Inverse problems in the wave equation:

from stability to parameter reconstruction

Lucie Baudouin

Toulouse, June 2025

The goal of this lecture is to detail methodologies based on Carleman estimates for several questions related to inverse problems for evolution partial differential equations. The focus will be the recovery of parameters in the wave equation posed in a bounded domain with Dirichlet boundary conditions.

Carleman estimates are a type of weighted energy inequalities and are particularly useful in proving for instance unique continuation properties, controllability results, or stability of inverse problems. Here, we will mainly consider the determination of the potential (a zeroth order term) in the whole domain from measurements of the Neumann boundary value of the solution of the wave equation.

We will first present the proof of the uniqueness and stability of the inverse problem, that indeed came first historically, more than 25 years ago, using local and then global Carleman estimates. Nevertheless, our main goal will be the proposition of a reconstruction algorithm, designed quite more recently, that uses the same technical tools. This algorithm has the particularity of being constructed using the strategy of the proof of stability of the inverse problem to get its convergence. It is indeed relying on the minimization of a functional based on the corresponding Carleman weight function. In a setting where the inverse problem is a non-linear one, conventional methods can encounter local minima, which is not the case with this approach, that is globally convergent. We will outline the milestones of demonstrating the convergence of this algorithm in reconstructing a time-independent potential in the wave equation. The overcome numerical difficulties will also be presented. We will conclude with several illustrations of this Carleman-based Reconstruction algorithm.

We will also take this opportunity to show numerical simulations in two extensions of this approach to the reconstruction of parameters in other cases : reconstruction of the potentials terms in a network of string equations, but also reconstruction of the speed of wave propagation, both problems of more direct applied interest.

The results presented here were developed over many years, in close collaboration with Maya de Buhan and Sylvain Ervedoza, but also Axel Osses, Emmanuelle Crépeau and Julie Valein and I warmly thank each of them for our fruitful and long-standing interactions.

Contents

1 Introduction **3**

2 Setting - The wave equation in a bounded domain **4**
2.1 Basic existence and regularity properties 4
2.2 A toy inverse problem 8
2.3 Inverse problem and stability results 10

3 Main tool - Carleman estimates for the waves **12**
3.1 A 2-parameters global Carleman estimate 13
3.2 A 1-parameter specific global Carleman estimate 14
3.3 Proof of the 1-parameter Carleman estimate 15

4 Proof of Lipschitz Stability for the inverse problem **21**

5 Carleman-based reconstruction algorithm (C-bRec) **25**
5.1 Main ideas of the C-bRec algorithm 26
5.2 Statement of the reconstruction algorithm 28
5.3 Convergence result: statement and proof 30
5.3.1 Landmarks 30
5.3.2 Proof of the convergence of the C-bRec algorithm 32

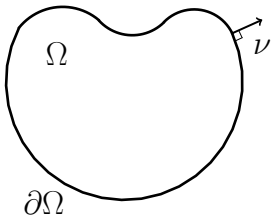
6 Numerics and further applications **35**
6.1 Numerical tractability 35
6.2 Numerical simulations of Algorithm 1 in 1-D 37
6.3 Some other related settings where C-bRec was succesfully applied 40
6.3.1 Reconstruction of a potential on a network of string equations 40
6.3.2 Reconstruction of the speed in a wave equation 44

1 Introduction

Assume that we are interested in the study of a dynamic phenomenon that is modeled by a partial differential equation (PDE). The first topic of interest then would be the forward (or direct) problem, where, knowing all the physical parameters or coefficients, source terms and initial conditions, we want to prove the existence of a solution and study its behavior and properties.

In science, an inverse problem is the reverse process of studying from a set of observations, the causal factors that produced them. Thus, for instance, one could be interested in an inverse problem related to PDEs, such as the determination of some physical coefficient from a local measurement of the behavior of the solution. besides, inverse problems are very relevant mathematical problems because they focus on the study of parameters that cannot be directly observed.

Let Ω be an open smooth bounded domain of $\mathbb{R}^d$, with $d \geq 1$, $\partial\Omega$ denotes its boundary, and we call ν the outward unit normal vector on $\partial\Omega$. Let also T be a strictly positive number.



This lecture focuses on the determination of the potential in a wave equation according to the following inverse problem:

> Given the in-domain f and boundary f_∂ source terms and the initial position and velocity data (w_0, w_1), considering the solution of the wave partial differential equation system
>
> $$\begin{cases} \partial_t^2 w - \Delta w + q^\star w = f, & \text{in } (0,T)\times\Omega, \\ w = f_\partial, & \text{on } (0,T)\times\partial\Omega, \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases} \tag{1}$$
>
> can we determine the unknown potential $q^\star = q^\star(x)$, assumed to depend only on $x \in \Omega$, from the additional knowledge of the flux of the solution $\partial_\nu w$ through a part Γ_0 of the boundary $\partial\Omega$, namely
>
> $$\mathscr{M} = \partial_\nu w|_{(0,T)\times\Gamma_0}? \tag{2}$$

In the sense of J. Hadamard [Had02], an appropriate mathematical modeling of physical phenomena has to ensure that the solution exists, is unique and is stable (i.e. satisfies a continuous dependance with respect to the data in a reasonable topology). This is not a straightforward question regarding direct problems for PDEs, where all the data (physical coefficients, source terms and initial data) are supposed to be known and we are looking for the solution of the system. Indeed, the proof of the existence of solutions in appropriate functional

spaces can be a challenge in itself, and their uniqueness or stability with respect to the data as well. It may be even more complicated when considering an inverse problem such as the one presented above. Indeed, inverse problems are often ill-posed, meaning usually that either the solution is not unique and/or small variability in the observation or known data may lead to large errors in the solution.

Actually, when considering such an inverse problem, several questions naturally arise, that can be summed up by the ideas of *identifiability* of the unknown coefficient, *identification / reconstruction*, and finally the *convergence* of the numerical reconstruction toward the physical coefficient.

More precisely, first, the well-posedness of the problem itself answers the question of *identifiability*, addressing the uniqueness and stability of the solution of the inverse problem (see for instance [Yam99] and [Bau10]). Then, as soon as this first difficulty can be overcome, the proposition of a *reconstruction* algorithm for the unknown coefficient $q^\star$, from the extra knowledge of $\mathscr{M}$ will be our main goal, taking also care of *convergence* issues.

In this lecture, we will first recall basic existence and regularity properties of solutions of the wave equation (1), and detail a statement of the Lipschitz stability result for the inverse problem under consideration in Section 2. In order to prove stability, Section 3 will present and prove some global Carleman estimates for the wave operator, and then Section 4 details the proof of stability. Section 5 will detail the design of a Carleman-based Reconstruction (C-bRec) algorithm for the potential and give the proof of its convergence. Finally, numerical simulations will be presented in Section 6, along with various other cases of the C-bRec approach for coefficient reconstructions.

All these results were obtained in collaborations with several colleagues, starting by very fruitful interactions with Maya de Buhan and Sylvain Ervedoza, as highlighted in [BdBE13], [BE13], [BdBE17], then with Axel Osses [BEO15], Emmanuelle Crépeau and Julie Valein [BdBCV25]. Along the document, we will give several landmarks in the existing literature. However, we do not mean to conduct a complete survey on the topic and refer the reader to the bibliography of the articles mentioned previously.

2 Setting - The wave equation in a bounded domain

2.1 Basic existence and regularity properties

This section is very much inspired by the unpublished notes of the lectures given by Jean-Pierre Puel in june 2004 during the advanced school *Control of Solids and Structures: Mathematical Modelling and Engineering Applications* at the C.I.S.M. in Udine, Italy. It's also partly close to [Zua24, Chapter 2] that focuses on controllability questions for the wave equation.

We will need to consider wave equation's systems taking the generic shape (see [DL90] for

a complete study of the wave equations):

$$\begin{cases} \partial_t^2 v(t,x) - \Delta v(t,x) + q(x)v(t,x) = f(t,x), & t \in (0,T),\ x \in \Omega \\ v(t,x) = f_\partial(t,x), & t \in (0,T),\ x \in \partial\Omega \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{3}$$

where (v_0, v_1) are the initial position and velocity, and f and f_∂ are the in-domain and boundary source terms.

In the whole lecture, we will denote, for some $m > 0$,

$$L^\infty_{\leq m}(\Omega) = \{q \in L^\infty(\Omega), \|q\|_{L^\infty(\Omega)} \leq m\}.$$

Let us first recall here well known results about the existence of finite energy solutions for the wave equation, starting with the case of homogeneous boundary conditions ($f_\partial = 0$):

Theorem 1 *Let $\Omega \subset \mathbb{R}^d$ be an open smooth bounded domain, $T > 0$ and $m > 0$. Assuming that $q \in L^\infty_{\leq m}(\Omega)$, $f \in L^1(0,T;L^2(\Omega))$, $v_0 \in H^1_0(\Omega)$, $v_1 \in L^2(\Omega)$ and $f_\partial = 0$, the Cauchy problem (3) is well-posed. Indeed, system (3) admits a unique weak solution*

$$v \in C([0,T], H^1_0(\Omega)) \cap C^1([0,T], L^2(\Omega))$$

such that there exists a constant $C = C(\Omega, T, m) > 0$ ensuring that for all $t \in (0,T)$, the energy of the system

$$E_v(t) = \frac{1}{2}\left(\|\partial_t v(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)}\right)$$

satisfies

$$E_v(t) \leq C\left(\|v_0\|^2_{H^1_0(\Omega)} + \|v_1\|^2_{L^2(\Omega)} + \|f\|^2_{L^1(0,T,L^2(\Omega))}\right). \tag{4}$$

Moreover, the normal derivative $\partial_\nu v$ belongs to $L^2(0,T;L^2(\partial\Omega))$ and satisfies

$$\|\partial_\nu v\|^2_{L^2(0,T;L^2(\partial\Omega))} \leq C\left(\|v_0\|^2_{H^1_0(\Omega)} + \|v_1\|^2_{L^2(\Omega)} + \|f\|^2_{L^1(0,T,L^2(\Omega))}\right). \tag{5}$$

Proof. This result is very classical and we refer to [LM72] (Chapter 3) for the proof of the existence and uniqueness of solution to equation (3). Estimate (4) can formally be deduced from multiplication of (3) by $\partial_t v$, and the integration of this equality on Ω, using some integrations by parts. Concerning estimate (5), we refer to [Lio88] (Chapter 1). This estimate is a hidden regularity result which can be obtained by multiplier's technique. Anyway, we shall give here some more details about these proofs.

• The proof of existence and regularity of the solution is based on an energy estimate. The formal calculations that will be given here can actually be justified on approximate equations and one can process then by convergence toward the solution. Indeed, more rigorously, a Galerkin method, based on a Fourier decomposition of the solutions allows to prove this lemma.

In order to prove (4), we can multiply (3) by $\partial_t v$, and integrate over Ω, or we can calculate the time derivative of $t \mapsto E_v(t)$. Anyhow, using (3) and integrating by parts, one gets, from Cauchy-Schwarz inequality,

$$\frac{d}{dt}E_v(t) \leq \|q\|_{L^\infty(\Omega)}\|v(t)\|_{L^2(\Omega)}\|\partial_t v(t)\|_{L^2(\Omega)} + \|f(t)\|_{L^2(\Omega)}\|\partial_t v(t)\|_{L^2(\Omega)}.$$

Since Ω is bounded, the Poincaré inequality shows that there exists $C_\Omega > 0$ such that

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega)}. \tag{6}$$

Therefore, we can write

$$\frac{d}{dt}E_v(t) \leq C_\Omega \|q\|_{L^\infty(\Omega)} E_v(t) + \|f(t)\|_{L^2(\Omega)}\sqrt{2E_v(t)},$$

so that

$$\frac{d}{dt}\sqrt{E_v(t)} \leq \frac{C_\Omega}{2}\|q\|_{L^\infty(\Omega)}\sqrt{E_v(t)} + \frac{1}{\sqrt{2}}\|f(t)\|_{L^2(\Omega)}.$$

Finally, using Gronwall's Lemma, we obtain that for all $t \in (0,T)$,

$$\sqrt{E_v(t)} \leq \left(\sqrt{E_v(0)} + \frac{1}{\sqrt{2}}\|f\|_{L^1(0,T,L^2(\Omega))}\right) e^{\frac{C_\Omega}{2}\|q\|_{L^\infty(\Omega)}T},$$

meaning, for $C = C(\Omega, T, m)$,

$$\|\partial_t v(t)\|_{L^2(\Omega)} + \|\nabla v(t)\|_{L^2(\Omega)} \leq C\left(\|v_0\|_{H_0^1(\Omega)} + \|v_1\|_{L^2(\Omega)} + \|f\|_{L^1(0,T,L^2(\Omega))}\right).$$

This estimate is equivalent to (4), which gives an explicit bound on the energy E_v in terms of the initial data, the source term and the potential of the equation, as well as a formal argument to assess the existence of a unique solution of (3) in the energy space $C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$.

