e_i \rangle_{E_h} = \int_0^T \langle (\mathcal{B}_h \otimes B) v_h(s), e^{-(T-s)\mathcal{L}_h^\star} \phi_k^h \otimes e_i \rangle_{E_h} \, ds.$$

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$$\mathcal{L}_h^\star = \mathcal{A}_h \otimes \mathrm{Id}_n + \mathrm{Id}_{E_h} \otimes C^\star$$

$$B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & & & & 0 \\ 1 & \ddots & & \iddots & \vdots \\ 0 & \ddots & \iddots & & \\ \vdots & \iddots & & & \\ 0 & & 0 & 1 & 0 \end{pmatrix}, \quad B^\star = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad C^\star = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ \vdots & & \ddots & \iddots & \vdots \\ & \iddots & & & 0 \\ & & & \iddots & 1 \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

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$$-\langle e^{-T\mathcal{L}_h} y_0^h, \phi_k^h \otimes e_i \rangle_{E_h} = \int_0^T \left\langle v_h(s), e^{-(T-s)\lambda_k^h} \sum_{j=1}^{i} \frac{(s-T)^{i-j}}{(i-j)!} (\mathcal{B}_h^\star \phi_k^h) \otimes (B^* e_j) \right\rangle_{U_h} ds.$$

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We test the control problem against each eigenelement $\phi_k^h \otimes e_i$

$$\underbrace{-\langle e^{-T\mathcal{L}_h} y_0^h, \phi_k^h \otimes e_i \rangle_{E_h}}_{:= v_{k,i}^h} = (\mathcal{B}_h^\star \phi_k^h) \int_0^T v_h(s) e^{-(T-s)\lambda_k^h} \frac{(s-T)^{i-1}}{(i-1)!}\, ds.$$

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$$\mathcal{L}_h^\star = \mathcal{A}_h \otimes \mathrm{Id}_n + \mathrm{Id}_{E_h} \otimes C^\star$$

$$B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \iddots & \vdots \\ 0 & \iddots & 0 \\ \vdots & \iddots & 1 \\ 0 & \cdots & 0 \end{pmatrix}, \quad B^\star = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, \quad C^\star = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ & & \iddots & \iddots & 0 \\ & & & \iddots & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}.$$

We test the control problem against each eigenelement $\phi_k^h \otimes e_i$

$$\underbrace{-\langle e^{-T\mathcal{L}_h} y_0^h, \phi_k^h \otimes e_i \rangle_{E_h}}_{:= v_{k,i}^h} = (\mathcal{B}_h^\star \phi_k^h) \int_0^T v_h(s) e^{-(T-s)\lambda_k^h} \frac{(s-T)^{i-1}}{(i-1)!}\, ds.$$

We consider now a biorthogonal family $(q_{l,j}^h)_{l,j}$ a biorthogonal family in $L^2(0,T)$ of the family of functions

$$\left(s \mapsto e^{-(T-s)\lambda_k^h} \frac{(-(T-s))^{i-1}}{(i-1)!} \right)_{k,i}.$$

We obtain a suitable relaxed control $v_h(s) = \sum_{\lambda_l^h \leq C/h^2} \frac{\mathcal{B}_h^\star \phi_l^h}{\|\mathcal{B}_h^\star \phi_l^h\|^2} \sum_{j=0}^{n-1} v_{l,j}^h q_{l,j}^h(s)$ with appropriate bound.

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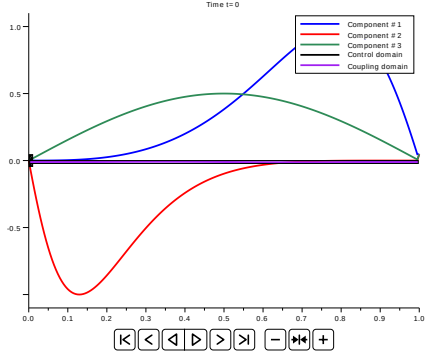
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A 3 equation system with mixed boundary/distributed controls

(Olive, '14)

$$\partial_t y - \partial_x^2 y + \begin{pmatrix} -2 & -6 & -2 \\ -4 & 0 & 2 \\ -2 & 3/2 & -2 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ \mathbf{1}_\omega(x) \end{pmatrix} \cancel{v_1}, \quad y(t,0) = \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix}$$

$$y_0(x) = \left(\sin(\pi x^3), -\sin(\pi(1-x)^5), 0.5\sin(\pi x)\right)^t, \ \omega =]0.3, 0.7[,$$

