Problems

Problem 1.1

Let $a \in L^\infty(0,1)$ be a function such that for a certain $\varepsilon_0 > 0$ one has $a \geq \varepsilon_0$ a.e. in $(0,1)$. We wish to obtain some informations about the function a by solving the differential equation

$$\text{(1.1)} \qquad \begin{cases} -(au')' = 0 & \text{in } (0,1) \\ u(0) = \alpha_0 \\ u(1) = \alpha_1 \end{cases}$$

1) Write explicitely the solution u in terms of a and α_0, α_1.

Hint : Do you really need a hint ?…

2) For $\alpha_0 := 1$ and $\alpha_1 := 0$, one measures also $\beta_0 := (au')(0)$. What can be determined about the conductivity a, in terms of β_0 ?

Hint : Use the previous question.

3) Would it be useful to consider other values of α_0, α_1 in the hope of obtaining more informations about a ?

Problem 1.2

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. We shall denote $\|\cdot\|$ the norm of $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle$ the duality between $H^{1/2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$.
With the notations of section 2 of Chapter 2 of the lecture notes, if $\sigma_0 \in \partial\Omega$ we consider a sequence of functions $(\psi_m)_{m\geq 1}$ in $H^{1/2}(\partial\Omega)$ whose supports concentrate around σ_0 and such that $\|\psi_m\|_{H^{1/2}} = 1$, and for $0 \leq s < 1$

$$c_1 m^{-(1+2s)/2} \leq \|\psi_m\|_{H^{-s}(\partial\Omega)} \leq c_2 m^{-(1+2s)/2}.$$

We assume that for $j = 0, 1$ we have two sets of coefficients $a_j, q_j \in L^\infty(\Omega)$ satisfying $a_j \geq \varepsilon_0$ and $q_j \geq 0$. We denote

$$\Lambda_j(\varphi) := a_j \frac{\partial u_j}{\partial \mathbf{n}}$$

when $\varphi \in H^{1/2}(\partial\Omega)$ and $u_j \in H^1(\Omega)$ is solution to $-\mathrm{div}(a_j \nabla u_j) + q_j u_j = 0$ and $u_j = \varphi$ on $\partial\Omega$.
We denote by $(L_j, D(L_j))$ the self-adjoint operator with compact resolvent defined by

$$L_j u := -\mathrm{div}(a_j \nabla u_j) + q_j u, \qquad D(L_j) := \left\{ u \in H_0^1(\Omega) \,;\, \mathrm{div}(a_j \nabla u_j) \in L^2(\Omega) \right\}.$$

1) Let $a, q \in L^\infty(\Omega)$ two functions such that $q \geq 0$ and $a \geq \varepsilon_0$ for some given $\varepsilon_0 > 0$. Verify that for $m \geq 1$ the equation

(1.2)
$$\begin{cases} -\operatorname{div}(a\,\nabla u_m) + q u_m = 0 & \text{in } \Omega \\ u_m = \psi_m & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u_m \in H^1(\Omega)$, obtained by minimizing a quadratic functional, and that there exists $c > 0$ such that for all $m \geq 1$ one has $\|u_m\| \leq c\, m^{-1}$.

Hint : Consider the functional $J : H^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$J(v) := \int_\Omega a(x)\,|\nabla v(x)|^2\,dx + \int_\Omega q(x)\,|v(x)|^2\,dx,$$

on the convex set $K := \left\{ v \in H^1(\Omega)\ ;\ v = \psi_m \text{ on } \partial\Omega \right\}$. Show that the functional J is still strictly convex and $J(v_j) \to +\infty$ if $v_j \in K$ and $\|v_j\|_{H^1} \to +\infty$, so that u_m is obtained as the minimum of J on K, and then use Lemma 1.2.6.

2) We consider again the equation (1.2) where we do not assume anymore $q \geq 0$, but we assume that there exists $\lambda_1 > 0$ such that $(Lv|v) \geq \lambda_1 \|v\|^2$ for all $v \in D(L)$. Show that the results of Question 1) are still valid.

