

C1: Theoretical background and basic results. Elliptic PDEs

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- 1 Linear elliptic problems.
- 2 Maximum Principle.
- 3 Eigenvalue problems.
- 4 Sub-supersolution method. Applications.
- 5 Stability and uniqueness.
- 6 Bifurcation method. Applications.

Why elliptic equations?

There are several biological and physical phenomena that can be modeled by PDEs

$$u_t(x, t) - \Delta u(x, t) = f(x, u(x, t))$$

$x \in \Omega$, bounded regular domain of $\mathbb{R}^N$, $t > 0$, $-\Delta$ the Laplacian (linear second order elliptic operator):

$$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}.$$

Many times $u(x, t) \rightarrow u^*(x)$ as $t \rightarrow \infty$, where u^* is solution of the elliptic problem

$$-\Delta u = f(x, u).$$

The main objective of these sections is to provide tools, methods, etc ... for the study of existence, non-existence, regularity, uniqueness, multiplicity of the elliptic equation

$$\begin{cases} -\Delta u = f(\lambda, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $f : \mathbb{R} \times \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Caratheodory function.

Other elliptic equations

- 1 Others boundary conditions can be considered

$$\frac{\partial \mathbf{u}}{\partial n} = 0 \quad (Neumann) \quad \frac{\partial \mathbf{u}}{\partial n} + \beta \mathbf{u} = 0 \quad (Robin)$$

- 2 The nonlinearity can depend on the gradient of $\mathbf{u}$:

$$-\Delta \mathbf{u} = f(\lambda, x, \mathbf{u}, \nabla \mathbf{u}).$$

- 3 Non-local equations:

$$-\Delta \mathbf{u} = f\left(\lambda, x, \mathbf{u}, \int_{\Omega} \mathbf{u} \, dx\right).$$

- ① u is a **classical solution** of (1) if $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$-\Delta u(x) = f(\lambda, x, u(x)),$$

for all $x \in \Omega$ and $u(x) = 0$ for all $x \in \partial\Omega$.

- ② u is a **strong solution** of (1) if $u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$-\Delta u(x) = f(\lambda, x, u(x)),$$

a.e. $x \in \Omega$.

- ③ u is a **weak solution** of (1) if $u \in H_0^1(\Omega)$,

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f(\lambda, x, u) v, \quad \text{for all } v \in H_0^1(\Omega).$$

Linear equations: classical solutions

$$\begin{cases} -\Delta \mathbf{u} + c(x)\mathbf{u} = f(x) & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

Question: given $c, f \in C(\overline{\Omega})$, is there a unique solution $\mathbf{u} \in C^2(\overline{\Omega})$ to (2)? The answer is **NO**.

Instead of considering merely continuous functions, we consider **Hölder spaces**, defined as follows: given $\alpha \in (0, 1)$

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ f \in C(\overline{\Omega}) : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \right\}.$$

Theorem

Assume that $c, f \in C^{0,\alpha}(\overline{\Omega})$, $c \geq 0$. Then, (2) possesses a unique solution $\mathbf{u} \in C^{2,\alpha}(\overline{\Omega})$ which satisfies

$$\|\mathbf{u}\|_{2,\alpha} \leq C(\|f\|_\alpha) \quad (3)$$

with a positive constant C independent of f .

Linear equations: strong solutions

Question: What happens if f has less regularity? For example, suppose that

$$f \in L^p(\Omega).$$

Theorem

Assume that $c \in L^\infty(\Omega)$, $c \geq 0$, $f \in L^p(\Omega)$ for some $p \in (1, +\infty)$. Then, (2) possesses a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ which satisfies

$$\|u\|_{W^{2,p}} \leq C(\|f\|_p). \quad (4)$$

$$W^{2,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq 2\}.$$

Weak derivative

Let $f \in L^p(\Omega)$, We say that $g \in L^p(\Omega)$ is the weak partial derivative of f with respect to x_i if

$$\int_{\Omega} f(x) \frac{\partial \varphi}{\partial x_i}(x) dx = - \int_{\Omega} g(x) \varphi(x) dx, \quad \forall \varphi \in C_c^\infty(\Omega).$$

In this case, we write

$$\boxed{\frac{\partial f}{\partial x_i} = g}$$

in the weak sense.

Weak solutions

Suppose that we cannot require $\mathbf{u} \in W^{2,p}(\Omega)$.

