Problems

Problem 2.1

Let $\rho_0 > 0$ be a constant, and $\Omega \subset \mathbb{R}^3$ a domain filled with a material having density ρ_0. It is known that the Newtonian potential u created by this object is given as the unique solution of the Poisson equation

$$-\Delta u = 4\pi\rho_0 1_\Omega \qquad \text{and} \qquad \lim_{|x|\to+\infty} u(x) = 0.$$

1) We assume that $\Omega \subset \mathbb{R}^3$ is bounded and Lipschitz, and that a certain function $f \in L^2(\mathbb{R}^3)$ with compact support is given. Show that if $u \in H^1_{\rm loc}(\mathbb{R}^3)$ satisfies the equation $-\Delta u = f$ in $\mathbb{R}^3$, then for all harmonic functions v defined on $\mathbb{R}^N$ we have

$$\int_\Omega f(x)\, v(x)\, dx = \int_{\partial\Omega} \Big(\frac{\partial v(\sigma)}{\partial \mathbf{n}} u(\sigma) - \frac{\partial u(\sigma)}{\partial \mathbf{n}} v(\sigma) \Big)\, d\sigma.$$

2) Let Ω_j, for $j = 1, 2$, two bounded domains of $\mathbb{R}^3$ and consider u_j solutions of the Poisson equations $-\Delta u_j = 4\pi\rho_0 1_{\Omega_j}$ verifying moreover $\lim_{|x|\to+\infty} u_j(x) = 0$. Assume that a certain smooth domain Ω is given such that $\overline{\Omega_1} \cup \overline{\Omega_2} \subset \Omega$ and $u_1 = u_2$ on $\partial\Omega$.
Denoting $u := u_1 - u_2$, prove that $u \equiv 0$ in Ω^c. (Use the maximum principle for the function u which is harmonic in the domain $\Omega_R := B(0, R) \setminus \Omega$, for R large enough, remembering that u tends to zero at infinity and $u = 0$ on $\partial\Omega$).
Infer that for any function v harmonic on $\mathbb{R}^N$ one has

$$\int_{\Omega_1} v(x)\, dx = \int_{\Omega_2} v(x)\, dx.$$

3) The domain Ω being as above, show that if $u_1 = u_2$ on $\partial\Omega$, then Ω_1 and Ω_2 have the same volume and the same center of gravity. What can be said if one knows that Ω_j are balls ?

4) Instead of assuming $u_1 = u_2$ on $\partial\Omega$, we assume here $\nabla u_1 = \nabla u_2$ on $\partial\Omega$. Show that the above conclusions still hold. (Note that if $u := u_1 - u_2$, then ∇u is harmonic, equal to zero $\partial\Omega$, and show that ∇u tends to zero at infinity, then apply the maximum principle and finally show that $u \equiv 0$ in Ω^c).

Problem 2.2

Let $N \geq 2$ be an integer and $\Omega \subset \mathbb{R}^N$ a smooth bounded domain. One considers a function $\boldsymbol{b} \in (W^{1,\infty}(\Omega))^N$, and for $\varphi, \psi \in H^{1/2}(\partial\Omega)$ one wihes to solve the following elliptic equations

$$(2.1) \qquad \begin{cases} -\Delta u + \boldsymbol{b}\cdot\nabla u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta v - \operatorname{div}(v\boldsymbol{b}) = 0 & \text{in } \Omega \\ v = \psi & \text{on } \partial\Omega. \end{cases}$$

1) In a first step we assume that $\operatorname{div}(\boldsymbol{b}) = 0$. Prove that each of the above equations has a unique solution, and that the mappings $\varphi \mapsto u$ and $\psi \mapsto v$ are continuous from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$.

2) When $\operatorname{div}(\boldsymbol{b}) = 0$, one defines the Poincaré–Steklov operators $\Lambda_{\boldsymbol{b}}$ and $\widetilde{\Lambda}_{\boldsymbol{b}}$ by setting

$$\Lambda_{\boldsymbol{b}}(\varphi) \coloneqq \frac{\partial u}{\partial \boldsymbol{n}} \qquad \text{and} \qquad \widetilde{\Lambda}_{\boldsymbol{b}}(\psi) \coloneqq \frac{\partial v}{\partial \boldsymbol{n}} + (\boldsymbol{b}\cdot\boldsymbol{n})\,\psi.$$

Show that these operators are well defined and bounded from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$, and that $\Lambda_{\boldsymbol{b}}^{*} = \widetilde{\Lambda}_{\boldsymbol{b}}$.

3) For $j = 0, 1$ one considers two functions $\boldsymbol{b_j} \in (W^{1,\infty}(\Omega))^N$ such that $\operatorname{div}(\boldsymbol{b_j}) = 0$, and for $\varphi_j \in H^{1/2}(\partial\Omega)$ one solves the problems

(2.2)
$$\begin{cases} -\Delta u_j + \boldsymbol{b_j}\cdot\nabla u_j = 0 & \text{in } \Omega \\ u_j = \varphi_j & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v_j - \operatorname{div}(v_j \boldsymbol{b_j}) = 0 & \text{in } \Omega \\ v_j = \varphi_j & \text{on } \partial\Omega. \end{cases}$$

Show that

$$\Big\langle \big(\Lambda_{\boldsymbol{b_0}} - \Lambda_{\boldsymbol{b_1}}\big)\,\varphi_0, \varphi_1 \Big\rangle = \int_{\Omega} ((\boldsymbol{b_0}(x) - \boldsymbol{b_1}(x))\cdot\nabla u_0(x))\, v_1(x)\, dx.$$

4) Assuming $\operatorname{div}(\boldsymbol{b_j}) = 0$ for $j = 0, 1$, one denotes $\mathbb{E}$ the vector subspace of $\big(L^1(\Omega)\big)^N$ generated by the products $v_1\nabla u_0$ where u_0 and v_1 satisfy (2.2) for $j = 0$ and $j = 1$ respectively. Show that if $\Lambda_{\boldsymbol{b_0}} = \Lambda_{\boldsymbol{b_1}}$ and if $\mathbb{E}$ is dense in $\big(L^1(\Omega)\big)^N$, then one has $\boldsymbol{b_0} = \boldsymbol{b_1}$ in Ω.