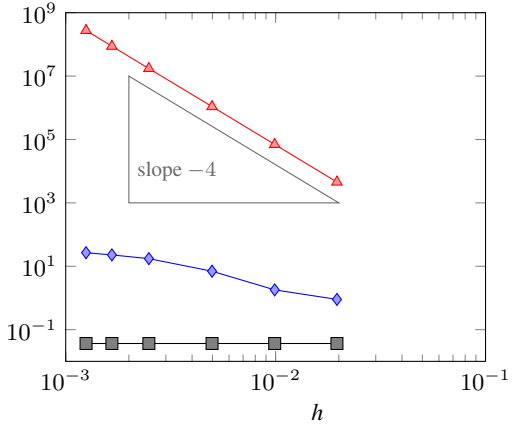


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Cost of the control
Size of the target
Optimal energy

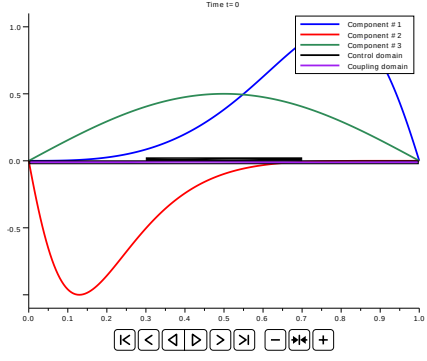
App. Cont. ✖
Null Cont. ✖

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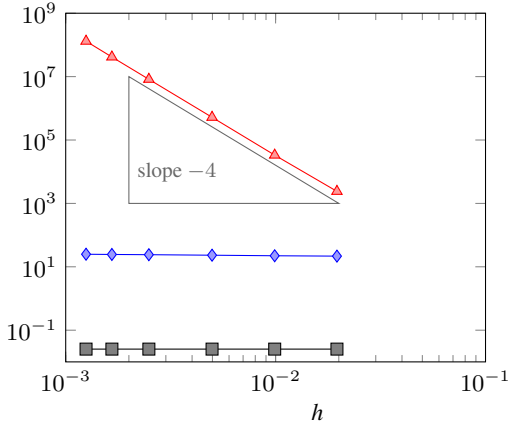


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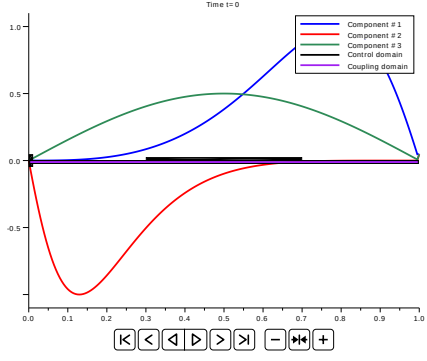
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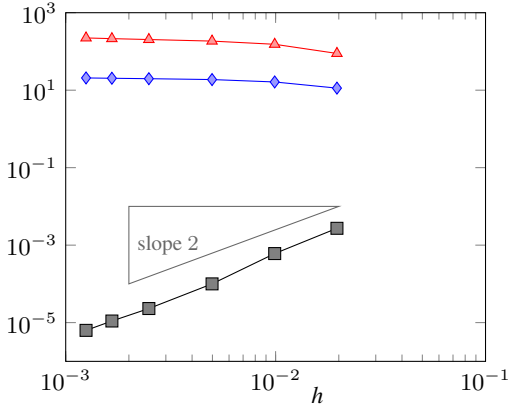


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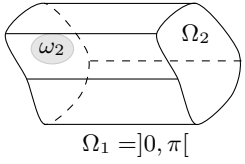
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Multi-D cascade system with boundary control

$$(S_B)\begin{cases}\partial_t y-\Delta y+Cy=0, & \text{in } \Omega=]0,\pi[\times\Omega_2\\ y=\mathbf{1}_{\{0\}\times\omega_2}Bv, & \text{on } \partial\Omega.\end{cases}$$



THEOREM ((Benabdallah–B.–Gonzalez-Burgos-Olive, '14))

System (S_B) is null-controllable at time $T>0$ if and only if the 1D system

$$(S_B^{1D})\begin{cases}\partial_t y-\Delta y+Cy=0, & \text{in } \Omega_1=]0,\pi[\\ y=1_{\{0\}}Bv, & \text{on } \partial\Omega_1,\end{cases}$$

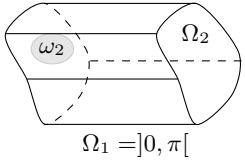
is null-controllable.