• The proof of the hidden regularity result on the normal derivative is not a consequence of the regularity of the solution proved above. Indeed, trace estimates of $v(t) \in H_0^1(\Omega)$ do not yield $\partial_\nu v(t) = \nabla v(t)\cdot\nu \in L^2(\partial\Omega)$. Here, we will rely on the following lemma that states a multiplier's identity, where computations are done for regular solutions. One can read details in the book [Lio88] of J.-L. Lions or the one of V. Komornik [Kom94].

Lemma 1 *Let $\mu \in C^1(\overline{\Omega}; \mathbb{R}^d)$ be a vector field. For any solution of*

$$\begin{cases} \partial_t^2 v - \Delta v = f, & \text{in } (0,T)\times\Omega \\ v = 0, & \text{in } (0,T)\times\partial\Omega \\ v(0) = v_0, \quad \partial_t v(0) = v_1, & \text{in } \Omega, \end{cases} \tag{7}$$

with smooth initial and source data, the following multiplier's identity holds true:

$$\frac{1}{2}\int_0^T\int_{\partial\Omega} |\partial_\nu v|^2 (\mu\cdot\nu)\, d\sigma dt = \left[\int_\Omega \partial_t v\, \mu\cdot\nabla v\, dx\right]_0^T + \frac{1}{2}\int_0^T\int_\Omega (\mathrm{div}\mu)\left(|\partial_t v|^2 - |\nabla v|^2\right) dxdt$$
$$+ \int_0^T\int_\Omega \sum_{i,j=1}^n \partial_{x_i} v\, \partial_{x_i}\mu_j\, \partial_{x_j} v\, dxdt - \int_0^T\int_\Omega f\,\mu\cdot\nabla v\, dxdt.$$

The proof of this result stems from the multiplication of $\partial_t^2 v - \Delta v = f$ in (7) by $\mu \cdot \nabla v$ and integration over $(0,T)\times\Omega$. Indeed, forgetting (t,x) variables, usual integrations by parts bring first, thanks to the homogeneous boundary condition in (7),

$$\int_0^T \int_\Omega \partial_t^2 v\, \mu \cdot \nabla v\, dxdt = \left[\int_\Omega \partial_t v\, \mu \cdot \nabla v\, dx\right]_0^T + \frac{1}{2}\int_0^T \int_\Omega (\mathrm{div}\mu)|\partial_t v|^2\, dxdt$$

and second,

$$\int_0^T \int_\Omega -\Delta v\, \mu \cdot \nabla v\, dxdt = \int_0^T \int_\Omega \sum_{i,j=1}^n \partial_{x_i} v\, \partial_{x_i}\mu_j\, \partial_{x_j} v\, dxdt$$
$$- \frac{1}{2}\int_0^T \int_\Omega (\mathrm{div}\mu)|\nabla v|^2\, dxdt - \frac{1}{2}\int_0^T \int_{\partial\Omega} |\partial_\nu v|^2\, (\mu\cdot\nu)\, d\sigma dt.$$

Summing up the two equations and using $\partial_t^2 v - \Delta v = f$, one gets the desired identity.

Remark 1 *This kind of multiplier's identity has two main uses, corresponding to different choices of vector field μ : besides the proof of hidden regularity result (see below), it is also applied to get observability estimates when proving controllability.*
This multiplier's technique is usually first applied for regular solutions and then validated by density arguments.

Let us apply this result to equation (3), where $\partial_t^2 v - \Delta v = f - qv$ and $f_\partial = 0$, and with a specific vector field $\mu = \mu(x) \in C^1\left(\overline{\Omega}\times[0,T],\mathbb{R}^p\right)$ such that $\mu = \nu$ on $\partial\Omega$, of class C^1. Note that reference [Lio88] contains a rigorous demonstration of the existence of such an extension $\mathbb{R}^p$ of a field of normal vectors at the boundary of a bounded open set of class C^1.

Here, $\mu\cdot\nu = 1$ on $(0,T)\times\partial\Omega$, and one gets, for a constant $C = C(m) > 0$

$$\int_0^T \int_{\partial\Omega} |\partial_\nu v|^2\, d\sigma dt \le C\left(E_v(0) + E_v(T) + \int_0^T E_v\, dt + \int_0^T \int_\Omega |f||\nabla v|\, dxdt\right)$$

bringing, from (4), the desired result (5). □

Let us finally consider now a wave equation with non homogeneous Dirichlet boundary condition $f_\partial \neq 0$. A transposition method, not detailed here, allows to prove the well-posedness of the direct problem presented in system (3) as stated below.

Theorem 2 *Assume that $q \in L^\infty_{\le m}(\Omega)$.*
If $f \in L^1(0,T;H^{-1}(\Omega))$, $f_\partial \in L^2((0,T)\times\partial\Omega)$, $(v_0,v_1)\in L^2(\Omega)\times H^{-1}(\Omega)$, then the initial boundary value problem (3) is well-posed in

$$C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$$

and the solution v is such that $\partial_\nu v \in L^2(0,T;H^{-1}(\partial\Omega))$.

Similarly, if $f \in L^1(0,T;L^2(\Omega))$, $f_\partial \in H^1((0,T)\times\partial\Omega)$, $(v_0,v_1) \in H^1(\Omega)\times L^2(\Omega)$, *and assuming the compatibility condition* $f_\partial(0,x) = v_0(x)$ *for all* $x \in \partial\Omega$, *then* (3) *is well-posed in*

$$C^0([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$$

and the solution v *is such that* $\partial_\nu v \in L^2(0,T;L^2(\partial\Omega))$.

Moreover, if now $f \in L^1(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$, $f_\partial \in H^2((0,T)\times\partial\Omega)$, $(v_0,v_1) \in H^2(\Omega)\times H^1(\Omega)$, *and assuming compatibility conditions* $f_\partial(0,x) = v_0(x)$ *and* $\partial_t f_\partial(0,x) = v_1(x)$ *for all* $x \in \partial\Omega$, *then* (3) *is well-posed in*

$$C^0([0,T];H^2(\Omega)) \cap C^1([0,T];H^1(\Omega)) \cap C^2([0,T];L^2(\Omega))$$

and the solution v *is such that* $\partial_\nu v \in L^2(0,T;H^1(\partial\Omega))$.

Proof. These results, proved by transposition, are also quite classical and we can refer to [Lio88, Kom94, Zua24], and for the two higher regularity statements, see [LLT86]. One should also know that here again, the regularity of the normal derivative of the solution is not a consequence of the functional spaces of the well-posedness, but regularity results in themselves. □

2.2 A toy inverse problem

As a first inverse problem for the wave equation, one could consider the following one :

Considering the solution of the wave partial differential equation system

$$\begin{cases} \partial_t^2 w - \Delta w = 0, & \text{in } (0,T)\times\Omega, \\ w = 0, & \text{on } (0,T)\times\partial\Omega, \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases} \tag{8}$$

can we determine the unknown initial position and velocity data (w_0, w_1) from the additional knowledge of the flux of the solution through a part Γ_0 of the boundary $\partial\Omega$, namely

$$\mathscr{M} = \partial_\nu w|_{(0,T)\times\Gamma_0}? \tag{9}$$

Since the application $(w_0,w_1)|_\Omega \mapsto \partial_\nu w|_{(0,T)\times\Gamma_0}$, that associates the initial data with the boundary observation, is linear, this inverse problem is linear as well. This makes it simpler to study with respect to the one we presented in the introduction that is non linear and will be the main object of interest in this course.

Let us consider more closely the question of the well-posedness applied to this linear inverse problem. Considering that the existence of initial data corresponding to the measurement we have access to is implicit, the main difficulty for the well-posedness lies actually in the proof of positive answers, under possible restrictive conditions on the setting, to the two following questions:

Uniqueness : Does $\partial_\nu w = 0$ on $(0,T)\times\Gamma_0$ imply $(w_0,w_1) = (0,0)$ on Ω ?

Stability : Is it possible to estimate some norm of (w_0,w_1) on Ω, by a suitable norm of $\partial_\nu w$ on $(0,T)\times\Gamma_0$?

First, as one can find in the literature, the **uniqueness** question has a positive answer back to Holmgren's uniqueness theorem (see [Hor63] e.g.). Indeed, the following unique continuation result holds true, for every Lipschitz domain Ω and open and non-empty subset Γ_0 of $\partial\Omega$ and for $T > T_0(\Omega, \Gamma_0)$:

> If w solution of (8) satisfies $\partial_\nu w = 0$ on $(0,T) \times \Gamma_0$,
> then $w \equiv 0$ and therefore $(w_0, w_1) = (0,0)$ on Ω

thanks to Holmgren's theorem (see also [Lio88, Section I.8]).

Second, an answer about the **stability** can be written for $(w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\partial_\nu w \in L^2((0,T) \times \Gamma_0)$, that is called *Observability inequality*, or also *inverse inequality*, and writes

$$\|(w_0, w_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq C \|\partial_\nu w\|_{L^2((0,T)\times\Gamma_0)}.$$

It's a well known result, used in the theory of controllability of the wave equation from the boundary, and it holds under some geometric and time condition for instance [Lio88], where T and Γ_0 have to be large enough in some specific sense.

This estimate can indeed be proved using the multiplier's identity given in Lemma 1, choosing the vector field $\mu(x) = x - x_0$ for $x_0 \in \mathbb{R}^d$, that suggest to choose

$$\Gamma_0 = \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\} \quad \text{and} \quad T > \sup_{x\in\Omega} |x - x_0|.$$

We will not prove here this result, but the interested reader can find all needed details in [Zua24]. And obviously, we can deduce uniqueness from the stability result.

To conclude this quick overview of this toy inverse problem, we can point out that the main quality of the multiplier method is that it is very accessible, relying essentially on a few integration by parts. But it cannot tackle lower order terms in general, and a systematic method for that is provided by Carleman estimates, which are a sort of "sophisticated version of the multiplier method" (borrowing these words from [TW09]), the multiplier being constructed from an appropriate weight function. We shall work this way in the next sections.

Another classical type of linear inverse problem is the determination of the source term of a partial differential equation. When considering Carleman estimates approaches of this question, there is a huge literature available on this topic that we do not intend to present here. Nevertheless, let us mention that concerning the wave equation, stability of the inverse source problem from extra boundary data goes back to [PY96] with a combination of a local Carleman estimate, observability result and a the compactness-uniqueness argument. The recent article [BdBDIM26] studies the identification of a space dependent source term multiplied by a known time and space dependent function, from internal velocity or field measurements. There, the stability of the inverse problem is proved from global Carleman estimates and a sequential reconstruction strategy with numerical simulations are also given. Finally, in a one-dimensional wave equation with variable coefficients, [CM25] propose and analyze the numerical approximation of a source determination from a boundary observation, using mixed finite element method and dedicated uniform boundary observability inequality.

2.3 Inverse problem and stability results

Coming back to the main topic of this lecture, let us denote by $w[q^\star]$ the weak solution of system (1)

$$\begin{cases} \partial_t^2 w(t,x) - \Delta w(t,x) + q^\star(x) w(t,x) = f(t,x), & \text{in } (0,T)\times\Omega, \\ w(t,x) = f_\partial(t,x), & \text{on } (0,T)\times\partial\Omega, \\ w(0,x) = w_0(x), \quad \partial_t w(0,x) = w_1(x), & \text{in } \Omega. \end{cases}$$

Before going further, let us state that according to Lemma 2, with $q^\star \in L^\infty(\Omega)$, $f \in L^1(0,T;L^2(\Omega))$, $f_\partial \in H^1((0,T)\times\partial\Omega)$, $w_0 \in H^1(\Omega)$ and $w_1 \in L^2(\Omega)$, and assuming the compatibility condition $f_\partial(0,x) = w_0(x)$ for all $x \in \partial\Omega$, the initial boundary value problem (1) has a unique solution

$$w[q^\star] \in C^0([0,T];H^1(\Omega)) \cap C^1([0,T];L^2(\Omega)).$$

Moreover, $\partial_\nu w[q^\star]$ is well-defined as an element of $L^2((0,T)\times\partial\Omega)$, see [LLT86].