We multiply

$$-\Delta \mathbf{u} + c \mathbf{u} = f$$

by a test function v and integrate:

$$-\int_{\Omega} \Delta \mathbf{u} v \, dx + \int_{\Omega} c(x) \mathbf{u} v \, dx = \int_{\Omega} f v \, dx.$$

After integration by parts,

$$\int_{\Omega} \nabla \mathbf{u} \cdot \nabla v \, dx + \int_{\Omega} c(x) \mathbf{u} v \, dx = \int_{\Omega} f v \, dx.$$

Thus, we only need first derivatives of $\mathbf{u}$:

$$\mathbf{u} \in W_0^{1,2}(\Omega).$$

The **weak formulation** is

$$\int_{\Omega} \nabla \mathbf{u} \cdot \nabla v \, dx + \int_{\Omega} c(x) \mathbf{u} v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in W_0^{1,2}(\Omega).$$

Lax–Milgram theorem

Define the bilinear form

$$a(\mathbf{u}, v) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla v \, dx + \int_{\Omega} c(x) \mathbf{u} v \, dx$$

and the linear functional

$$L(v) = \int_{\Omega} f v \, dx.$$

The weak formulation can then be written as

$$a(\mathbf{u}, v) = L(v), \quad \forall v \in W_0^{1,2}(\Omega).$$

Question: Does this weak formulation admit a unique solution?

The answer is provided by the **Lax–Milgram theorem**.

Lax–Milgram theorem

Let V be a Hilbert space and let

$$a : V \times V \rightarrow \mathbb{R}$$

be a continuous and coercive ($a(v, v) \geq C\|v\|^2$) bilinear form.
Let

$$L : V \rightarrow \mathbb{R}$$

be a continuous linear functional.

Then there exists a unique $u \in V$ such that

$$a(u, v) = L(v), \quad \forall v \in V.$$

Application to linear elliptic equation

For the Poisson problem

$$\begin{cases} -\Delta \mathbf{u} + c(x)\mathbf{u} = f(x) & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

we take

$$V = W_0^{1,2}(\Omega),$$
$$a(\mathbf{u}, v) = \int_{\Omega} \nabla \mathbf{u} \cdot \nabla v \, dx + \int_{\Omega} c(x)\mathbf{u}v \, dx,$$

and

$$L(v) = \int_{\Omega} f v \, dx.$$

Under the assumption on $f \in L^2(\Omega)$, the Lax–Milgram theorem yields a unique weak solution

$$\mathbf{u} \in W_0^{1,2}(\Omega) \quad \text{and} \quad \|\mathbf{u}\|_{1,2} \leq C\|f\|_2.$$

Summary

Type solution	Assumption on f	Equation	Regularity of u
Classical sol.	$f \in C^\alpha(\overline{\Omega})$	$-\Delta u = f$ pointwise	$u \in C^{2,\alpha}(\overline{\Omega})$
Strong sol.	$f \in L^p(\Omega)$	$-\Delta u = f$ a.e. in Ω	$u \in W^{2,p}(\Omega)$
Weak sol.	$f \in L^2(\Omega)$	$\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} c u v \, dx = \int_{\Omega} f \varphi \, dx$	$u \in H_0^1(\Omega)$

Table: Different notions of solution for the Poisson equation $-\Delta u = f$.

Classical solution $\implies$ Strong solution $\implies$ Weak solution

Weak maximum principle

$$Lu := -\Delta u + c(x)u$$

$$u^+ := \max\{u, 0\}, \quad u^- := \min\{u, 0\}.$$

Theorem

- ① If $c \equiv 0$ and $Lu \geq 0$, then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

- ② If $c \geq 0$ and $Lu \geq 0$, then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-.$$

- ③ If $c \geq 0$ and $Lu = 0$, then

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|.$$

Theorem

Assume that $Lu \geq 0$ in Ω .

- 1 Assume $c \equiv 0$. Then, if u attains its minimum in Ω , u is constant.
- 2 Assume $c \geq 0$. Then, if u attains its non-positive minimum in Ω , u is constant.

Theorem

Assume $c \geq 0$ and

$$\begin{cases} Lu \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega. \end{cases}$$

- 1 Then, $u \geq 0$ and $u > 0$ unless $Lu = 0$ and $u = 0$.
- 2 If $u \not\equiv 0$, then $u(x) > 0$, $\forall x \in \Omega$.
- 3 If $u(x_0) = 0$ for some $x_0 \in \partial\Omega$, then

$$\frac{\partial u}{\partial n}(x_0) < 0.$$

Maximum Principle and Comparison

Consider two functions u, v satisfying

$$-\Delta u \geq -\Delta v \quad \text{in } \Omega$$

and

$$u \geq v \quad \text{on } \partial\Omega.$$

Then

$$u \geq v \quad \text{in } \Omega.$$

This is the **comparison principle**.

$$\text{Maximum principle} \implies \text{Comparison principle.}$$

Why is it Important?

The maximum principle has several important consequences for elliptic PDEs.