OK FOR THE USUAL CASCADE EXAMPLE

$$B=\begin{pmatrix}1\\0\\|\\0\end{pmatrix},\quad C=\begin{pmatrix}0&&&&0\\1&\ddots&&&|\\0&&\ddots&&|\\|&&&\ddots&|\\0&0&1&0&0\end{pmatrix}.$$

Multi-D cascade system with boundary control

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SKETCH OF PROOF :

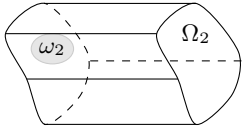
- Use the time slicing procedure *a la Lebeau-Robbiano* in Ω_2 on a sequence of **infinite dimensional subspaces**.
- Each step of the procedure needs to solve a 1D control problem in Ω_1 that looks like (S_B^{1D}) with a **suitable estimate** of the control cost in $e^{C/T}$.

MAIN INGREDIENT OF THIS STRATEGY

Take advantage of the Cartesian product structure of the domain Ω and of the underlying operator !

Multi-D cascade system with boundary control

$$(S_B)\begin{cases}\partial_t y + (-\Delta_1)\otimes \mathrm{Id}\otimes \mathrm{Id}\, y + \mathrm{Id}\otimes(-\Delta_2)\otimes \mathrm{Id}\, y + \mathrm{Id}\otimes \mathrm{Id}\otimes C\, y \\ \qquad\qquad = (\Delta_1 x_1)\otimes\big((1_{\omega_2}\otimes B)v\big), \ \text{in } \Omega\end{cases}$$



THEOREM ((Benabdallah–B.–Gonzalez-Burgos-Olive, '14))

System (S_B) is null-controllable at time $T > 0$ if and only if the 1D system

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is null-controllable.

Main objective

Question

Can we prove the uniform **boundary** $\phi(h)$-null controllability of the following **2D** semi-discrete coupled system ?

$$\begin{cases} \partial_t \alpha_{i,j} + \dfrac{-\alpha_{i-1,j} + 2\alpha_{i,j} - \alpha_{i+1,j}}{h^2} + \dfrac{-\alpha_{i,j-1} + 2\alpha_{i,j} - \alpha_{i,j+1}}{h^2} = 0, \quad \forall i,j, \\ \partial_t \beta_{i,j} + \dfrac{-\beta_{i-1,j} + 2\beta_{i,j} - \beta_{i+1,j}}{h^2} + \dfrac{-\beta_{i,j-1} + 2\beta_{i,j} - \beta_{i,j+1}}{h^2} + \alpha_{i,j} = 0, \quad \forall i,j, \\ \alpha_{i,0} = \beta_{i,0} = \alpha_{i,N+1} = \beta_{i,N+1} = 0, \quad \forall i, \\ \alpha_{N+1,j} = \beta_{0,j} = \beta_{N+1,j} = 0, \quad \forall j, \\ \alpha_{0,j} = v_j 1_{\omega_2}(jh), \quad \forall j. \end{cases}$$

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More compact form $\alpha^h = (\alpha_{i,j})_{i,j}, \beta^h = (\beta_{i,j})_{i,j} \in E_h = M_{N,N}(\mathbb{R})$

$$\partial_t \begin{pmatrix} \alpha^h \\ \beta^h \end{pmatrix} + \begin{pmatrix} \mathcal{A}_h^1 + \mathcal{A}_h^2 & 0 \\ 0 & \mathcal{A}_h^1 + \mathcal{A}_h^2 \end{pmatrix} \begin{pmatrix} \alpha^h \\ \beta^h \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha^h \\ \beta^h \end{pmatrix} = \begin{pmatrix} \mathcal{B}_h v \\ 0 \end{pmatrix}.$$

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Even more compact form $y^h = \alpha^h \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta^h \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in E_h \otimes \mathbb{R}^2$

$$\partial_t y^h + (\mathcal{A}_h^1 + \mathcal{A}_h^2) \otimes \mathrm{Id}_2 \, y^h + \mathrm{Id}_{E_h} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y^h = \mathcal{B}_h \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(t).$$