We are interested in the following non-linear inverse problem:

> Is it possible to retrieve the potential $q^\star = q^\star(x),\ x \in \Omega$ from measurement of the normal derivative $\partial_\nu w[q^\star]|_{(0,T)\times\Gamma_0}$ where Γ_0 is a part large enough of the boundary $\partial\Omega$ and the observation time T is also large enough ?

The non-linearity of this inverse problem stems from the non-linearity of the application $q^\star \mapsto \partial_\nu w[q^\star]$ and leads us to find a local answer to this question. Assuming that $q^\star \in L^\infty(\Omega)$ is an unknown potential, we are thus concerned with the stability around $q^\star$ and then by its reconstruction. In this direction, we will answer to two more precise problems.

Uniqueness : Under geometrical conditions on Γ_0 and T, does the equality $\partial_\nu w[q^\star] = \partial_\nu w[q]$ on $(0,T)\times\Gamma_0$ imply $q^\star = q$ on Ω ?

Stability : Under geometrical conditions on Γ_0 and T, is it possible to estimate $\|q^\star - q\|_{L^2(\Omega)}$ or better, a stronger norm of $(q^\star - q)$, by a suitable norm of $\partial_\nu w[q^\star] - \partial_\nu w[q]$ on $\Gamma_0\times(0,T)$?

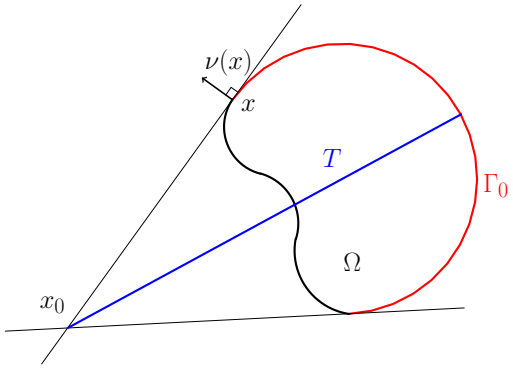
These questions for the wave equation have all already received positive answers since the uniqueness result for the linear inverse problem has been proved by M.V. Klibanov in [Kli92] and Lipschitz stability results (for both linear and non-linear inverse problems) of M. Yamamoto, deriving from it, can be read in [Yam99]. The proof in [Yam99] is based on a local Carleman estimate for the wave operator and a compactness-uniqueness argument in order to conclude to the stability from the uniqueness result and an observability estimate. We aim here at giving a direct proof of a Lipschitz stability estimate from a global Carleman estimate, result also interesting by itself. Another gain of this new proof of the precise result given below is the weakened assumptions on the solution of the wave equation under study. Besides, from the Carleman estimate we prove in the sequel, we directly obtain that a measurement of the flux of the solution on a suitable part Γ_0 of the boundary (instead of the whole boundary $\partial\Omega$)

is sufficient.

Our results will all require the following intertwined geometric and time conditions (sometimes called "multiplier condition"): $\exists\, x_0 \notin \overline{\Omega}$, such that

$$\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\}, \tag{H1}$$

$$T > \sup_{x\in\Omega} |x - x_0|. \tag{H2}$$



Space and time conditions (H1)–(H2) are natural from the observability point of view, and appear naturally in the context of the multiplier techniques developed in [Ho86, Lio88]. They are more restrictive than the well-known observability results [BLR92] by Bardos Lebeau Rauch based on the behavior of the rays of geometric optics, but the geometric conditions (H1)–(H2) yield much more robust results, and this will be of primary importance in our approach.

In fact, under the regularity assumption

$$w[q^\star] \in H^1(0,T;L^\infty(\Omega)),\ \text{with } \|w[q^\star]\|_{H^1(0,T;L^\infty(\Omega))} \leq K \tag{H3}$$

the positivity condition on the initial position

$$\exists r > 0 \text{ such that } |w_0| \geq r \text{ in } \Omega, \tag{H4}$$

the knowledge of an *a priori* bound $m > 0$ such that

$$\|q^\star\|_{L^\infty(\Omega)} \leq m, \quad i.e. \quad q^\star \in L^\infty_{\leq m}(\Omega) = \{q \in L^\infty(\Omega), \|q\|_{L^\infty(\Omega)} \leq m\}, \tag{H5}$$

and the multiplier conditions (H1)–(H2), the results in [Bau10] (and in [Yam99] under more regularity hypothesis) state the Lipschitz stability of the inverse problem consisting in the determination of the potential $q^\star$ in (1) from the measurement of the flux $\mathscr{M}$ in (9). This is precisely stated in the result below.

Theorem 3 ([Bau10]) *Let $m > 0$, $K > 0$ and $r > 0$ be such that assumptions* (H1)-(H2)-(H3)-(H4)-(H5) *are satisfied by, respectively, Γ_0, T, $w[q^\star]$, w_0 and $q^\star$. Then for all $q \in L^\infty_{\leq m}(\Omega)$, $\partial_\nu w[q] - \partial_\nu w[q^\star]$ belongs to $H^1(0,T;L^2(\Gamma_0))$ and there exists a constant $C = C(m,T,K,r) > 0$ such that for any $q \in L^\infty_{\leq m}(\Omega)$,*

$$C^{-1}\|q - q^\star\|_{L^2(\Omega)} \leq \|\partial_\nu w[q] - \partial_\nu w[q^\star]\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|q - q^\star\|_{L^2(\Omega)}. \tag{10}$$

The first inequality in (10) is the Lipschitz stability of the inverse problem while the second gives the continuous dependance of the derivatives of the solutions with respect to the potentials.

The proof of Theorem 3 is based on the use of a global Carleman estimate and is very close to the approach of [IY01b], that concerns the wave equation with Neumann boundary condition and Dirichlet observation for the inverse problem of retrieving a potential. Actually, it also follows some steps of [Yam99] but requirering less regularity conditions.

The use of Carleman estimate to prove uniqueness in inverse problems was introduced in [BK81] by A. L. Bukhgeim and M. V. Klibanov. Concerning inverse problems for hyperbolic equations with a single observation, we can refer to [PY96], [PY97], where the method relies on uniqueness results obtained by local Carleman estimates (see e.g. [Ima02], [Isa06]) and compactness-uniqueness arguments based on observability inequalities (see also [Zha00]). Related references [IY01b] and [IY01a] also use Carleman estimates, but rather consider the case of interior or Dirichlet boundary data observation.

Remark 2 *The condition $w[q^\star] \in H^1(0,T;L^\infty(\Omega))$ can be guaranteed uniformly for $q^\star \in L^\infty_{\leq m}(\Omega)$ with more constraints on the data $(w_0,w_1), f, f_\partial$ in (1). Indeed, if the dimension is $d=1$, assuming $(w_0,w_1)\in H^2(\Omega)\times H^1(\Omega)$, $f\in W^{1,1}(0,T;L^2(\Omega))$ and $f_\partial \in H^2(0,T;H^2(\Omega))$, with the compatibility conditions $f_\partial(x,0)=w_0(x)$ and $\partial_t f_\partial(x,0)=w_1(x)$ for all $x\in\partial\Omega$ then $w[q^\star]$ solution of (1) is such that $\partial_t w[q^\star]\in C^0([0,T];H^1(\Omega))\cap C^1([0,T];L^2(\Omega))$, with estimates depending only on m and the norms of $(w_0,w_1), f, f_\partial$ in these spaces. Therefore, due to Sobolev's imbedding, $\partial_t w[q^\star]\in L^2(0,T;L^\infty(\Omega))$. And for $d=2$ or 3, one needs extra orders of regularity.*

Remark 3 *The condition on w_0 requires compatibility conditions for w_0 and f_∂ on $\partial\Omega\times\{0\}$. In particular, $|f_\partial(x,0)|\geq r>0,\ x\in\partial\Omega$ must be satisfied since $|w_0(x)|\geq r>0,$ ae in $\overline{\Omega}$.*

Once knowing this inverse problem stability result, our goal will be to design a reconstruction algorithm that allows, from the extra knwoledge of $\mathscr{M}=\partial_\nu w|_{(0,T)\times\Gamma_0}$, the calculation of the unknown potential $q^\star=q^\star(x)$. The approach for such a design will rely on two main pillars: the understanding of the technical structure of the stability proof, and the construction of the most appropriate Carleman estimate.

We shall give first in the next section some details about Carleman estimates for this inverse problem. Then we will give the proof of Theorem 3, before describing the C-bRec agorithm and giving it's convergence proof.

3 Main tool - Carleman estimates for the waves

Local Carleman estimates for regular functions with compact support were introduced in [Car39] and have been developed e.g. in [Hor63]. Carleman estimates are weighted energy estimates build with respect to various operators and in this lecture we are interested in proving global Carleman estimates (meaning for functions considered up to the boundary of the domain) corresponding to the wave operator.

Let us define here, for $x_0\notin\overline{\Omega}$, $\lambda>0$ and $\beta\in(0,1)$, and for all $(t,x)\in(-T,T)\times\Omega$:

$$\varphi(t,x)=|x-x_0|^2-\beta t^2 \quad \text{and} \quad \psi(t,x)=e^{\lambda(\varphi(t,x)+C_0)} \tag{11}$$

where $C_0 > 0$ is chosen so that $\varphi + C_0 \geq 1$ on $\Omega \times (-T, T)$.

3.1 A 2-parameters global Carleman estimate

Theorem 4 ([Bau10]) *Assume that Γ_0 satisfies* (H1). *There exists two parameters $\lambda_0 > 0$ and $s_0 > 0$ and a constant $M = M(s_0, \lambda_0, T, \Omega, \beta, x_0)$ such that for all $\lambda > \lambda_0$, $s > s_0$,*

$$s\lambda \int_{-T}^{T}\int_{\Omega} e^{2s\psi}(|\partial_t v|^2 + |\nabla v|^2)\,dxdt + s^3\lambda^3 \int_{-T}^{T}\int_{\Omega} e^{2s\psi}|v|^2\,dxdt \\ \leq M \int_{-T}^{T}\int_{\Omega} e^{2s\psi}|\partial_t^2 v - \Delta v|^2\,dxdt + Ms\lambda \int_{-T}^{T}\int_{\Gamma_0} e^{2s\psi}\left|\partial_\nu v\right|^2\,d\sigma dt \quad (12)$$

for any $v \in H^1(-T, T; H_0^1(\Omega))$ satisfying

$$\begin{cases} \partial_t^2 v - \Delta v \in L^2((-T, T) \times \Omega), \\ v(\pm T, x) = \partial_t v(\pm T, x) = 0, \ \forall x \in \Omega. \end{cases}$$

We do not give here extensive references about Carleman estimates for hyperbolic equations, but one can have a look at [Ima02] and find references therein. One can read other versions of global Carleman estimates for hyperbolic equations in [Zha00] and [Tat96].

Remark 4 *Estimate* (12) *is easily adapted to the operator $\partial_t^2 - \Delta + p$, uniformly with respect to p for $p \in L^\infty_{\leq m}(\Omega)$. This is one of the specific interest of using such a tool, compared to a simpler multiplier approach. Moreover, this Carleman estimate is proved for any arbitrary time T, but is assuming that the function v and its time derivative are vanishing at time $\pm T$.*

Main ideas of the proof. From (11), let us set, for $s > 0$,

$$w(t, x) = v(t, x)e^{s\psi(t,x)},$$

and proceed with the formal calculation of what can be called a conjugate operator P of the wave operator $\partial_t^2 - \Delta$:

$$Pw = e^{s\psi}(\partial_t^2 - \Delta)(e^{-s\psi}w),$$

that obviously brings

$$e^{-s\psi}Pw = (\partial_t^2 - \Delta)v.$$

We have easily

$$\begin{aligned} Pw \ = \ & \partial_t^2 w - 2s\lambda\psi(\partial_t w \partial_t \psi - \nabla w.\nabla\psi) + s^2\lambda^2\psi^2 w(|\partial_t\psi|^2 - |\nabla\psi|^2) - \Delta w \\ & - s\lambda\psi w(\partial_t^2\psi - \Delta\psi) - s\lambda^2\psi w(|\partial_t\psi|^2 - |\nabla\psi|^2) \end{aligned}$$

and if we set, choosing the parameter α appropriately,

$$\begin{aligned} P_1 w &= \partial_t^2 w - \Delta w + s^2\lambda^2\psi^2 w(|\partial_t\psi|^2 - |\nabla\psi|^2) \\ P_2 w &= (\alpha - 1)s\lambda\psi w(\partial_t^2\psi - \Delta\psi) - s\lambda^2\psi w(|\partial_t\psi|^2 - |\nabla\psi|^2) - 2s\lambda\psi(\partial_t w\partial_t\psi - \nabla w.\nabla\psi) \quad (13) \\ Rw &= -\alpha s\lambda\psi w(\partial_t^2\psi - \Delta\psi) \end{aligned}$$

we get $P_1w + P_2w = Pw - Rw$, so that

$$\int_{-T}^{T}\int_{\Omega}\left(|P_1w|^2 + |P_2w|^2\right)\,dxdt + \,2\int_{-T}^{T}\int_{\Omega} P_1wP_2w\,dxdt = \int_{-T}^{T}\int_{\Omega}|Pw - Rw|^2\,dxdt$$

The main goal of the proof will be to minimize the cross-term in P_1wP_2w by positive and dominant terms looking similar to the one of the left hand side of (12) and negative boundary terms that will be moved to the right hand side of the estimate (see [Bau10]). The so-called dominant terms have the quality of depending on powers of the parameters s and λ that allow to absorb dominated terms when choosing s and λ large enough. This will also be the procedure to eliminate the term Rw in the right hand side of the estimate. All this will lead to (12).