- **Uniqueness:** Dirichlet problems have at most one solution.
- **Stability:** small changes in the boundary data lead to controlled changes in the solution.
- **Comparison:** solutions can be ordered using their boundary values and the differential inequality.
- **Qualitative information:** one can obtain bounds on u without explicitly solving the PDE.

Strong maximum principle

- The condition $c \geq 0$ is necessary in the above results.
- Is this condition optimal?

Eigenvalue problem: self-adjoint case

$$\begin{cases} -\Delta u + c(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5)$$

Definition

We say that λ is an eigenvalue on (5), if there exists a function $u \neq 0$ solution of (5).

Remark

If u is a eigenfunction associated to λ , then Ku is also eigenfunction for all $K \neq 0$.

Eigenvalue problem: self-adjoint case

Assume that $c \geq 0$ and consider the operator

$$T : L^2(\Omega) \mapsto L^2(\Omega) \quad f \mapsto \textcolor{red}{u} = T(f),$$

where $\textcolor{red}{u}$ is the unique solution of (2). Then,

- ❶ T is well-defined.
- ❷ $\mu \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of T if and only if $1/\mu$ is an eigenvalue of (5).
- ❸ T is self-adjoint:

$$(Tf, g)_2 = (f, Tg)_2.$$

- ❹ T is compact.

Eigenvalue problem: self-adjoint case

Theorem (Spectral Theorem for Compact Self-Adjoint Operators)

Let H be a Hilbert space and let $T : H \rightarrow H$ be a compact self-adjoint operator. Then:

- 1 *T has infinitely many non-zero real eigenvalues, and they form a sequence $(\mu_n)_{n \geq 1}$ such that*

$$\mu_n \in \mathbb{R}, \quad \mu_n \rightarrow 0.$$

- 2 *There exists an orthonormal family of eigenfunctions $\{e_n\}_{n \geq 1} \subset H$ that forms an orthonormal basis in H .*

Theorem

The spectrum of (5) consists in an increasing sequence of real numbers, $\{\lambda_n\}$, $\lambda_n \rightarrow +\infty$. Moreover, the eigenfunctions $\{\varphi_n\}$ form an orthonormal basis in $L^2(\Omega)$.

Regularity eigenfunction: bootstrapping argument

Consider $\varphi \in L^2(\Omega)$ an eigenfunction, then

$$-\Delta\varphi = \lambda_n\varphi \in L^2(\Omega) \implies \varphi \in W^{2,2}(\Omega).$$

Hence,

$$\varphi \in L^{q_1}(\Omega), \quad \text{for } q_1 > 2.$$

Then

$$-\Delta\varphi = \lambda_n\varphi \in L^{q_1}(\Omega) \implies \varphi \in W^{2,q_1}(\Omega),$$

hence,

$$\varphi \in L^{q_2}(\Omega), \quad \text{for } q_2 > q_1.$$

After several steps, $\varphi \in W^{2,q_n}(\Omega)$ and then $\varphi \in C^{1,\alpha}(\overline{\Omega})$. Again,

$$-\Delta\varphi = \lambda_n\varphi \in C^{1,\alpha}(\overline{\Omega})$$

hence,

$$\varphi \in C^{2,\alpha}(\overline{\Omega}).$$

Proposition

It holds:

$$\lambda_1(c) = \inf \left\{ \int_{\Omega} (|\nabla \mathbf{u}|^2 + c(x)\mathbf{u}^2), \quad \mathbf{u} \in H_0^1(\Omega), \quad \|\mathbf{u}\|_2 = 1 \right\}.$$

Moreover, if $w \in H_0^1(\Omega)$ and

$$\lambda_1(c) = \frac{\int_{\Omega} |\nabla w|^2 + c(x)w^2}{\int_{\Omega} w^2},$$

then w is an eigenfunction associated $\lambda_1(c)$.

Corollary

- 1 $\lambda_1(c)$ is simple and its corresponding eigenfunctions do not change sign.
- 2 Reciprocally, if an eigenfunction has definite sign, it corresponds to $\lambda_1(c)$.
- 3 If $c(x) \equiv 0$, then $\lambda_1 > 0$.
- 4 If $c_1 \leq c_2$, then $\lambda_1(c_1) \leq \lambda_1(c_2)$
- 5 If $\Omega_1 \subset \Omega_2$, then $\lambda_1^{\Omega_1}(c) > \lambda_1^{\Omega_2}(c)$.

