Problems

Problem 3.1

Let Ω be a smooth bounded domain of $\mathbb{R}^N$. For $j = 1, 2$we consider the linear heat equation :

$$\begin{cases} \partial_t u_j - \Delta u_j = 0 & \text{in } (0,\infty)\times\Omega \\ u_j(t,\sigma) = 0 & \text{on } (0,\infty)\times\partial\Omega \\ u_j(0,x) = u_{0j}(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where the initial data $u_{0j} \in L^2(\Omega)$ are given. We shall denote by $(\cdot\,|\,\cdot)$ the scalar product of $L^2(\Omega)$ and by $\|\cdot\|$ its associated norm.

1) Let $(\varphi_k)_{k\geq 1}$ and $(\lambda_k)_{k\geq 1}$ be the eigenfunctions and eigenvalues of the Laplace operator $-\Delta$ with Dirichlet boundary conditions. Verify that the solution of (3.1) may be written as

$$u_j(t) = \sum_{k\geq 1} \alpha_{jk}\, e^{-\lambda_k t}\varphi_k,$$

where the coefficients α_{jk}, as well as the meaning of the convergence of the series will be given.

2) Let $T > 0$ be given and for $0 < t < T$ denote $\theta := \tau/T$. Prove that

$$\sum_{k\geq 1} |\alpha_k|^2 e^{-\lambda_k \tau} \leq \left(\sum_{k\geq 1} |\alpha_k|^2\right)^{1-\theta} \left(\sum_{k\geq 1} |\alpha_k|^2 e^{-\lambda_k T}\right)^{\theta}.$$

3) Show that if for a certain $T > 0$ one has $u_1(T) = u_2(T)$ then $u_{01} \equiv u_{02}$ in Ω.

4) Let $0 < \tau < T$ and define the space

$$H := H(\tau) := \left\{ v\ ;\ \exists\, u_0 \in L^2(\Omega) \text{ such that } v = \sum_{k\geq 1} (u_0|\varphi_k)\, e^{-\lambda_k \tau}\varphi_k \right\}.$$

If $v \in H$ and $v = \sum_{k\geq 1}(u_0|\varphi_k)\, e^{-\lambda_k\tau}\varphi_k$, we define $\|v\|_H := \|u_0\|$. Verify that $v \mapsto \|v\|_H$ is a well defined norm on the vector space H.

5) If $v \in H$ and $v = \sum_{k\geq 1}(u_0|\varphi_k)\, e^{-\lambda_k\tau}\varphi_k$, consider the operator $v \mapsto Bv$ defined by

$$Bv := \sum_{k\geq 1} (u_0|\varphi_k)\, e^{-\lambda_k T}\varphi_k.$$

For fixed $\varepsilon, \delta > 0$, we define $E(R, \delta, \tau, T)$ by setting

$$E(R, \varepsilon, \tau, T) := \sup\{\|v\|\ ;\ v \in H,\ \|v\|_H \leq R,\ \|Bv\| \leq \varepsilon\}.$$

Using Question 2), prove that if $\theta := \tau/T$ then one has

$$E(R,\varepsilon,\tau,T) \le R^{1-\theta}\,\varepsilon^{\theta}.$$

6) Assume that the initial data u_{0j} in (3.1) satisfy $\|u_{01}-u_{02}\| \le R$. Show that for $0<\tau<T$ one has $\|u_1(\tau)-u_2(\tau)\| \le E(R,\varepsilon,\tau,T) \le R^{1-\theta}\,\|u_1(T)-u_2(T)\|^{\theta}$.

Problem 3.2

With the notations of the lecture notes, recall that if $u^{\mathrm{in}}(x) := \mathrm{e}^{\mathrm{i}k\theta\cdot x}$, the solution of the Helmholtz equation

$$\begin{cases} -\Delta u = k^2 n(x)\,u & \text{in } \mathbb{R}^3 \\ \dfrac{x}{|x|}\cdot\nabla u^{\mathrm{s}}(x) - \mathrm{i}k u^{\mathrm{s}}(x) = O\Big(\dfrac{1}{|x|^2}\Big) & \text{lorsque } |x|\to+\infty \end{cases} \tag{3.2}$$

can be written as

$$u(x) = u(x,\theta) = u^{\mathrm{in}}(x) + \frac{1}{|x|}\,\mathrm{e}^{\mathrm{i}k|x|}\,u_\infty\Big(\frac{x}{|x|},\theta\Big) + O\Big(\frac{1}{|x|^2}\Big),$$

with the convention that $u^{\mathrm{s}} := u - u^{\mathrm{in}}$. Assume that there exists a function $f:[-1,+1]\longrightarrow\mathbb{C}$ such that for all $\sigma,\theta\in S^2$ one has

$$u_\infty(\sigma,\theta) = f(\sigma\cdot\theta). \tag{3.3}$$

We wish to show that the function n is spherically symmetric, that is $n(x)=\widetilde{n}(|x|)$ for a certain function $\widetilde{n}:(0,\infty)\longrightarrow\mathbb{C}$, or equivalently $n(Mx)=n(x)$ for all rotation matrices M, that is matrices verifying $M^*M=I$, the identity matrix in $\mathbb{R}^3$.

1) Let M be a rotation matrix, that is $M^*M=I$, the identity matrix in $\mathbb{R}^3$. Set $n_1(x) := n(Mx)$ for $x\in\mathbb{R}^3$. Denoting by $u_1(x,\theta)$ the solution of the Helmholtz equation (3.2) where n is replaced with n_1, determine $u_{1,\infty}(\sigma,\theta)$ in terms of u.

2) Deduce that n is spherically symmetric.

3) When n is spherically symmetric does (3.3) hold ?

4) Denote by u^{B} the Born approximation of the solution to the Helmholz equation. Determine the relation existing between the spherical symmetry of n and the property stating that $u^{\mathrm{B}}_\infty(\sigma,\theta)=f(\sigma\cdot\theta)$.