We will not go further here, except for mentioning that if trying to prove (12) but for operator $\partial_t^2 - \Delta + p$, dealing with the presence of the potential p is only a matter of estimating pv the same way the terms of Rw were handled. Indeed, on $(-T,T)\times\Omega$,

$$|(\partial_t^2 - \Delta)v|^2 \leq 2|(\partial_t^2 - \Delta + p)v|^2 + 2\|p\|_{L^\infty_{\leq m}(\Omega)}|v|^2 \leq 2|(\partial_t^2 - \Delta + p)v|^2 + 2m|v|^2.$$

and choosing s_0 (or λ_0) large enough, one can absorb the extra term $2m\displaystyle\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|v|^2\,dxdt$ by the left hand side of (12) and obtain the same estimate for $\partial_t^2 - \Delta + p$ with slightly different constants. □

Nevertheless, pursuing the idea that we will use a Carleman estimate both for proving stability of the inverse problem and designing a reconstruction algorithm, we will give below another Carleman estimate using a single parameter s instead of two parameters s and λ. This will come at the cost made precise in the next section.

3.2 A 1-parameter specific global Carleman estimate

Here, we will state, and prove, for the wave operator, a global Carleman estimate build with a single parameter weight. Before stating this new result, we shall define the following restricted set $\mathcal{O}$, based on the definition of the weight function φ in (11), for which we now assume that $\beta \in (0,1)$ satisfies

$$\beta T > \sup_{x\in\Omega}|x - x_0|, \tag{14}$$

which is possible since condition (H2) holds for some $x_0 \notin \overline{\Omega}$. We then set:

$$\mathcal{O} = \{(t,x)\in(0,T)\times\Omega,\ \beta t > |x - x_0|\} = \{(t,x)\in(0,T)\times\Omega,\ |\partial_t\varphi(t,x)| \geq |\nabla\varphi(t,x)|\}, \tag{15}$$

which is depicted in Figure 1.

Theorem 5 ([BdBE17]) *Assume the multiplier conditions* (H1)-(H2) *and* $\beta \in (0,1)$ *as in* (14). *Define the weight function* φ *as in* (11). *Then there exist* $s_0 > 0$ *and a constant* $M > 0$ *such that for all* $s \geq s_0$*:*

$$\begin{aligned} s\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}\left(|\partial_t z|^2 + |\nabla z|^2 + s^2|z|^2\right)\,dxdt \leq M\int_{-T}^{T}\int_{\Omega} e^{2s\varphi}|\partial_t^2 z - \Delta z|^2\,dxdt \\ + Ms\int_{-T}^{T}\int_{\Gamma_0} e^{2s\varphi}\,|\partial_\nu z|^2\,d\sigma dt + Ms^3\iint_{(|t|,x)\in\mathcal{O}} e^{2s\varphi}|z|^2\,dxdt, \end{aligned} \tag{16}$$

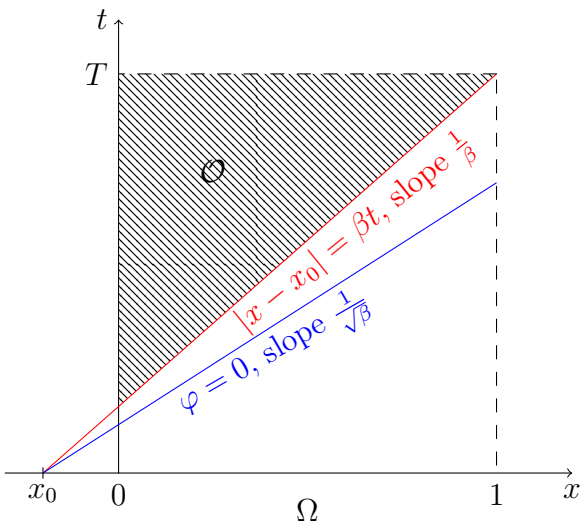


Figure 1: Illustration of domain $\mathcal{O}$ in the case $\Omega = (0, 1)$.

for all $z \in C^0([-T, T]; H_0^1(\Omega)) \cap C^1([-T, T]; L^2(\Omega))$ *with* $\partial_t^2 z - \Delta z \in L^2((-T, T) \times \Omega)$*, where the set* $\mathcal{O}$ *is given by* (15)*.*
Furthermore, if $z(0, x) = 0$ *for all* $x \in \Omega$*, one can add to the left hand-side of* (16)*, the following term:*

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t z(0)|^2 \, dx. \tag{17}$$

The Carleman estimate (16) of Theorem 5 is quite classical and can be found in the literature in several places, among which [BK12, LRS86, Isa06, Zha00, FYZ07, LTZ00]. For the convenience of the reader, we briefly sketch the proof in Section 3.3.

However, the fact that (17) can be added in the left hand side of (16) when $z(0, \cdot) = 0$ in Ω is not explicitly written in the aforementioned references, although this is one of the important point of the proof of the stability result in [IY01a, IY01b]. Nevertheless, the idea can be adapted easily from [BdBE13], as we will detail below.

Besides, one should be aware that as soon as the Carleman estimate should be applied, the concerned variable will need to be set on $(-T, T)$, thus usually needing an extension to negative times.

3.3 Proof of the 1-parameter Carleman estimate

Since a lot of different references, several of them mentioned right above, present detailed proof of Carleman estimates for the wave equation, we only give here the main calculations yielding the result presented in Theorem 5.

Recall that $\varphi(t, x) = |x - x_0|^2 - \beta t^2$, and let us set, for $s > 0$,

$$y(t, x) = z(t, x) e^{s\varphi(t,x)},$$

and proceed with the formal calculation of what can be called a conjugate operator P of the wave operator $\partial_t^2 - \Delta$:

$$Py = e^{s\varphi}(\partial_t^2 - \Delta)(e^{-s\varphi}y),$$

bringing $e^{-s\varphi}Py = (\partial_t^2 - \Delta)z$. Easy computations give

$$\begin{aligned} Py &= \partial_t^2 y - \Delta y + s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)y - 2s\partial_t y\partial_t\varphi + 2s\nabla y\cdot\nabla\varphi - s(\partial_t^2\varphi - \Delta\varphi)y \\ &= P_1 y + P_2 y + Ry, \end{aligned} \tag{18}$$

with

$$\begin{aligned} P_1 y &= \partial_t^2 y - \Delta y + s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)y \\ P_2 y &= -2s\partial_t y\partial_t\varphi + 2s\nabla y\cdot\nabla\varphi + \alpha s y \\ Ry &= -s(\partial_t^2\varphi - \Delta\varphi)y - \alpha s y, \end{aligned}$$

where we have set, d being the space dimension :

$$\alpha = 2d - 2.$$

Based on the estimate

$$\begin{aligned} 2\int_{-T}^{T}\int_\Omega P_1 y P_2 y\,dxdt &\le \int_{-T}^{T}\int_\Omega \left(|P_1 y|^2 + |P_2 y|^2\right)\,dxdt + 2\int_{-T}^{T}\int_\Omega P_1 y P_2 y\,dxdt \\ &\le 2\int_{-T}^{T}\int_\Omega |Py|^2\,dxdt + 2\int_{-T}^{T}\int_\Omega |Ry|^2\,dxdt, \end{aligned} \tag{19}$$

the main part of the proof consists in the computation and bound from below of the cross-term

$$I = \int_{-T}^{T}\int_\Omega P_1 y\, P_2 y\,dxdt$$

by positive and dominant terms, similar to the one of the left hand side of (16), and a negative boundary term, that will be moved to the right hand side of the estimate. For the sake of clarity, we will divide the proof in several steps.

All the computations below are done for smooth functions z (equivalently, y). Then, by a classical density argument, we can extend the results to any $z \in C^0([-T,T];H_0^1(\Omega)) \cap C^1([-T,T];L^2(\Omega))$ satisfying $\partial_t^2 z - \Delta z \in L^2((-T,T)\times\Omega)$ and $\partial_\nu z \in L^2((-T,T)\times\partial\Omega)$.

Step 1. Explicit calculations.

We set

$$I = \int_{-T}^{T}\int_\Omega P_1 y\, P_2 y\,dxdt = \sum_{i,k=0}^{3} I_{i,k}$$

where $I_{i,k}$ is the integral of the product of the ith-term in $P_1 y$ and the kth-term in $P_2 y$. We mainly use integrations by parts and the property of y that reads $y = 0$ on $(-T,T)\times\partial\Omega$. We shall also persistently use the fact that $\partial_t\varphi$ does not depend on x and $\nabla\varphi$ does not depend on t, and thus

$$\partial_t\nabla\varphi = \partial_t^2\nabla\varphi = \mathbf{0}, \qquad \partial_t\Delta\varphi = \partial_t(|\nabla\varphi|^2) = 0.$$

An integrations by part in time gives easily

$$\begin{aligned} I_{11} &= \int_{-T}^{T}\int_\Omega \partial_t^2 y\,(-2s\partial_t y\,\partial_t\varphi)\,dxdt = -s\int_{-T}^{T}\int_\Omega \partial_t((\partial_t y)^2)\partial_t\varphi\,dxdt \\ &= s\int_{-T}^{T}\int_\Omega (\partial_t y)^2\partial_t^2\varphi\,dxdt - s\left[\int_\Omega (\partial_t y)^2\partial_t\varphi\,dx\right]_{-T}^{T}. \end{aligned}$$

Similarly, using integrations by part in time or space, one has

$$\begin{aligned} I_{12} &= \int_{-T}^{T}\int_\Omega \partial_t^2 y\,(2s\nabla y\cdot\nabla\varphi)\,dxdt \\ &= -2s\int_{-T}^{T}\int_\Omega \partial_t y\,(\nabla\partial_t y\cdot\nabla\varphi)\,dxdt + 2s\left[\int_\Omega \partial_t y\,(\nabla y\cdot\nabla\varphi)\,dx\right]_{-T}^{T} \\ &= -s\int_{-T}^{T}\int_\Omega \nabla(\partial_t y)^2\cdot\nabla\varphi\,dxdt + 2s\left[\int_\Omega \partial_t y\,(\nabla y\cdot\nabla\varphi)\,dx\right]_{-T}^{T} \\ &= s\int_{-T}^{T}\int_\Omega (\partial_t y)^2\Delta\varphi\,dxdt + 2s\left[\int_\Omega \partial_t y\,(\nabla y\cdot\nabla\varphi)\,dx\right]_{-T}^{T}, \end{aligned}$$

and

$$I_{13} = \int_{-T}^{T}\int_\Omega \partial_t^2 y\,(\alpha s y)\,dxdt = -\alpha s\int_{-T}^{T}\int_\Omega (\partial_t y)^2\,dxdt + \alpha s\left[\int_\Omega \partial_t y\,y\,dx\right]_{-T}^{T}.$$

We compute in the same way

$$\begin{aligned} I_{21} &= \int_{-T}^{T}\int_\Omega (-\Delta y)\,(-2s\partial_t y\,\partial_t\varphi)\,dxdt = -2s\int_{-T}^{T}\int_\Omega \nabla y\cdot\partial_t\nabla y\,\partial_t\varphi\,dxdt \\ &= -s\int_{-T}^{T}\int_\Omega \partial_t(|\nabla y|^2)\,\partial_t\varphi\,dxdt = s\int_{-T}^{T}\int_\Omega |\nabla y|^2\,\partial_t^2\varphi\,dxdt - s\left[\int_\Omega |\nabla y|^2\,\partial_t\varphi\,dx\right]_{-T}^{T}, \end{aligned}$$

and

$$I_{23} = \int_{-T}^{T}\int_\Omega (-\Delta y)(\alpha s y)\,dxdt = \alpha s\int_{-T}^{T}\int_\Omega |\nabla y|^2\,dxdt.$$