The principal eigenvalue and the maximum principle

- L verifies the strong maximum principle (SMP) if for any $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that

$$\begin{cases} L(u) \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases}$$

with some inequality strict, it verifies

1 $u > 0$ in Ω and

2

$$\frac{\partial u}{\partial n}(x_0) < 0, \quad \forall x_0 \in \partial\Omega \text{ such that } u(x_0) = 0.$$

- A function $h \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ is called a positive strict supersolution of L if $h > 0$ in Ω and one of the following conditions holds
 - $L(h) > 0$ in Ω
 - $L(h) \geq 0$ in Ω and $h > 0$ on $\partial\Omega$

The principal eigenvalue and the maximum principle

Theorem

The following statements are equivalent:

- 1 *L admits a positive strict supersolution.*
- 2 *L verifies (SMP).*
- 3 *It holds $\lambda_1(c) > 0$.*

Remark

If $c \geq 0$, then $h = C > 0$ is supersolution

$$Lh = -\Delta h + c(x)h \geq 0 \quad \text{in } \Omega, \quad h = C > 0 \quad \text{on } \partial\Omega,$$

then,

$$\lambda_1(c) > 0.$$

The principal eigenvalue and the maximum principle

What happens if T is non self-adjoint???

We can use the Krein-Rutman Theorem: If

- T is a linear,
- T is compact, and
- T is a positive operator,

then there exists at least a real eigenvalue (simple) with eigenfunctions do not change sign.

- ① All the theory developed above can be generalized to an elliptic operator of the form

$$\mathcal{L}(u) = - \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

with

$$a_{ij} = a_{ji}, \quad \sum_{i,j=1}^N a_{ij} \xi_i \xi_j \geq C |\xi|^2 > 0, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

❶ Other problems (Stokes equation)

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = f(x) & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad (6)$$

where $\mathbf{u}$ and f are vector-valued functions, that is,

$$\mathbf{u} = (u_1, \dots, u_N) \quad \text{and} \quad f = (f_1, \dots, f_N),$$

and p is a scalar-valued function.

Here, $\mathbf{u}$ represents the velocity of the particles in the fluid, while p represents the pressure acting on the particles. The condition

$$\nabla \cdot \mathbf{u} = 0$$

is known as the incompressibility condition, meaning that the fluid is assumed to be incompressible.

To solve (6), one works with a weak formulation of the problem, namely,

$$\int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} = \int_{\Omega} f \mathbf{v} \quad \forall \mathbf{v} \in V, \quad (7)$$

where V is the following space:

$$V := \left\{ \mathbf{u} \in H_0^1(\Omega)^N : \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \right\}.$$

It can be shown that, in fact,

$$V = \overline{\mathcal{V}}^{H_0^1(\Omega)^N},$$

where

$$\mathcal{V} := \left\{ \varphi \in D(\Omega)^N : \nabla \cdot \varphi = 0 \right\}.$$

And also, the eigenvalue associated problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \lambda \boldsymbol{u} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. \end{cases} \quad (8)$$

- 1 More general operators:

$$\mathcal{K}(u) := - \int_{\Omega} K(x, y) u(y) dy + a(x) u(x)$$

$K(x, y)$ is interpreted as a transition probability density or a jump rate, depending on the model.

For example, if

$$K(x, y) = e^{-|x-y|^2},$$

then points y close to x have a strong influence, whereas points farther away have a weaker influence.

- 1 More general operators:

$$-\Delta u - \mathcal{K}(u) + a(x)u = -\Delta u - \int_{\Omega} K(x, y)u(y)dy + a(x)u$$

or even

$$-\chi_A \Delta u - \chi_{\Omega \setminus A} \int_{\Omega} K(x, y)u(y)dy$$

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (9)$$

where $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ a nonlinear measurable function.

Different methods:

- 1 Monotonicity methods; sub-supersolution method.
- 2 Bifurcation methods: local and global.
- 3 Compactness methods: fixed points.
- 4 Galerkin methods.
- 5 Variational methods: minimization, mountain pass Theorem, Nehari manifold,...
- 6 Topological methods: Leray-Schauder degree.
- 7

The sub-supersolution method

$$\begin{cases} -\Delta \underline{u} = f(x, \underline{u}) & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad (10)$$

where $f \in C^1(\overline{\Omega} \times \mathbb{R})$.

Definition

A pair of functions $(\underline{u}, \overline{u}) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is called a pair of sub-supersolution of (10) if:

- 1 $\underline{u}(x) \leq \overline{u}(x), \quad \forall x \in \Omega,$
- 2 $\underline{u}(x) \leq 0 \leq \overline{u}(x), \quad \forall x \in \partial\Omega,$
- 3 $-\Delta \underline{u}(x) \leq f(x, \underline{u}(x)), \quad -\Delta \overline{u}(x) \geq f(x, \overline{u}(x)) \quad \forall x \in \Omega.$

Theorem

Assume $f \in C^1(\overline{\Omega} \times \mathbb{R})$ and that (10) admits a sub-supersolution. Then, there exist two classical solutions $u_*, u^* \in C^2(\overline{\Omega})$ of (10).