5) Let $\boldsymbol{b} \in \big(W^{1,\infty}(\Omega)\big)^N$ such that for a certain $\varepsilon > 0$ one has

$$\operatorname{div}(\boldsymbol{b}) \le 2\lambda_1 - \varepsilon \quad \text{where } \lambda_1 \coloneqq \inf\left\{ \int_{\mathbb{R}^N} |\nabla\varphi|^2 dx \;;\; \varphi \in H_0^1(\Omega),\ \int_{\Omega} |\varphi|^2 dx = 1 \right\}.$$

Prove that both equations (2.1) have unique solutions, and that the Poincaré–Steklov operators $\Lambda_{\boldsymbol{b}}$ and $\widetilde{\Lambda}_{\boldsymbol{b}}$ are well defined.

6) We assume that for $j = 0$ or $j = 1$, the functions $\boldsymbol{b_j}$ satisfy the assumptions of Question 5). Show that the result of Question 4) is still valid.

7) In this question we assume that $\boldsymbol{b_0}, \boldsymbol{b_1} \in \mathbb{R}^N$ are constants. Let $\xi \in \mathbb{R}^N \setminus \{0\}$ be fixed and set $\zeta_j \coloneqq \mathrm{i}\xi + \eta_j$. Prove that one may choose $\eta_j \in \mathbb{R}^N$ in such a way that the functions

$$u_0(x) \coloneqq \exp(\zeta_0\cdot x), \qquad v_1(x) \coloneqq \exp(\zeta_1\cdot x)$$

satisfy the equations

$$-\Delta u_0 + \boldsymbol{b_0}\cdot\nabla u_0 = 0, \qquad -\Delta v_1 - \operatorname{div}(v_1\boldsymbol{b_1}) = 0.$$

If $\Lambda_{\boldsymbol{b_0}} = \Lambda_{\boldsymbol{b_1}}$, can one deduce that $\boldsymbol{b_0} = \boldsymbol{b_1}$?

8) Let $\theta \in W^{2,\infty}(\Omega)$ and $\boldsymbol{b} \coloneqq \nabla\theta$. Verify that there exists $q \in L^{\infty}(\Omega)$ such that if a function $u \in H^1(\Omega)$ satisfies the equation

$$-\Delta u + \boldsymbol{b}\cdot\nabla u = 0,$$

then the function w défined by $u = \exp(\theta/2)\, w$ is solution to

$$-\Delta w + q w = 0.$$

9) For $j = 0, 1$, assume that $\boldsymbol{b_j} := \nabla\theta_j$ where $\theta_j \in W^{2,\infty}(\Omega)$ is a function satisfying $\theta_j = 0$ on the boundary $\partial\Omega$. In a first step, assume that $\Delta\theta_j \leq 2\lambda_1 - \varepsilon$, for some $\varepsilon > 0$. Using the previous question show that if $\Lambda_{\boldsymbol{b_0}} = \Lambda_{\boldsymbol{b_1}}$ then one has

$$\frac{\Delta \exp(-\theta_0/2)}{\exp(-\theta_0/2)} = \frac{\Delta \exp(-\theta_1/2)}{\exp(-\theta_1/2)}.$$

Deduce that $\theta_0 = \theta_1$, and thus $\boldsymbol{b_0} = \boldsymbol{b_1}$.

10) Assume again that $\boldsymbol{b_j} := \nabla\theta_j$ where $\theta_j \in W^{2,\infty}(\Omega)$ is a function satisfying $\theta_j = 0$ on the boundary $\partial\Omega$. Noting that the equation $-\Delta u + \boldsymbol{b} \cdot \nabla u = 0$ is equivalent to $\operatorname{div}(a(x)\,\nabla u) = 0$ for a well chosen function a, show that if $\Lambda_{\boldsymbol{b_0}} = \Lambda_{\boldsymbol{b_1}}$ then $\boldsymbol{b_0} = \boldsymbol{b_1}$.

11) Let $V \in L^\infty(\Omega)$. When $\boldsymbol{b_j} := \nabla\theta_j$ where $\theta_j \in W^{2,\infty}(\Omega)$ is a function satisfying $\theta_j = 0$ on the boundary $\partial\Omega$. We are interested in the inverse problem of determining $\boldsymbol{b_j}$, using the Poincaré–Steklov opertaor $\Lambda_{\boldsymbol{b_j}}$ which is defined by $\Lambda_{\boldsymbol{b_j}}(\varphi) := \partial u_j/\partial \boldsymbol{n}$ where $u_j \in H^1(\Omega)$ is solution to

$$-\Delta u_j + \boldsymbol{b_j} \cdot \nabla u_j + V u_j = 0, \qquad u_j = \varphi \text{ on } \partial\Omega.$$

If $\Lambda_{\boldsymbol{b_0}} = \Lambda_{\boldsymbol{b_1}}$, can one deduce that $\boldsymbol{b_0} = \boldsymbol{b_1}$ in Ω ?