Main objective

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Even even more compact form $E_h = M_{N,N}(\mathbb{R}) \approx \mathbb{R}^N \otimes \mathbb{R}^N = E_h^1 \otimes E_h^2 \Rightarrow y^h \in E_h^1 \otimes E_h^2 \otimes \mathbb{R}^2$.

$$\partial_t y^h + \mathcal{A}_h \otimes \mathrm{Id}_h \otimes \mathrm{Id}_2 y^h + \mathrm{Id}_h \otimes \mathcal{A}_h \otimes \mathrm{Id}_2 y^h + \mathrm{Id}_{E_h} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y^h = \mathcal{B}_h \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(t).$$

multi-D cascade system with boundary control

... in the discrete world

The multi-D semi-discrete system

$$(S_B^h) \begin{cases} \partial_t y + \mathcal{A}_{1h} \otimes \mathrm{Id}_h \otimes \mathrm{Id}\, y + \mathrm{Id}_h \otimes \mathcal{A}_{2h} \otimes \mathrm{Id}\, y + \mathrm{Id}_h \otimes \mathrm{Id}_h \otimes C\, y \\ \qquad = \mathcal{B}_h \otimes \big((1_{\omega_2} \otimes B)v\big), \end{cases}$$

The associated 1D semi-discrete system

$$(S_B^{1D,h}) \begin{cases} \partial_t y + \mathcal{A}_{1h} \otimes \mathrm{Id}\, y + \mathrm{Id}_h \otimes C\, y \\ \qquad = \mathcal{B}_h \otimes (Bv), \end{cases}$$

Theorem

Assume that

1. *System* $(S_B^{1D,h})$ *is null-controllable at any time* $T > 0$*, with a cost of the control that behaves like* $e^{C/T}$*,* **uniformly in** h
2. *The operator* $\mathcal{A}_{2h}$ *satisfies a discrete Lebeau-Robbiano spectral inequality with respect to* 1_{ω_2}*,* **uniformly in** h*.*
3. *The control operator* $\mathcal{B}_h$ *satisfies some admissibility property,* **uniformly in** h*.*

Then, the multi-D system (S_B^h) *is null-controllable,* **uniformly in** h*.*

This is almost surely an empty theorem ...

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... in the discrete world

The multi-D semi-discrete system

$$(S_B^h)\begin{cases}\partial_t y + \mathcal{A}_{1h}\otimes \mathrm{Id}_h\otimes \mathrm{Id}\, y + \mathrm{Id}_h\otimes \mathcal{A}_{2h}\otimes \mathrm{Id}\, y + \mathrm{Id}_h\otimes \mathrm{Id}_h\otimes C\, y \\ \qquad\qquad = \mathcal{B}_h\otimes\big((1_{\omega_2}\otimes B)v\big),\end{cases}$$

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Theorem ((Allonsius-B., '18))

Assume that (+ uniform admissibility conditions for $\mathcal{B}_h$ relative to $\mathcal{A}_{1h}$)

1. *There exists $C_1>0, \mu_1>0$ such that for any $T>0$ and any $y_{0,h}$ there exists a control v_h for $(S_B^{1D,h})$ such that*
$$[\![v_h]\!]_{L^2(0,T;U_h)} \le e^{C_1\left(1+\frac{1}{T}\right)}\left\|y_{0,h}\right\|_h, \quad \text{and} \quad \|y_h(T)\|_h \le e^{C_1\left(1+\frac{1}{T}\right)}e^{-\mu_1 T}\left\|y_{0,h}\right\|_h.$$
2. *There exists $C_2>0$, μ_2, such that for any $\mu\le\mu_2$ we have*
$$\|\phi\|_h \le e^{C_2(1+\sqrt{\mu})}\left\|1_{\omega_2}\phi\right\|_h, \ \ \forall \phi\in \mathrm{span}(\phi_{j,2}^h, \lambda_{j,2}^h\le\mu).$$

Then, for $\mu^*=\min(\mu_1,\mu_2)$ there exists $C>0$, depending only on C_1, C_2 such that, for any $T>0$ and any $y_{0,h}$, there exists a control v_h for system (S_B^h) satisfying

$$[\![v_h]\!]_{L^2(0,T;U_h)} \le e^{C\left(1+T+\frac{1}{T}\right)}\left\|y_{0,h}\right\|_h, \ \ \|y_h(T)\|_h \le e^{C\left(1+T+\frac{1}{T}\right)}e^{-\mu^* T/4}\left\|y_{0,h}\right\|_h.$$

multi-D cascade system with boundary control

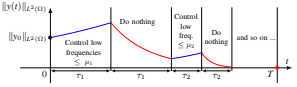
... in the discrete world

The multi-D semi-discrete system

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The associated 1D semi-discrete system