Moreover:

- 1 u_*, u^* are limits of monotone sequence.
- 2 Any other solution $u \in C^2(\overline{\Omega})$ of (10) such that

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x),$$

it also verifies

$$u_*(x) \leq u(x) \leq u^*(x).$$

Remark

- 1 u_* and u^* are called *minimal and maximal solutions*, respectively.
- 2 The conditions of Theorem could be relaxed (less regularity of f , of $\underline{u}$ and $\overline{u}$,...)
- 3 Condition (a) ($\underline{u}(x) \leq \overline{u}(x)$) is essential for the validity of Theorem 0.12.
- 4 The result is true for other boundary conditions.
- 5 One could obtain existence of solution and, however, sub-supersolutions do not exist.
- 6 The method is constructive.

Idea of the proof I:

Define the sequences $\{u_n\}$ and $\{u^n\}$ with $u_0 = \underline{u}$ and $u^0 = \bar{u}$ and for $n \geq 1$

$$\begin{cases} -\Delta u_n + K u_n = f(x, u_{n-1}) + K u_{n-1} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (11)$$

$$\begin{cases} -\Delta u^n + K u^n = f(x, u^{n-1}) + K u^{n-1} & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega. \end{cases} \quad (12)$$

Then,

$$u_n \nearrow u_*, \quad u^n \searrow u^*$$

where $u_* \leq u^*$ are classical maximal and minimal solutions.

Idea of the proof II:

Define the map $T : [\underline{u}, \bar{u}] \mapsto [\underline{u}, \bar{u}]$, by $T(w) = u$, where

$$\begin{cases} -\Delta u + Ku = f(x, w) + Kw & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (13)$$

Then:

- 1 T is well-defined.
- 2 T is compact.
- 3 $T([\underline{u}, \bar{u}]) \subset [\underline{u}, \bar{u}]$.

Hence, T possesses a fixed point $T(u) = u$, that is, u is solution.

Application I: biochemical reaction

In a biochemical reaction, the concentration of a certain enzyme is governed by the following equation

$$\begin{cases} -\Delta u = -\sigma \frac{u}{1 + au} + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (14)$$

where $\sigma, a > 0$ are parameters related to the reaction and $g \in C^1(\overline{\Omega})$, $g > 0$ in $\overline{\Omega}$.

Theorem

There exists at least one positive solution (14) for $\sigma, a > 0$.

Application II: logistic equation

Here $u(x)$ represents the population density of a species inhabiting in Ω :

$$\begin{cases} -\Delta u = \lambda u - bu^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (15)$$

with $b > 0$, $\lambda \in \mathbb{R}$.

- λ represents the growth rate of the species.
- The term $-bu^2$ represents the crowding effect.

Theorem

There exists at least a positive solution of (31) if and only if $\lambda > \lambda_1$.

Application II (bis): heterogeneous habitat

$$\begin{cases} -\Delta u = \lambda m(x)u - u^2, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

- $\lambda m(x)u$: local growth rate;
- $m(x) > 0$: favorable habitat ;
- $m(x) < 0$: unfavorable habitat.

Question

Can a population persist if only part of the habitat is favorable?

Answer

There exists a positive solution if λ is large.

A sufficiently favorable refuge can sustain a positive population.

Application III: Holling-Tanner equation

In this case, the population follows the equation

$$\begin{cases} -\Delta u = \lambda u + \frac{u}{1+u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (16)$$

Theorem

There exists at least one positive solution if and only if $\lambda \in (\lambda_1 - 1, \lambda_1)$.

Application IV: A non-uniqueness example

$$\begin{cases} -\Delta u = \lambda u - bu^3 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (17)$$

with $b > 0$, $\lambda \in \mathbb{R}$.

Theorem

- 1 If $\lambda \leq \lambda_1$, (17) admits only the trivial solution.
- 2 If $\lambda > \lambda_1$, (17) possesses at least two solutions, one positive and another negative.

$$\begin{cases} -\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (18)$$

with $\lambda \in \mathbb{R}$ and $0 < q < 1$.

Theorem

(18) has at least one positive solution if and only if $\lambda > 0$.

Application VI: Logistic equation with non-linear diffusion

$$\begin{cases} -\Delta(u^m) = \lambda u - bu^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (19)$$

with $b > 0$, $\lambda \in \mathbb{R}$ and $m > 1$.

The parameter $m > 1$ represents a non-linear diffusion, in this case slow diffusion.

Under the change of variable

$$u^m = w$$

equation (19) transforms into

$$\begin{cases} -\Delta w = \lambda w^q - bw^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (20)$$

with

$$0 < q < 1, \quad q < p.$$

$$\begin{cases} -\Delta w = \lambda w^q - bw^p & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (21)$$

with

$$0 < q < 1, \quad q < p.$$

Theorem

(21) has a positive solution if and only if $\lambda > 0$.