Using the fact that $y|_{(-T,T)\times\partial\Omega} = 0$ implies $\nabla y = (\partial_\nu y)\nu$, that $|\nabla y|^2 = |\partial_\nu y|^2$ on $(-T,T)\times\partial\Omega$, and since $\nabla^2\varphi = 2\mathrm{Id}$, we can compute:

$$\begin{aligned} I_{22} &= \int_{-T}^{T}\int_\Omega (-\Delta y)\,(2s\nabla y\cdot\nabla\varphi)\,dxdt \\ &= 2s\int_{-T}^{T}\int_\Omega \nabla y\cdot\nabla(\nabla y\cdot\nabla\varphi)\,dxdt - 2s\int_{-T}^{T}\int_{\partial\Omega} (\nabla y\cdot\nu)(\nabla y\cdot\nabla\varphi)\,d\sigma dt \\ &= s\int_{-T}^{T}\int_\Omega \nabla(|\nabla y|^2)\cdot\nabla\varphi\,dxdt + 2s\int_{-T}^{T}\int_\Omega \nabla^2\varphi\cdot\nabla y\cdot\nabla y\,dxdt - 2s\int_{-T}^{T}\int_{\partial\Omega} |\partial_\nu y|^2\partial_\nu\varphi\,d\sigma dt \\ &= -s\int_{-T}^{T}\int_\Omega |\nabla y|^2\Delta\varphi\,dxdt + 4s\int_{-T}^{T}\int_\Omega |\nabla y|^2\,dxdt - s\int_{-T}^{T}\int_{\partial\Omega} |\partial_\nu y|^2\partial_\nu\varphi\,d\sigma dt. \end{aligned}$$

This term I_{22} together with I_{12}, that correspond to $(\partial_t^2 y - \Delta y)$ multiplied by $\nabla\varphi \cdot \nabla y$ recall the multiplier's calculations proving Lemma 1, in a more intricate version.
Finally, some integrations by part enable to obtain

$$\begin{aligned} I_{31} &= \int_{-T}^{T}\int_{\Omega} (s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)y)(-2s\partial_t y\, \partial_t\varphi)\, dxdt \\ &= -s^3 \int_{-T}^{T}\int_{\Omega} ((\partial_t\varphi)^2 - |\nabla\varphi|^2)\partial_t\varphi\partial_t(y^2)\, dxdt \\ &= s^3 \int_{-T}^{T}\int_{\Omega} \partial_t\left(((\partial_t\varphi)^2 - |\nabla\varphi|^2)\partial_t\varphi\right) y^2\, dxdt - s^3\left[\int_{\Omega}((\partial_t\varphi)^2 - |\nabla\varphi|^2)\partial_t\varphi\, y^2\, dx\right]_{-T}^{T}, \end{aligned}$$

$$\begin{aligned} I_{32} &= \int_{-T}^{T}\int_{\Omega} (s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)y)(2s\nabla y\cdot\nabla\varphi)\, dxdt \\ &= s^3 \int_{-T}^{T}\int_{\Omega} ((\partial_t\varphi)^2 - |\nabla\varphi|^2)\nabla(y^2)\cdot\nabla\varphi\, dxdt \\ &= -s^3 \int_{-T}^{T}\int_{\Omega} \nabla\cdot\left(\nabla\varphi((\partial_t\varphi)^2 - |\nabla\varphi|^2)\right) y^2\, dxdt, \end{aligned}$$

and

$$I_{33} = \int_{-T}^{T}\int_{\Omega} (s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)y)(\alpha s y)\, dxdt = \alpha s^3 \int_{-T}^{T}\int_{\Omega} ((\partial_t\varphi)^2 - |\nabla\varphi|^2)y^2\, dxdt.$$

Gathering all the terms that have been computed, we get

$$\begin{aligned} I &= s\int_{-T}^{T}\int_{\Omega} |\partial_t y|^2(\partial_t^2\varphi + \Delta\varphi - \alpha)\, dxdt + s\int_{-T}^{T}\int_{\Omega} |\nabla y|^2\,(\partial_t^2\varphi - \Delta\varphi + \alpha + 4)\, dxdt \\ &+ s^3\int_{-T}^{T}\int_{\Omega} |y|^2\left[\partial_t\left(\partial_t\varphi(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\right) + \alpha(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\ -\nabla\cdot\left(\nabla\varphi(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\right)\right] dxdt \\ &- s\left[\int_{\Omega}\left(|\partial_t y|^2 + |\nabla y|^2\right)\partial_t\varphi\, dx\right]_{-T}^{T} + 2s\left[\int_{\Omega}\partial_t y\,(\nabla y\cdot\nabla\varphi)\, dx\right]_{-T}^{T} \\ &- s^3\left[\int_{\Omega} y^2(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\partial_t\varphi\ dx\right]_{-T}^{T} + \alpha s\left[\int_{\Omega}\partial_t y\, y\, dx\right]_{-T}^{T} \\ &- s\int_{-T}^{T}\int_{\partial\Omega} |\partial_\nu y|^2\partial_\nu\varphi\, d\sigma dt. \end{aligned}$$

In the rest of the proof, $M > 0$ corresponds to a generic constant depending at least on Ω and T but independent of s.

Step 2. Dominating terms.

Let us now briefly explain how each term can be estimated.

• We focus on the terms in $s|\partial_t y|^2$ and $s|\nabla y|^2$ in order to insure that they are strictly positive. Taking $\alpha = 2d-2$, this yields

$$\partial_t^2\varphi + \Delta\varphi - \alpha = -2\beta + 2d - \alpha = 2(1-\beta) \quad \text{and}$$
$$\partial_t^2\varphi - \Delta\varphi + \alpha + 4 = -2\beta - 2d + \alpha + 4 = 2(1-\beta),$$

that are positive thanks to the assumption $\beta \in (0,1)$.

• The terms in $s^3|y|^2$ can be rewritten as follows (since $\nabla^2\varphi = 2\mathrm{Id}$):

$$\begin{aligned}
&\partial_t\left(\partial_t\varphi(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\right) + \alpha(|\partial_t\varphi|^2 - |\nabla\varphi|^2) - \nabla\cdot\left(\nabla\varphi(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\right)\\
&= (\partial_t^2\varphi - \Delta\varphi + \alpha)(|\partial_t\varphi|^2 - |\nabla\varphi|^2) + 2|\partial_t\varphi|^2\partial_t^2\varphi + 2\nabla^2\varphi\cdot\nabla\varphi\cdot\nabla\varphi\\
&= (-6\beta - 2d + \alpha)(|\partial_t\varphi|^2 - |\nabla\varphi|^2) + 4(1-\beta)|\nabla\varphi|^2\\
&= -(2+6\beta)(|\partial_t\varphi|^2 - |\nabla\varphi|^2) + 4(1-\beta)|\nabla\varphi|^2.
\end{aligned}$$

This quantity is bounded from below by a strictly positive constant in the region of $(-T,T)\times\Omega$ in which

$$|\partial_t\varphi(t,x)|^2 - |\nabla\varphi(t,x)|^2 \le 0 \Longleftrightarrow \beta t \le |x - x_0|,$$

i.e. the complementary of the set $\big\{(t,x) \in (-T,T)\times\Omega \text{ with } (|t|,x) \in \mathcal{O}\big\}$ where $\mathcal{O}$ satisfies (15).

• We now estimate the boundary terms in time appearing at time $t = T$ and $t = -T$. We focus on the terms at time T, as the ones at time $-T$ can be handled similarly. Let us first collect them:

$$\begin{aligned}
I_T := 2s\beta T\int_\Omega \left(|\partial_t y(T)|^2 + |\nabla y(T)|^2\right)\,dx + 8s^3\beta T\int_\Omega |y(T)|^2(\beta^2T^2 - |x-x_0|^2)\,dx\\
+ 4s\int_\Omega \partial_t y(T)\left(\nabla y(T)\cdot(x-x_0) + \frac{\alpha}{4}y(T)\right)\,dx.
\end{aligned}$$

The first and second terms are obviously positive (under Condition (14) for the second one), so we only need to check that they are sufficiently positive to absorb the last term, whose sign is unknown. We remark that

$$\begin{aligned}
&\int_\Omega \left|\nabla y(T)\cdot(x-x_0) + \frac{\alpha}{4}y(T)\right|^2\,dx\\
&= \int_\Omega |\,\nabla y(T)\cdot(x-x_0)|^2\,dx + \frac{\alpha}{4}\int_\Omega (x-x_0)\cdot\nabla\left(|y(T)|^2\right)\,dx + \frac{\alpha^2}{16}\int_\Omega |y(T)|^2\,dx\\
&= \int_\Omega |\,\nabla y(T)\cdot(x-x_0)|^2\,dx + \left(\frac{\alpha^2}{16} - \frac{\alpha d}{4}\right)\int_\Omega |y(T)|^2\,dx\\
&\le \sup_\Omega\left\{\,|x-x_0|^2\right\}\int_\Omega |\,\nabla y(T)|^2\,dx,
\end{aligned}$$

since $\alpha = 2d-2$ gives $\alpha^2 - 4\alpha d = -4(d-1)(d+1) \le 0$. This inequality allows to deduce, by

Cauchy-Schwarz inequality, that

$$4s\int_\Omega \partial_t y(T)\left(\nabla y(T)\cdot(x-x_0)+\frac{\alpha}{4}y(T)\right)\,dx \\ \leq 2s\sup_\Omega\{\,|x-x_0|\}\left(\int_\Omega\left(|\partial_t y(T)|^2+|\nabla y(T)|^2\right)\,dx\right).$$

Using again Condition (14), we easily obtain $I_T \geq 0$.

Step 3. Conclusion and back to the z variable.

Gathering these informations, and using the geometric condition (H1) on Γ_0, it yields that there exists a constant $M>0$ independent of s such that

$$\int_{-T}^T\int_\Omega P_1yP_2y\,dxdt \geq Ms\int_{-T}^T\int_\Omega\left(|\partial_t y|^2+|\nabla y|^2+s^2|y|^2\right)\,dxdt \\ -Ms\int_{-T}^T\int_{\Gamma_0}|\partial_\nu y|^2\,d\sigma dt - Ms^3\iint_{(|t|,x)\in\mathcal{O}}|y|^2\,dxdt.$$

From (19), we easily derive, since $Ry=-s(\partial_t^2\varphi-\Delta\varphi+\alpha)y$,

$$s\int_{-T}^T\int_\Omega\left(|\partial_t y|^2+|\nabla y|^2+s^2|y|^2\right)\,dxdt+\int_{-T}^T\int_\Omega\left(|P_1y|^2+|P_2y|^2\right)\,dxdt \\ \leq M\int_{-T}^T\int_\Omega|Py|^2\,dxdt+Ms^2\int_{-T}^T\int_\Omega|y|^2\,dxdt \\ +\ Ms\int_{-T}^T\int_{\Gamma_0}|\partial_\nu y|^2\,d\sigma dt+Ms^3\iint_{(|t|,x)\in\mathcal{O}}|y|^2\,dxdt.$$

We take now s_0 large enough in order to make sure that the term in $s^2|y|^2$ of the right hand side is absorbed by the dominant term in $s^3|y|^2$ of the left hand side as soon as $s\geq s_0$ and we obtain

$$s\int_{-T}^T\int_\Omega\left(|\partial_t y|^2+|\nabla y|^2+s^2|y|^2\right)\,dxdt+\int_{-T}^T\int_\Omega\left(|P_1y|^2+|P_2y|^2\right)\,dxdt \\ \leq M\int_{-T}^T\int_\Omega|Py|^2\,dxdt+Ms\int_{-T}^T\int_{\Gamma_0}|\partial_\nu y|^2\,d\sigma dt+Ms^3\iint_{(|t|,x)\in\mathcal{O}}|y|^2\,dxdt \quad (20)$$

Finally, going back to the variable z, since $z=ye^{-s\varphi}$, we have $\partial_t z=(\partial_t y-sy\partial_t\varphi)e^{-s\varphi}$ so that

$$e^{2s\varphi}(\partial_t z)^2\leq 2(\partial_t y)^2+2s^2(\partial_t\varphi)^2y^2\leq 2(\partial_t y)^2+Ms^2y^2,\ \text{in }\Omega\times(-T,T).$$

Similarly, we get

$$e^{2s\varphi}|\nabla z|^2\leq 2|\nabla y|^2+2s^2|\nabla\varphi|^2y^2\leq 2|\nabla y|^2+Ms^2y^2,\ \text{in }\Omega\times(-T,T)$$

and, using $y=0$ on $\partial\Omega$,

$$e^{2s\varphi}\,|\partial_\nu z|^2=|\partial_\nu y|^2\,,\ \text{on }\partial\Omega\times(-T,T).$$

We eventually deduce (16) by substituting $y = ze^{s\varphi}$ in the right hand side of (20).