Hint : Same procedure as the one described for Question 1) : begin by proving that if $(v_{0j})_{j\geq 1}$ is a sequence in $H_0^1(\Omega)$ such that $\|\nabla v_{0j}\| \to \infty$ as $j \to \infty$, then $(Lv_{0j}|v_{0j}) \to +\infty$.

3) We assume here $q_j \geq 0$ and $a_j \in C(\overline{\Omega})$. Assume that for all $\varphi \in H^{1/2}(\partial\Omega)$ one has

(1.3)
$$\langle \Lambda_0(\varphi), \varphi\rangle \geq \langle \Lambda_1(\varphi), \varphi\rangle .$$

Taking your inspiration from what we have done in the lecture notes, prove that one has necessarily $a_0 \geq a_1$ on $\partial\Omega$. Can one find a neighbourhood of $\partial\Omega$ on which $a_0 \geq a_1$?

Hint : Assuming that at some point $\sigma_0 \in \partial\Omega$ one has $a_0(\sigma_0) < a_1(\sigma_0)$, following the proof of Theorem 2.2.4 show that for $m \geq 1$ large enough one has $\langle \Lambda_0(\psi_m), \psi_m\rangle < \langle \Lambda_1(\psi_m), \psi_m\rangle$. To this end, repeat the proof of Lemma 2.2.2 and Lemma 2.2.3.

4) Prove that if $\Lambda_0 = \Lambda_1$ then $a_0 = a_1$ on $\partial\Omega$.

Hint : Do you really need one ?...

5) Let $q_j \in L^\infty(\Omega)$, for $j = 0, 1$, be such that there exists $\lambda_{1j} > 0$ such that $(Lv|v) \geq \lambda_{1j}\|v\|^2$ for all $v \in D(L_j)$. We assume here $a_j \in C^1(\overline{\Omega})$ and that $\Lambda_0 = \Lambda_1$. Prove that $\nabla a_0 = \nabla a_1$ on $\partial\Omega$.

Hint : Follow the proof of Lemma 2.2.5 to show that for some constants $c_3, c_4 > 0$

$$\int_\Omega a(x)\,|\nabla u_m|^2 \varphi_1(x)\,dx \geq \frac{c_3}{m} - \frac{c_4}{m^2}.$$

and then proceed as in the proof of Theorem 2.2.6.

6) With the assumptions of Question 5), assume $N \geq 3$ and moreover that $a := a_0 = a_1 \in W^{2,\infty}(\Omega)$. Show that if $\Lambda_0 = \Lambda_1$ then $q_0 = q_1$ in Ω.

Hint : Use the Liouville transform as in the proof of Proposition 2.3.5.

7) Let $V, q \in L^\infty(\Omega)$ and assume that for $j = 0, 1$, we have two function $z_j \in H^1(\Omega) \cap L^\infty(\Omega)$ such that for some $\varepsilon_0 > 0$ we have $z_j \geq \varepsilon_0$ a.e. in Ω. Show that if

$$-\Delta z_j + V z_j = \frac{q}{z_j} \quad \text{in } \Omega, \qquad z_0 = z_1 \quad \text{on } \partial\Omega, \qquad \frac{\partial z_0}{\partial n} = \frac{\partial z_1}{\partial n} \quad \text{on } \partial\Omega,$$

then one has $z_0 = z_1$ on Ω.

Hint : Set $z := z_0 - z_1$ and write the equation satisfied by z. Then think of the unique continuation theorem.

8) With the assumptions of Question 5), assume $N \geq 3$ and moreover that $q := q_0 = q_1$ and $a_j \in W^{2,\infty}(\Omega)$ for $j = 0, 1$. Show that if $\Lambda_0 = \Lambda_1$ then $a_0 = a_1$ in Ω.

Hint : Use the fact that $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on $\partial\Omega$, and then use the Liouville transform as in the proof of Proposition 2.3.5.