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (22)$$

with $\lambda \in \mathbb{R}$ and $0 < q < 1 < p$.

Theorem

There exists $\lambda_0 > 0$ such that (22) possesses a positive solution if $0 < \lambda < \lambda_0$.

Application VIII: Tumor growth and treatment

Let $u(x)$ denote the density of tumor cells in a tissue Ω .

$$\begin{cases} -\Delta u = \lambda u \left(1 - \frac{u}{K}\right) - q(x)u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

- $\lambda u(1 - u/K)$: logistic tumor growth;
- $q(x)u$: effect of a spatially distributed treatment;
- K : carrying capacity of the tissue.

Existence

There exists a positive solution (tumor presistence) if and only if $\lambda > \lambda_1(q)$

Tumor persistence $\longleftrightarrow$ growth vs. diffusion vs. treatment

We present two uniqueness results:

Theorem

Assume that $f(x, u)$ is decreasing in u . Then, there exists at most a solution of (10).

Theorem

Assume that the map

$$t \mapsto \frac{f(x, t)}{t} \quad \text{is decreasing for all } x \in \Omega. \quad (23)$$

Then, there exists at most a positive solution of (10).

Stability (local):

Consider the parabolic problem

$$\begin{cases} u_t - \Delta u = f(x, u) & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (24)$$

- An stationary solution u^* of (10) is stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that for any $u_0 \in C(\overline{\Omega})$ verifying $\|u_0 - u^*\|_\infty < \delta$, it holds

$$\|u(t, \cdot) - u^*\|_\infty < \epsilon \quad \forall t > 0, \quad (25)$$

where $u(t, x)$ is solution of (24). If moreover,

$$\lim_{t \rightarrow +\infty} \|u(t, \cdot) - u^*\|_\infty = 0,$$

u^* is asymptotically stable.

- u^* is unstable if it is not stable.

Theorem

- ① Assume that

$$\lambda_1(-f_u(x, u^*)) > 0,$$

then u^* is asymptotically stable.

- ② Assume that

$$\lambda_1(-f_u(x, u^*)) < 0,$$

then u^* is unstable.

The logistic equation revisited:

$$\begin{cases} -\Delta u = \lambda u - bu^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (26)$$

with $b > 0$, $\lambda \in \mathbb{R}$.

Theorem

- *The trivial solution exists for all λ , it is stable for $\lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$.*
- *If $\lambda > \lambda_1$ there exists a unique positive solution of (26) which is stable.*

The logistic equation revisited:

$$\begin{cases} -\Delta u = \lambda u - bu^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (27)$$

with $b > 0$, $\lambda \in \mathbb{R}$.

Then:

- $u \equiv 0$ is solution for all $\lambda \in \mathbb{R}$.
- There exists a positive solution if and only if $\lambda > \lambda_1$.
Moreover, the positive solution is unique, denoted by $u^* > 0$.

Furthermore, it is globally stable, that is,

- 1 If $\lambda < \lambda_1$ we have that $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$,
- 2 If $\lambda > \lambda_1$ we have that $u(x, t) \rightarrow u^*(x)$ as $t \rightarrow \infty$.

The logistic equation revisited:

Consequence (with respect to the spatial dependence): fixed a growth rate of the species, the species coexist if the domain Ω is large, and goes to the extinction if Ω is small.

Larger islands should be easier to find and colonize, and they should support larger populations which are less susceptible to extinction.

Problem: calculate λ_1 .

- 1 When $\Omega = (0, L)$, then $\lambda_1 = (\pi/L)^2$;
- 2 When $\Omega = B(0, R)$, then $\lambda_1 = \mu_1/R^2$, where μ_1 is the eigenvalue of $B(0, 1)$.

For other domains..... only estimates and numerical approximations are available.

The logistic equation revisited:

$$\begin{cases} -\Delta u = \lambda u - b(x)u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (28)$$

with $\lambda \in \mathbb{R}$ and $b(x)$ describes the effects of crowding, for example due to limitations of resources (food),

$$B_+ := \{x \in \Omega : b(x) > 0\}, \quad B_0 := \Omega \setminus \overline{B_+},$$

in this context, B_0 is called refuge.

- Any non-negative and non-trivial solution, it is positive.
- There exists a positive solution if and only if $\lambda \in (\lambda_1, \lambda_1^{B_0})$.
- There exists a unique positive solution, u_λ .