Step 4. Extra term in the left hand side.

Furthermore, under the additional condition $z(0,\cdot) = 0$ in Ω, we get $y(0,\cdot) = 0$ in Ω. We then choose $\rho : t \mapsto \rho(t)$ a smooth function such that $\rho(0) = 1$ and ρ vanishes close to $t = -T$. We multiply $P_1 y$ by $\rho\partial_t y$ and integrate over $(-T,0)\times\Omega$, to get

$$\begin{aligned}
\int_{-T}^{0}\int_\Omega P_1 y\,\rho\partial_t y\,dxdt &= \int_{-T}^{0}\int_\Omega \left(\partial_t^2 y - \Delta y + s^2((\partial_t\varphi)^2 - |\nabla\varphi|^2)y\right)\,\rho\partial_t y\,dxdt \\
&= \frac{1}{2}\int_{-T}^{0}\int_\Omega \rho\partial_t\left(|\partial_t y|^2 + |\nabla y|^2\right)\,dxdt + \frac{s^2}{2}\int_{-T}^{0}\int_\Omega \rho(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\partial_t(y^2)\,dxdt \\
&= \frac{1}{2}\int_\Omega |\partial_t y(0)|^2\,dx - \frac{1}{2}\int_{-T}^{0}\int_\Omega \partial_t\rho\left(|\partial_t y|^2 + |\nabla y|^2\right) + s^2\partial_t\left(\rho(|\partial_t\varphi|^2 - |\nabla\varphi|^2)\right)y^2 dxdt \\
&\geq \frac{1}{2}\int_\Omega |\partial_t y(0)|^2\,dx - M\int_{-T}^{0}\int_\Omega \left(|\partial_t y|^2 + |\nabla y|^2 + s^2|y|^2\right)\,dxdt.
\end{aligned}$$

By Cauchy-Schwarz inequality, this implies

$$s^{1/2}\int_\Omega |\partial_t y(0)|^2\,dx \leq \int_{-T}^{T}\int_\Omega |P_1 y|^2\,dxdt + Ms\int_{-T}^{T}\int_\Omega \left(|\partial_t y|^2 + |\nabla y|^2 + s^2|y|^2\right)\,dxdt.$$

From (20) and $y = ze^{s\varphi}$, we easily deduce the estimate of term (17) and conclude the proof of Theorem 5. □

From this proof of Theorem 5, we can directly exhibit (see (20)) the following "conjugate" Carleman estimate, of practical interest later on:

Corollary 1 *Assume the multiplier condition* (H1)-(H2) *and* $\beta \in (0,1)$ *as in* (14). *Define the weight function* φ *as in* (11). *Then there exist constants* $M > 0$ *and* $s_0 > 0$ *such that for all* $s \geq s_0$,

$$\begin{aligned}
s\int_{-T}^{T}\int_\Omega \left(|\partial_t y|^2 + |\nabla y|^2 + s^2|y|^2\right)\,dxdt + \int_{-T}^{T}\int_\Omega \left(|P_1 y|^2 + |P_2 y|^2\right)\,dxdt \\
\leq M\int_{-T}^{T}\int_\Omega |Py|^2\,dxdt + Ms\int_{-T}^{T}\int_{\Gamma_0} |\partial_\nu y|^2\,d\sigma dt + Ms^3\iint_{(|t|,x)\in\mathcal{O}} |y|^2\,dxdt \quad (21)
\end{aligned}$$

for all $y \in C^0([-T,T];H_0^1(\Omega))\cap C^1([-T,T];L^2(\Omega))$, *with* $Py \in L^2((-T,T)\times\Omega)$, *where* P, P_1 *and* P_2 *are defined in* (18).

Furthermore, if $y(0,\cdot) = 0$ *in* Ω, *one can add the term* $s^{1/2}\displaystyle\int_\Omega |\partial_t y(0)|^2\,dx$ *to the left hand-side of* (21).

4 Proof of Lipschitz Stability for the inverse problem

We aim here at proving the Lipschitz Stability result of Theorem 3 using the 2-parameters global Carleman estimate stated in Theorem 4. The demonstration method is often called by the name of Bukgheim-Klibanov method, dating back to their seminal result in [BK81].

The goal of this section is twofold : explaining the main technical steps of the method and exhibiting the elements on which we will build the reconstruction algorithm and its convergence proof.

Proof of Theorem 3 - Let us set $m > 0$, $K > 0$ and $r > 0$ be such that Γ_0 satisfies assumption (H1), T is large enough to have (H2), $w[q^\star]$ is regular enough so that (H3) holds, we also have the positivity assumption (H4) for w_0, and the a priori bound (H5) for $q^\star$ and q. The objective is to prove that there exists a constant $C = C(m, T, K, r) > 0$ such that

$$C^{-1}\|q - q^\star\|_{L^2(\Omega)} \leq \|\partial_\nu w[q] - \partial_\nu w[q^\star]\|_{H^1(0,T;L^2(\Gamma_0))} \leq C\|q - q^\star\|_{L^2(\Omega)}.$$

The formulation of the result itself suggests the definition of

$$v = \partial_t w[q] - \partial_t w[q^\star]$$

that satisfies the following system :

$$\left\{ \begin{array}{ll} \partial_t^2 v - \Delta v + qv = (q^\star - q)\partial_t w[q^\star], & \text{in } (0,T)\times\Omega, \\ v = 0, & \text{on } (0,T)\times\partial\Omega, \\ v(0) = 0, \quad \partial_t v(0) = (q^\star - q)w_0, & \text{in } \Omega, \end{array} \right. \tag{22}$$

Remark 5 *There is a clear link here with the Observability inequality presented in Section 2.2. Indeed, compared to the multiplier technique used in the former linear inverse problem, the Carleman estimate will allow us to balance the informations coming from initial data with respect to source terms.*

One can notice here one of the interest of this new PDE system : the term $(q^\star - q)$ we want to estimate appears both in the source term of the equation and in the initial velocity of the state. In the following, let us propose several steps for the proof of the awaited estimates (10) that now write

$$C^{-1}\|q - q^\star\|_{L^2(\Omega)} \leq \|\partial_\nu v\|_{L^2(0,T;L^2(\Gamma_0))} \leq C\|q - q^\star\|_{L^2(\Omega)}.$$

Step 1. Classical regularity and hidden regularity results - Let us start by applying Theorem 1 to equation (22) in order to verify the existence of the solution v and get useful regularity results. Noticing that

$$v_0 = 0,\ v_1 = (q^\star - q)w_0 \in L^2(\Omega) \ \text{ and } \ f = (q^\star - q)\partial_t w[q^\star] \in L^1(0,T,L^2(\Omega)),$$

since we assumed $w[q^\star] \in H^1(0,T;L^\infty(\Omega))$ and $(q^\star - q) \in L^\infty_{\leq m}(\Omega)$, we obtain a unique weak solution v such that

$$v \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)) \ \text{ and } \ \partial_\nu v \in L^2(0,T;L^2(\partial\Omega)).$$

More precisely, there exists a constant $C = C(\Omega, T, m) > 0$ such that for all $t \in (0,T)$,

$$\begin{array}{rcl} E_v(t) & = & \dfrac{1}{2}\left(\|\partial_t v(t)\|^2_{L^2(\Omega)} + \|\nabla v(t)\|^2_{L^2(\Omega)}\right) \\ & \leq & C\left(\|(q^\star - q)w_0\|^2_{L^2(\Omega)} + \|(q^\star - q)\partial_t w[q^\star]\|^2_{L^1(0,T,L^2(\Omega))}\right) \\ & \leq & C\|(q^\star - q)\|^2_{L^2(\Omega)}\|w[q^\star]\|^2_{H^1(0,T,L^2(\Omega))} \ \leq \ CK^2\|(q^\star - q)\|^2_{L^2(\Omega)}. \end{array} \tag{23}$$

and for the same reasons,

$$\|\partial_\nu v\|^2_{L^2(0,T;L^2(\partial\Omega))} \leq CK^2\|(q^\star - q)\|^2_{L^2(\Omega)}$$

bringing directly the second inequality in (10), that illustrates the continuous dependence of the normal derivative of v with respect to the difference of the two potentials. In a way, we have thus given meaning to the measurement of the flux of the solution at the boundary of the domain, information on which our inverse problem is based.

Step 2. Choice of the weight function - The definition in (11) of weight function φ by $\varphi(t,x) = |x - x_0|^2 - \beta t^2$ satisfies $\varphi(0,x) = |x - x_0|^2$ and $\varphi(\pm T, x) = |x - x_0|^2 - \beta T^2$ for all $x \in \Omega$. Therefore, since $T > \sup_{x\in\Omega} |x - x_0|$ in (H2), one can choose $\beta \in (0,1)$ such that $\beta T^2 > \sup_{x\in\Omega} |x - x_0|^2$ and $\eta > 0$ so that

$$\begin{cases} \varphi(0,x) + C_0 > C_0, & \forall\, x \in \Omega, \\ \varphi(t,x) + C_0 \leq C_0, & \forall\, (t,x) \in ([-T, -T+\eta] \cup [T-\eta, T]) \times \Omega. \end{cases}$$

In particular,

$$\begin{cases} \psi(0,x) \geq e^{\lambda C_0}, & \forall\, x \in \Omega, \\ \psi(t,x) \ \leq e^{\lambda C_0}, & \forall\, (t,x) \in ([-T, -T+\eta] \cup [T-\eta, T]) \times \Omega, \end{cases} \tag{24}$$

and in the following, we fix this β, and (λ_0, s_0) such that Carleman's inequality (12) can be applied.

Step 3. Extension in time truncation - Let us now extend system (22) in the time variable over $(-T,T)$ in an even manner, setting $v(t,x) = v(-t,x)$ for all $(t,x) \in (-T,0) \times \Omega$. We also extend $\partial_t w[q^\star]$ as an even function and keep the same notations for the new problem. Consequently,

$$v \in C([-T,T]; H^1_0(\Omega)) \cap C^1([-T,T]; L^2(\Omega)) \text{ et } \partial_t w[q^\star] \in L^2(-T,T; L^\infty(\Omega)).$$

Our goal is to apply the Carleman inequality (12) of Theorem 4 so that we can estimate a norm of $(q^\star - q)$ by a norm of $\partial_\nu v$. The solution v of the equation (22) extended to $(-T,T)$ is a good candidate but it is not zero in time $t = \pm T$.

We will therefore use a time cut-off function $\chi \in C^\infty(\mathbb{R}; [0,1])$ calibrated to work with the weight ψ and η (chosen in the previous step), defined by

$$\begin{cases} \chi(\pm T) = \chi'(\pm T) = 0 & \\ \chi(t) = 1 & \forall t \in [-T+\eta, T-\eta] \end{cases} \tag{25}$$

and illustrated by Figure 2.

We now set $u = \chi v$ that satisfies the system :

$$\begin{cases} \partial_t^2 u - \Delta u + qu = \chi(q^\star - q)\partial_t w[q^\star] + \chi'' v + 2\chi' \partial_t v, & \text{in } (-T,T) \times \Omega \\ u = 0, & \text{on } (-T,T) \times \partial\Omega \\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = (q^\star - q) w_0, & \text{in } \Omega \\ u(\pm T, \cdot) = 0, \quad \partial_t u(\pm T, \cdot) = 0, & \text{in } \Omega. \end{cases} \tag{26}$$

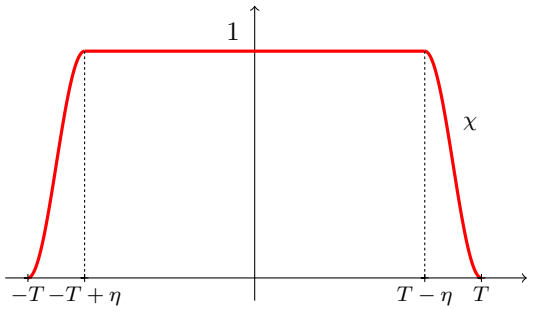


Figure 2: Cut-off function χ.