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Sketch of proof : Lebeau-Robbiano construction with some adaptation

- Finite number of steps (depending on μ^*)
- At each step we do not completely kill low frequencies. They are however sufficiently damped out.
- Uniformity w.r.t h of all the constants in the estimates.
- At the end of the process we only obtain a target of size

$$e^{C\left(1+T+\frac{1}{T}\right)} e^{-C\sqrt{\mu}} \|y_{0,h}\|$$

$\rightsquigarrow$ another additional control step on a large time interval $(T/2, T)$ to recover

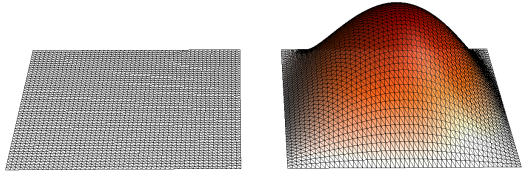
$$e^{C\left(1+T+\frac{1}{T}\right)} e^{-\mu T/4} \|y_{0,h}\|$$

Multi-D cascade system with boundary control

An example

Without control

$$\begin{cases} \partial_t y_1 - \gamma \Delta y_1 & = 0, \\ \partial_t y_2 - \gamma \Delta y_2 + y_1 & = 0. \end{cases}$$

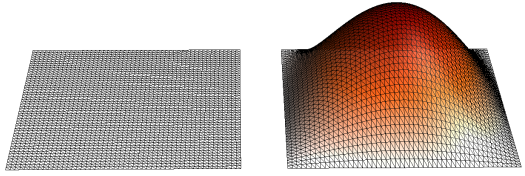


Multi-D cascade system with boundary control

An example

With a boundary control

$$\begin{cases} \partial_t y_1 - \gamma \Delta y_1 = 0, \\ \partial_t y_2 - \gamma \Delta y_2 + y_1 = 0. \end{cases}$$



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Boundary time-discrete control problems

Let $T > 0$ and $y_0 \in L^2(]0,1[)$ be given. Let $M \geq 1$ and set $\delta t = T/M$.

$$(S) \quad \begin{cases} \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A} y^{n+1} = 0, \quad \forall n \in [\![0, M[\![, \\ y^{n+1}(0) = 0, \quad y^{n+1}(1) = v^{n+1}, \quad \forall n \in [\![0, M[\![, \\ y^0 = y_0. \end{cases}$$

Problem formulation

Does there exist a family $(v^{n+1})_{n \in [\![0,M[\![} \subset \mathbb{R}$ such that the solution $(y^n)_{n \in [\![0,M[\![} \subset H^1(]0,1[)$ of (S) satisfies the null-controllability condition $\boxed{y^M = 0}$?

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Basic answer

This problem cannot be solved in general since

$$(v^n)_{n \in [\![0,M[\![} \in \mathbb{R}^M \quad \text{and} \quad \dim(H^1(]0,1[)) = +\infty.$$

Here also, we will need to relax the null-controllability condition.

Remark : for this particular problem we can use time-discrete Carleman stuff but this does not work anymore when dealing with cascade coupled systems.

(B.- Hernández–Santamaria, '20)

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Dual formulation

The final time solution y^M of the problem above satisfies

$$\langle y^M, \phi_k \rangle = p_{\delta t}^M[\lambda_k] \, \langle y_0, \phi_k \rangle - \left(\mathcal{B}^\star \phi_k\right) \sum_{n=0}^{M-1} \delta t \, v^{n+1} \, p_{\delta t}^{M-n}[\lambda_k], \quad \forall k \geq 1.$$

Here we have

$$\mathcal{B}^\star = -\partial_x\Big|_{x=1}, \quad \text{and} \quad p_{\delta t}^j[\lambda] = (1 + \delta t \lambda)^{-j}, \quad \forall j \geq 0.$$

The discrete functions $p_\bullet^\bullet[\lambda]$ play the role of the exponentials : $\sup_{j \in [\![0,M]\!]} \left| p_{\delta t}^j[\lambda] - e^{-\lambda(j\delta t)} \right| \leq C_\lambda \delta t.$