The time-dependent problem

$$\begin{cases} u_t - \Delta u = \lambda u - b(x)u^2 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega, \\ u(x, 0) = u_0(x) > 0 & \text{in } \Omega. \end{cases}$$

We have:

- 1 If $\lambda < \lambda_1$ we have that $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$,
- 2 If $\lambda \in (\lambda_1, \lambda_1^{B_0})$ we have that $u(x, t) \rightarrow u_\lambda(x)$ as $t \rightarrow \infty$,
- 3 If $\lambda > \lambda_1^{B_0}$ we have that $\|u(x, t)\|_\infty \rightarrow +\infty$ as $t \rightarrow \infty$.

$$\begin{cases} -\Delta u = \lambda u + f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \quad (29)$$

where $f(x, 0) = 0$.

In this case, the trivial solution $u \equiv 0$ is solution of (29) for all $\lambda \in \mathbb{R}$.

- Is there a value of λ , say λ_0 , from which emanates new non-trivial solutions?
- What happens to these new solutions next to $(\lambda_0, 0)$?
- Is there a global behaviour of these new solutions?

λ^* is called a bifurcation point from the trivial solution (29) if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times E$ with $u_n \neq 0$ of solutions of (29) such that

$$(\lambda_n, u_n) \rightarrow (\lambda^*, 0).$$

Proposition

Assume that $f_u(x, 0) = 0$. If λ^ is a bifurcation point, then $\lambda^* = \lambda_k$, where λ_k is an eigenvalue of $-\Delta$.*

Local Bifurcation: Crandall-Rabinowitz's Theorem (1971)

Let X, Y be real Banach spaces and $V \subset X \times \mathbb{R}$ a neighborhood of $(0, \lambda_0)$. Consider the abstract equation:

$$F(\underline{u}, \lambda) = 0, \quad \text{where } F \in C^2(V, Y) \quad \text{and} \quad F(0, \lambda) = 0 \quad \forall \lambda.$$

The 3 Fundamental Hypotheses of Crandall–Rabinowitz (1971)

There exists a critical parameter $\lambda_0 \in \mathbb{R}$ such that:

1 One-Dimensional Kernel:

$$\ker D_u F(0, \lambda_0) = \text{span}\{\phi_0\}, \quad \text{with } \phi_0 \neq 0.$$

2 Codimension One (Closed Range):

$$\text{codim} R(D_u F(0, \lambda_0)) = 1 \implies Y = R(D_u F(0, \lambda_0)) \oplus \text{span}\{\psi\}.$$

3 Transversality Condition:

$$D_{\lambda u} F(0, \lambda_0)[\phi_0] \notin R(D_u F(0, \lambda_0)).$$

Local Bifurcation: Crandall–Rabinowitz Theorem (II)

Then, $(0, \lambda_0)$ is a **bifurcation point** for $F(\mathbf{u}, \lambda) = 0$.

There exist $\delta > 0$ and a continuous C^1 non-trivial curve:

$$\{(\mathbf{u}(s), \lambda(s)) : s \in (-\delta, \delta)\} \subset V$$

such that $F(\mathbf{u}(s), \lambda(s)) = 0$, with $\mathbf{u}(0) = 0$, $\lambda(0) = \lambda_0$, and $\mathbf{u}(s) \neq 0$ for $s \neq 0$. Locally around $(0, \lambda_0)$, the solution set $F^{-1}(0)$ consists precisely of two C^1 curves intersecting transversally:

- The **trivial branch**: $\{(0, \lambda) : \lambda \in \mathbb{R}\}$.
- The **bifurcated branch**: parametrized by $s \in (-\delta, \delta)$ as:

$$\begin{cases} \mathbf{u}(s) = s\phi_0 + s\psi(s), & \text{with } \psi(0) = 0 \text{ and } \psi(s) \in Z \\ \lambda(s) = \lambda_0 + s\mu(s), & \text{with } \mu(s) \text{ continuous.} \end{cases}$$

Application: Semilinear Logistic Equation

Let $X = \{u \in C^{2,\alpha}(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$ and $Y = C^{0,\alpha}(\bar{\Omega})$. Define $F : X \times \mathbb{R} \rightarrow Y$ as

$$F(u, \lambda) = -\Delta u - \lambda u - f(x, u).$$

- $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$ (trivial branch).
- $D_u F(0, \lambda)[v] = -\Delta v - \lambda v$.
- **1D Kernel:** $\ker D_u F(0, \lambda_1) = \text{span}\{\varphi_1\}$,
- **Codimension 1:** By the Fredholm Alternative,