Step 4. Use of the Carleman estimate - As in Theorem 5 giving a 1-parameter Carleman estimate, one can prove in the same way that the 2-parameter Carleman inequality (12) can yield also the following estimate for any $u \in H^1(-T,T;H_0^1(\Omega))$ satisfying

$$\begin{cases} L_q u = \partial_t^2 u - \Delta u + qu \in L^2((-T,T)\times\Omega), \\ u(0,x) = 0, \ \forall x \in \Omega, \\ u(\pm T, x) = \partial_t u(\pm T, x) = 0, \ \forall x \in \Omega, \end{cases}$$

$$s^{1/2}\int_\Omega e^{2s\psi(0)}|\partial_t u(0)|^2\,dx \le M\int_{-T}^T\int_\Omega e^{2s\psi}|L_q u|^2\,dxdt + Ms\lambda\int_{-T}^T\int_{\Gamma_0} e^{2s\psi}\,|\partial_\nu u|^2\,d\sigma dt. \quad (27)$$

Thus, to take advantage of applying this estimate to the solution u of (26), we first need to estimate the source term of the equation $L_q u = \chi(q^\star - q)\partial_t w[q^\star] + \chi'' v + 2\chi'\partial_t v$. It brings

$$\begin{aligned}&\int_{-T}^T\int_\Omega e^{2s\psi}|L_q u|^2\,dxdt \\ &\qquad \le 2\int_{-T}^T\int_\Omega e^{2s\psi}|\chi(q^\star - q)\partial_t w[q^\star]|^2\,dxdt + 2\int_{-T}^T\int_\Omega e^{2s\psi}\left(|\chi'\partial_t v|^2 + |\chi'' v|^2\right)\,dxdt\end{aligned}$$

and the properties (25) of the cut-off function χ and (24) of the weight function ψ allow

$$\begin{aligned}&\int_{-T}^T\int_\Omega e^{2s\psi}|L_q v|^2\,dxdt \\ &\le M\int_{-T}^T\int_\Omega e^{2s\psi}|q^\star - q|^2|\partial_t w[q^\star]|^2\,dxdt + M\left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^T\right)\int_\Omega e^{2s\psi}\left(|\partial_t v|^2 + |v|^2\right)\,dxdt \\ &\le M\int_{-T}^T\int_\Omega e^{2s\psi(0)}|q^\star - q|^2|\partial_t w[q^\star]|^2\,dxdt \\ &\qquad\qquad + M\left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^T\right)\int_\Omega e^{2se^{\lambda C_0}}\left(|\partial_t v|^2 + |v|^2\right)\,dxdt \\ &\le MK^2\int_\Omega e^{2s\psi(0)}|q^\star - q|^2\,dx + \ Me^{2se^{\lambda C_0}}\left(\int_{-T}^{-T+\eta} + \int_{T-\eta}^T\right)E_v(t)dt.\end{aligned}$$

The energy estimate (23) and again the properties (24) of ψ, let finally

$$\begin{aligned}\int_{-T}^{T}\int_{\Omega} e^{2s\psi}|L_q v|^2\,dxdt &\leq MK^2\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2\,dx + M\eta K^2 e^{2se^{\lambda C_0}}\int_{\Omega}|q^\star-q|^2\,dx\\ &\leq M\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2\,dx.\end{aligned}$$

Thus, Carleman estimate (27) brings

$$s^{1/2}\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2|w_0|^2\,dx \leq M\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2\,dx + Ms\int_{-T}^{T}\int_{\Gamma_0} e^{2s\psi}|\partial_\nu u|^2\,d\sigma dt.$$

The positivity assumption (H4) on w_0 yields

$$\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2\,dx \leq \frac{M}{\sqrt{s}}\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2\,dx + M\sqrt{s}\int_{-T}^{T}\int_{\Gamma_0} e^{2s\psi}|\partial_\nu u|^2\,d\sigma dt,$$

and the choice of a parameter s large enough to have $(1-M/\sqrt{s})>0$ finally give

$$\int_{\Omega} e^{2s\psi(0)}|q^\star-q|^2\,dx \leq M\sqrt{s}\int_{-T}^{T}\int_{\Gamma_0} e^{2s\psi}|\partial_\nu u|^2\,d\sigma dt.$$

The proof of inequality (10) of Theorem 3 ends when getting back to the variables v then w, with the upper bound of ψ on $\Gamma_0\times(0,T)$ and the lower bound of $\psi(0)$ on Ω, since we then obtain the Lipschitz stability estimate

$$\int_{\Omega}|q^\star-q|^2\,dx \leq M\sqrt{s}\int_{0}^{T}\int_{\Gamma_0}|\partial_\nu\partial_t(w[q]-w[q^\star])|^2\,d\sigma dt.$$

□

5 Carleman-based reconstruction algorithm (C-bRec)

Now equipped with a better knowledge of the context, let us recall the inverse problem for which we want to propose a reconstruction algorithm.

> Given the in-domain f and boundary f_∂ source terms and the initial position and velocity data (w_0, w_1), considering the solution of the wave partial differential equation system
>
> $$\begin{cases}\partial_t^2 w - \Delta w + q^\star w = f, & \text{in } (0,T)\times\Omega,\\ w = f_\partial, & \text{on } (0,T)\times\partial\Omega,\\ w(0)=w_0, \quad \partial_t w(0)=w_1, & \text{in } \Omega,\end{cases} \tag{1}$$
>
> can we determine the unknown potential $q^\star = q^\star(x)$, assumed to depend only on $x\in\Omega$, from the additional knowledge of $\mathscr{M} = \partial_\nu w|_{(0,T)\times\Gamma_0}$?

After the positive answers given to uniqueness and stability of the inverse problem, solving the identifiability issue, we now consider the following identification issue :

Reconstruction : Given $\mathscr{M} = \partial_\nu w[q^\star]|_{(0,T)\times\Gamma_0}$, how to compute $q^\star|_\Omega$?

A classical reconstruction idea would be, from the measurement $\mathscr{M}$, to minimize

$$J(q) = \|\partial_\nu y[q] - \mathscr{M}\|.$$

But J is not convex in general and may have several local minima. Thus, the reconstruction algorithms based on such a cost functional is not guaranteed to converge to the global minimum q^*. Indeed, the presence of local minima implies that traditional minimization algorithms generally do not converge to the global minimum of J, unless they can be initialized in a neighbourhood of this minimum, and also requires regularization techniques.

5.1 Main ideas of the C-bRec algorithm

An algorithm is said to be globally convergent if its convergence towards the solution can be guaranteed from any initial data. Globally convergent methods of reconstruction remain rare, even if decisive works in this direction have been done in the last 30 years, in particular the ones of Klibanov and his co-authors, based on Carleman inequalities. The original idea of introducing Carleman weights in the cost function to make the problem convex goes back to [Kli97]. For further references on these questions, we suggest the book by Beilina and Klibanov [BK12].

Two main articles present our approach for a globally convergent reconstruction algorithm of the potential in such a wave equation. They are both based on the following core ideas : 1 - Build a cost functional from the appropriate Carleman estimate, and 2 - Use the structure of the proof of stability to prove the global convergence towards the determination of q^*.

In the first one [BdBE13], the idea was presented and the structure of the reconstruction algorithm was based on the 2-parameters Carleman estimate (12). In the second one, [BdBE17], several issues were addressed to improve the quality of the numerical implementations, among which the use of the 1-parameter Carleman estimate (16). Both are based on the design of the Bukhgeim-Klibanov method developped in the proof of the stability. Indeed, the rough idea can be summed up by saying that if we denote q^k the previous value of the iterative process to reconstruct q^*, then $v = \partial_t w[q^k] - \partial_t w[q^\star]$ satisfies system (22)

$$\begin{cases} \partial_t^2 v - \Delta v + q^k v = (q^\star - q^k)\partial_t w[q^\star], & \text{in } (0,T)\times\Omega, \\ v = 0, & \text{on } (0,T)\times\partial\Omega, \\ v(0) = 0, \quad \partial_t v(0) = (q^\star - q^k)w_0, & \text{in } \Omega, \end{cases}$$

where the source term $h := (q^\star - q^k)\partial_t w[q^\star]$ is "less relevant" than the initial velocity data $(q^\star - q^k)w_0$, thanks to the Carleman estimate and the large value of its parameter s, whereas

$$\mu := \partial_\nu v = \partial_t\partial_\nu w[q^k] - \partial_t\partial_\nu w[q^\star] \quad \text{on } (0,T)\times\Gamma_0 \quad \text{is known.}$$

Hence, the goal of our algorithm will be to try to fit v using this information. We will indeed define, for $s > 0$, $q^k \in L^\infty_{\leq m}(\Omega)$, $h \in L^2((0,T)\times\Omega)$ and $\mu \in L^2((0,T)\times\Gamma_0)$, the cost functional

$$K_s[q^k,h,\mu](v) = \frac{1}{2}\int_0^T\int_\Omega e^{2s\psi}|\partial_t^2 v - \Delta v + q^k v - h|^2\,dxdt + \frac{s}{2}\int_0^T\int_{\Gamma_0} e^{2s\psi}|\partial_\nu v - \mu|^2\,d\sigma dt, \quad (28)$$

and the choice of the weight function ψ makes this quadratic functional continuous, coercive, and strictly convex (see [BdBE13]). Therefore, for all h and μ, it admits a unique minimizer (which turns out to be the solution of (22))

$$\begin{aligned} v &= \arg\min_{v\in\mathcal{T}} K_s[q^k,h,\mu](v) \\ \text{in } \ \mathcal{T} &= \Big\{ v \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)), \\ &\qquad \partial_t^2 v - \Delta v \in L^2((0,T)\times\Omega), v(0,\cdot) = 0 \text{ in } \Omega\Big\}. \end{aligned} \quad (29)$$

Nevertheless, even if $\mu = \partial_t\left(\partial_\nu w[q^k] - \mathscr{M}\right)$ is known from a previous iteration of the algorithm, the source term h remains unknown. A solution is to minimize $K_s[q^k,0,\mu](v)$ instead of $K_s[q^k,h,\mu](v)$, knowing that we will be able to estimate the error made by replacing v by $v_k = \arg\min_{v\in\mathcal{T}} K_s[q^k,0,\mu](v)$ with respect to h.

Finally, recalling that w_0 satisfies assumption (H4), we remark that

$$\partial_t v(0) = (q^\star - q^k)w_0 \quad \Leftrightarrow \quad q^\star = q^k + \frac{\partial_t v(0)}{w_0}$$

inducing the idea to define

$$q^{k+1} = q^k + \frac{\partial_t v_k(0)}{w_0}.$$

Our goal will thus be to prove that q^{k+1} is closer from $q^\star$ than q^k was.

This algorithm presents the advantage of being convergent for any initial guess $q^0 \in L^\infty_{\leq m}(\Omega)$ without any a priori guess except for the knowledge of m. This is why we call this algorithm *globally convergent*. However, while this algorithm is theoretically satisfactory as at each iteration, it simply consists in the minimization of the strictly convex and coercive quadratic functional K_s, it nevertheless contains several flaws and drawbacks in its numerical implementation. In particular, we underline that the functional K_s involves two exponentials, namely

$$\exp(s\psi) = \exp(s\exp(\lambda(\varphi + C_0))),$$

with a choice of parameters s and λ large enough and whose sizes are difficult to estimate. In particular, for $s = \lambda = 3$ - which are not so large of course - $\Omega = (0,1)$, $x_0 \simeq 0^-$, $T \simeq 1^+$ and $\beta \simeq 1^-$, the ratio

$$\frac{\max\limits_{(0,T)\times\Omega}\{\exp(2s\psi)\}}{\min\limits_{(0,T)\times\Omega}\{\exp(2s\psi)\}} \quad (30)$$

is of the order of 10^{340} ! The numerical implementation therefore seems doomed.

We will give in the next subsection a first improvement of the idea of algorithm, based on the use of the weight φ and the associated 1-parameter Carleman estimate. Other technical solutions to improve the numerical simulations will be given in the last section of this document.

5.2 Statement of the reconstruction algorithm

Here, we will state properly and prove a new algorithm based on the 1-parameter Carleman estimate and a new cost functional. It is taken from [BdBE17], but will not incorporate all the technical improvements proposed in this article. The goal here is to propose the proof of the convergence of this algorithm in a reasonable and self contained manner.