Boundary time-discrete control problems

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A tentative time-discrete moment problem

The final time solution satisfies $y^M = 0$ if and only if

$$\frac{p_{\delta t}^M[\lambda_k]}{\mathcal{B}^\star \phi_k} \langle y_0, \phi_k \rangle = \sum_{n=0}^{M-1} \delta t \, v^{n+1} \, p_{\delta t}^{M-n}[\lambda_k], \quad \forall k \geq 1.$$

Relaxation

Solve only finitely many such equations, namely for $k \in [\![0, K_{\delta t}]\!]$ for some $K_{\delta t}$ such that

$$K_{\delta t} \xrightarrow{\delta t \to 0} +\infty.$$

Time discrete biorthogonal to "exponentials"

(B.-Hernández–Santamaria, '25)

Let $\Sigma = (\sigma_k)_k$ be an increasing sequence of distinct positive numbers satisfying

$$\sigma_1 \geq \rho, \qquad \sigma_{k+1} - \sigma_k \geq \rho, \ \forall k \geq 1, \qquad \sigma_k \leq \frac{k^2}{\rho}, \ \forall k \geq 1. \tag{H_ρ}$$

Theorem

There exists $\omega > 0$ and $\delta t_0 > 0$ depending only on ρ and T such that, for any $\delta t < \delta t_0$ there exists a family of time discrete functions $(q_{\delta t}[\mu])_{\substack{\mu \in \Lambda \\ \mu \delta t < \omega}}$ such that

$$\Big\langle q_{\delta t}[\mu], p_{\delta t}[\lambda] \Big\rangle_{L^2_{\delta t}(0,T)} := \sum_{n=1}^{M} \delta t \, q^n_{\delta t}[\mu] \, p^n_{\delta t}[\lambda] = \delta_{\lambda,\mu},$$

and satisfying the following estimate

$$\|q_{\delta t}[\lambda]\|^2_{L^2_{\delta t}(0,T)} \stackrel{\text{def}}{=} \sum_{n=1}^{M} \delta t \big| q^n_{\delta t}[\lambda] \big|^2 \leq C e^{\varepsilon(\lambda)\lambda}.$$

Application to our time-discrete control problem

There exists $C, \tilde{C}, \delta t_0 > 0$ such that, for any $\delta t < \delta t_0$ we can find a time-discrete control $v_{\delta t}$ satisfying

$$\|v_{\delta t}\|_{L^2_{\delta t}(0,T)} \leq C \|y_0\|_{L^2(\Omega)}, \quad \text{and} \quad \|y^M\|_{L^2(\Omega)} \leq C e^{-\frac{\tilde{C}}{\delta t}} \|y_0\|_{L^2(\Omega)}.$$

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Boundary time-discrete control problems for systems

$$(S)\quad \begin{cases} \dfrac{y^{n+1}-y^n}{\delta t} + \mathcal{A}\otimes \mathrm{Id}\, y^{n+1} + \mathcal{I}\otimes C\, y^{n+1} = 0, \quad \forall n \in [\![0, M[\![, \\ y^{n+1} = 1_{x=1}\otimes \left(B v^{n+1}\right), \quad \forall n \in [\![0, M[\![, \\ y^0 = y_0. \end{cases}$$

with

$$B = \begin{pmatrix} 1 \\ 0 \\ | \\ | \\ 0 \end{pmatrix} \in \mathbb{R}^n, \quad C = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \ddots & | \\ 0 & \ddots & | \\ | & \ddots & | \\ 0 & \cdots\ 0\ 1\ 0 & \end{pmatrix} \in M_n(\mathbb{R}).$$

Theorem ($\varphi(\delta t)$-null controllability)

Given any φ such that $\varphi(\delta t) \gtrapprox e^{-1/\delta t}$, there exists $C, \delta t_0 > 0$ such that, for any initial data $y_0 \in L^2(\Omega)$, there exists a time discrete control $v_{\delta t}$ such that

$$\|v_{\delta t}\|_{L^2_{\delta t}(0,T)} \leq C \|y_0\|_{L^2(\Omega)},$$

and the solution of (S) satisfies

$$\|y^M\|_{L^2(\Omega)} \leq C\sqrt{\varphi(\delta t)}\|y_0\|_{L^2(\Omega)}.$$

A similar fully discrete result can be obtained for free !

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Thanks for your attention.