$$R(D_u F(0, \lambda_1)) = \{y \in Y : \int_{\Omega} y \varphi_1 dx = 0\}.$$

$$\text{codim } R(D_u F(0, \lambda_1)) = \dim \ker D_u F(0, \lambda_1) = 1.$$

•

$$D_{\lambda u} F(0, \lambda_1)[\varphi_1] = -\varphi_1.$$

and then,

$$D_{\lambda u} F(0, \lambda_1)[\varphi_1] \notin R(D_u F(0, \lambda_1)).$$

Application: Semilinear Logistic Equation

Then, $(0, \lambda_1)$ is a **bifurcation point** of (29). That is, there exists $\delta > 0$ and a C^1 curve of non-trivial solutions $\{(u(s), \lambda(s)) : s \in (-\delta, \delta)\}$ with:

$$\begin{cases} u(s) = s\varphi_1 + s\psi(s), \text{ where } \psi(0) = 0 \text{ and } \int_{\Omega} \psi(s)\varphi_1 dx = 0, \\ \lambda(s) = \lambda_1 + s\mu(s). \end{cases}$$

- Non-trivial solutions branch off **exclusively** at the principal eigenvalue λ_1 .
- For $s > 0$ small, $u(s) \approx s\varphi_1 > 0$ yields a positive physical state.

Global Bifurcation: Rabinowitz's Theorem (1971)

Theorem

Assume $f(x, 0) = f_u(x, 0) = 0$ and let λ_k an eigenvalue of $-\Delta$ with odd multiplicity. Then, from λ_k emanates a component $\mathcal{C}$ (i. e. a maximal connected subset) of the closure of the set of nontrivial solutions of (29) such that either

- i) $\mathcal{C}$ is unbounded in $\mathbb{R} \times E$;
or
- ii) $\mathcal{C}$ meets at $u = 0$ in a point $(\mu, 0)$ with μ an eigenvalue of $-\Delta$ with $\mu \neq \lambda_k$.

Proposition

Assume $f(x, 0) = f_u(x, 0) = 0$. The point $(\lambda_1, 0)$ is a bifurcation point from the trivial solutions of positive solutions of (29). Moreover, the component $\mathcal{C}^+$ is unbounded in $\mathbb{R} \times E$.

$$\begin{cases} -\Delta u = \lambda u - bu^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (30)$$

with $b > 0$, $\lambda \in \mathbb{R}$.

Theorem

There exists at least a positive solution of (31) if and only if $\lambda > \lambda_1$.

Proof:

- There exists an unbounded continuum $\mathcal{C}$ of positive solutions emanating from the trivial solution at $\lambda = \lambda_1$.
- There do not exist positive solutions for $\lambda \leq \lambda_1$.
- For any positive solution u we have the a priori bound

$$u \leq \frac{\lambda}{b} \quad \text{in } \Omega.$$

Bifurcation method: Application II (nonlocal logistic equation)

$$\begin{cases} -\Delta u = u \left(\lambda - \int_{\Omega} b(x, y) u^p(y) dy \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (31)$$

with $p > 0$, $\lambda \in \mathbb{R}$ and $b(x, y) > 0$.

Theorem

There exists at least a positive solution of (31) if and only if $\lambda > \lambda_1$.

Theorem

Assume that

$$\lim_{t \rightarrow \infty} \frac{f(x, t)}{t^r} = h(x) \geq m > 0, \quad (32)$$

for some $1 < r < (N + 2)/(N - 2)$. Then, for any compact subset $\Lambda \subset \mathbb{R}$ there exists a constant C such that for any solution u of (29) with $\lambda \in \Lambda$, it holds

$$\|u\|_{\infty} \leq C.$$

$$\begin{cases} -\Delta u = \lambda u + bu^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (33)$$

with $b > 0$, $\lambda \in \mathbb{R}$.

Theorem

Assume that $1 < r < (N + 2)/(N - 2)$. There exists at least a positive solution of (31) if and only if $\lambda < \lambda_1$.

Proof:

- There exists an unbounded continuum $\mathcal{C}$ of positive solutions emanating from the trivial solution at $\lambda = \lambda_1$.
- There do not exist positive solutions for $\lambda \geq \lambda_1$.
- Since $1 < r < (N + 2)/(N - 2)$ there exists a priori bounds.

- 1 Bifurcation results can be applied to problems that are written in the general form

$$u = \lambda K(u) + N(\lambda, u),$$

where K is a compact operator and N is a completely continuous mapping such that $N(\lambda, u) = o(\|u\|)$ as $\|u\| \rightarrow 0$.

- 1 The sub-supersolution and bifurcation can be also applied to the Lotka-Volterra systems

$$\begin{cases} -\Delta u = u(\lambda - u \pm bv) & \text{in } \Omega, \\ -\Delta v = v(\mu - v \pm cu) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (34)$$

- 2 Similarly to the system arising from the tumor dynamics (chemotaxis)

$$\begin{cases} -\Delta u = -\alpha \operatorname{div}(u \nabla v) + u(\lambda - u) & \text{in } \Omega, \\ -\Delta v + v = cu & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (35)$$

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