In particular, instead of minimizing $K_s[q^k, 0, \mu]$ introduced before, we will perform a minimization process on a new functional based on (16) :

$$J_s[q,0,\tilde{\mu}](z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|\partial_t^2 z - \Delta z + qz|^2\, dxdt \\ + \frac{s}{2}\int_0^T \int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \tilde{\mu}|^2\, d\sigma dt + \frac{s^3}{2}\iint_{\mathcal{O}} e^{2s\varphi}|z|^2\, dxdt.$$

This new quadratic functional, under conditions (H1)–(H2)–(14), is strictly convex and coercive, therefore enjoying similar properties as $K_s[q^k, 0, \mu]$. Nevertheless, let us once more emphasize that $J_s[q, 0, \tilde{\mu}]$ will be less stiff than $K_s[q^k, 0, \mu]$ as now the weights are of the form $\exp(2s\varphi)$ instead of $\exp(2s\psi) = \exp(2s\exp(\lambda(\varphi + C_0)))$ in (28).

Before giving the detailed algorithm, let us make a few comments. The use of the notation $\tilde{\mu}$ instead of μ comes from the fact that the minimization of the functional $K_s[q^k, 0, \mu]$ leads to an optimal solution v satisfying

$$\partial_t^2 v - \Delta v + qv \simeq 0 \text{ in } (0,T)\times\Omega, \quad \text{and} \quad \partial_\nu v \simeq \mu \text{ in } (0,T)\times\Gamma_0,$$

whereas the functional $J_s[q, 0, \tilde{\mu}]$ leads to an optimal solution $\tilde{z}$ satisfying

$$\partial_t^2 \tilde{z} - \Delta \tilde{z} + q\tilde{z} \simeq 0 \text{ in } (0,T)\times\Omega, \quad \partial_\nu \tilde{z} \simeq \tilde{\mu} \text{ in } (0,T)\times\Gamma_0, \quad \text{and} \quad \tilde{z} \simeq 0 \text{ in } \mathcal{O}.$$

Therefore, since we are looking for $\tilde{z}$ small in $\mathcal{O}$, it is natural to introduce a smooth cut-off function $\eta \in C^2(\mathbb{R})$ such that $0 \le \eta \le 1$ and

$$\eta(\tau) = 0, \text{ if } \tau \le 0, \quad \text{and} \quad \eta(\tau) = 1, \text{ if } \tau \ge d_0^2 := d(x_0,\Omega)^2 = \inf_{x\in\Omega}|x - x_0|^2, \tag{31}$$

(recall that $d_0^2 > 0$ according to Assumption H1) see Figure 3. Next, the idea is that if

$$\tilde{\mu} = \eta(\varphi)\mu, \text{ in } (0,T)\times\Gamma_0.$$

and if z denotes the minimizer of the functional $J_s[q, 0, \mu]$, then the minimizer $\tilde{z}$ of $J_s[q, 0, \tilde{\mu}]$ should be close to $\eta(\varphi)z$ in $(0,T)\times\Omega$ and in particular at $t = 0$ this should yield, due to the choice of η in (31), $\partial_t\tilde{z}(0) \simeq \partial_t z(0)$ in Ω.

One can read at the end of the introduction of [BdBCV25] another nice explanation of the founding idea of the C-bRec method for the recovery of a potential in the case of a network of 1-D wave equations.

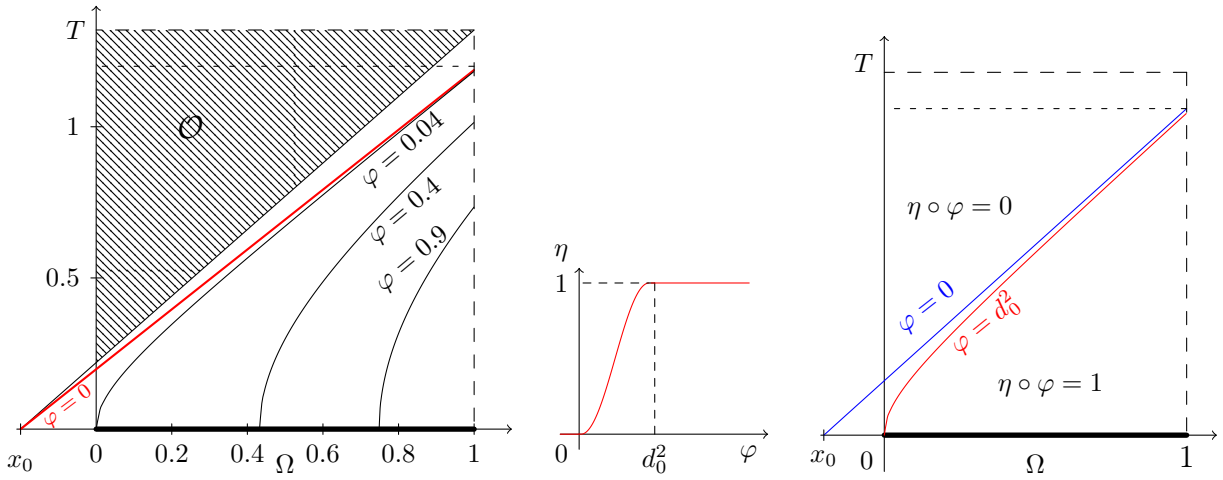


Figure 3: Isovalues of the function φ ($x_0 = -0.2$, $\beta = 0.99$). Definition and application of the cut-off function η. Case where $\Omega = (0, 1)$.

Algorithm 1
Initialization: *$q^0 = 0$ (or any guess $q^0 \in L^\infty_{\leq m}(\Omega)$).*
Iteration: ***From*** *k* ***to*** *$k+1$*
• *Step 1 - Given q^k, we set*

$$\tilde{\mu}^k = \eta(\varphi)\partial_t \left(\partial_\nu w[q^k] - \partial_\nu w[q^\star]\right), \qquad \text{on } (0,T)\times\Gamma_0, \tag{32}$$

where $w[q^k]$ denotes the solution of

$$\begin{cases} \partial_t^2 w - \Delta w + q^k w = f, & \text{in } (0,T)\times\Omega, \\ w = f_\partial, & \text{on } (0,T)\times\partial\Omega, \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases} \tag{33}$$

corresponding to (1) with the potential q^k and $\partial_\nu w[q^\star]$ is the measurement in (9).
• *Step 2 - For some $s > 0$ that will be chosen independently of k, we minimize the functional defined by*

$$J_s[q^k, 0, \tilde{\mu}^k](z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 z - \Delta z + q^k z|^2 \, dxdt + \frac{s}{2}\int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \tilde{\mu}^k|^2 \, d\sigma dt + \frac{s^3}{2}\iint_{\mathcal{O}} e^{2s\varphi}|z|^2 \, dxdt, \tag{34}$$

on the trajectories $z \in \mathcal{T}$ defined in (29), such that $\partial_\nu z \in L^2((0,T)\times\Gamma_0)$.
Let $\widetilde{Z}^k$ be the unique minimizer of the functional $J_s[q^k, 0, \tilde{\mu}^k]$.
• *Step 3 - Set*

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t \widetilde{Z}^k(0)}{w_0}, \qquad \text{in } \Omega, \tag{35}$$

where w_0 is the initial condition in (33)*, recalling the positivity assumption* (H4)*.*
• *Step 4 - Finally, set*

$$q^{k+1} = T_m(\tilde{q}^{k+1}), \qquad \textit{with } \; T_m(q) = \begin{cases} q, & \textit{if } |q| \leq m, \\ \textit{sign}(q)m, & \textit{if } |q| \geq m, \end{cases}$$

where m is the a priori bound of assumption (H5)*.*

5.3 Convergence result: statement and proof

We will prove the following result:

Theorem 6 *Let us fix $x_0 \notin \overline{\Omega}$ and assume that Γ_0 satisfies the geometric assumption* (H1)*, T is large enough to have* (H2)*, allowing to choose $\beta \in (0,1)$ such that* (14)*. Let us then set $K > 0$, $r > 0$ and $m > 0$ such that $w[q^\star]$ is regular enough to have* (H3)*, w_0 satisfies the positivity assumption* (H4)*, and $q^\star$, the a priori bound* (H5)*.*

Under these assumptions, there exist $C > 0$ and $s_0 > 0$ such that for all $s \geq s_0$, Algorithm 1 *is well-defined and convergent.*

Indeed, the iterates q^k constructed by Algorithm 1 *satisfy, for all $k \in \mathbb{N}$,*

$$\int_\Omega |q^k - q^\star|^2 e^{2s\varphi(0)}\, dx \leq \left(\frac{CK^2}{s^{1/2}r^2}\right)^k \int_\Omega |q^0 - q^\star|^2 e^{2s\varphi(0)}\, dx \tag{36}$$

and for s large enough, the sequence $\left(q^k\right)_{k\in\mathbb{N}}$ strongly converges towards $q^\star$ in $L^2(\Omega)$.

The proof of Theorem 6 will be given after some comments and details about the main ingredients, such as the properties of the cut-off function η and of the cost functional J_s.

5.3.1 Landmarks

Algorithms 1 is constructed from the fact that if $w[q^\star]$ is the solution of equation (1) and $w[q^k]$ solves (33), then

$$z^k = \partial_t \left(w[q^k] - w[q^\star]\right) \tag{37}$$

satisfies

$$\begin{cases} \partial_t^2 z^k - \Delta z^k + q^k z^k = g^k, & \text{in } (0,T)\times\Omega, \\ z^k = 0, & \text{on } (0,T)\times\partial\Omega, \\ z^k(0) = 0, \quad \partial_t z^k(0) = z_1^k, & \text{in } \Omega, \end{cases} \tag{38}$$

where $g^k = (q^\star - q^k)\partial_t w[q^\star]$, $z_1^k = (q^\star - q^k)w_0$, and $\partial_\nu z^k|_{(0,T)\times\Gamma_0} = \mu^k$.

In system (38), the source g^k and the initial data z_1^k are both unknown, and we are actually interested in finding a good approximation of z_1^k, which encodes the information on $q^\star - q^k$. In order to do so, we will try to fit "at best" the flux $\partial_\nu z$ with μ^k on the boundary, approximating the unknown source term g^k by 0.

This strategy works since we can prove that the source term g^k brings less information than μ^k does, and this is where the choice of the Carleman parameter s (large enough) will play a crucial role. Here, when considering the cost functional $J_s[q^k, 0, \eta(\varphi)\mu]$ defined in (34), we rather try to approximate $\tilde{z}^k = \eta(\varphi)z^k$, where the cut-off function $\eta \in C^2(\mathbb{R})$ is defined in (31) and φ is the Carleman weight function from which we also built J_s.

Properties of the cut-off function η - We calculate that

$$\tilde{z}^k = \eta(\varphi) z^k = \eta(\varphi)\partial_t \left(w[q^k] - w[q^\star] \right),$$

is the solution of

$$\begin{cases} \partial_t^2 z - \Delta z + q^k z = f^k, & \text{in } (0,T)\times\Omega, \\ z = 0, & \text{on } (0,T)\times\partial\Omega, \\ z(0) = 0, \quad \partial_t z(0) = \eta(\varphi(0))(q^\star - q^k) w_0, & \text{in } \Omega, \end{cases} \tag{39}$$

where $f^k := \eta(\varphi)(q^\star - q^k)\partial_t w[q^\star] - [\eta(\varphi), \partial_t^2 - \Delta]\,\partial_t \left(w[q^k] - w[q^\star] \right)$, using the Lie bracket notation $[A,B](z) = A(Bz) - B(Az)$.

The objective when constructing η is threefold. Indeed, we need to ensure that $\tilde{z}^k$ has the following properties, all of them pictured in Figure 3:

- Encoding $(q^\star - q^k)$, which is what we are looking for, through the initial velocity data $\partial_t \tilde{z}^k(0,\cdot) = \eta(\varphi(0))(q^\star - q^k) w_0$. For instance that $\eta(\varphi(0)) = 1$ would be appropriate.
- Vanishing on the domain $\mathcal{O}$ defined by (15) so that the last observation term in J_s integrated on $\mathcal{O}$ disappears. Thus one needs $\eta = 0$ on some domain greater than $\mathcal{O}$, in order to have $\tilde{z}^k|_{\mathcal{O}} = \eta(\varphi) z^k|_{\mathcal{O}} = 0$.
- Allowing the source term f^k of equation (39) to be *manageable*. Since one has a Lie bracket term in f^k, we will ask for $\eta(\varphi)$ to vary between 0 and 1 only in a small region of $(0,T)\times\Omega$. Actually, it will be allowed only where $0 < \varphi < d_0^2 := d(x_0,\Omega)^2$.

These ideas are actually behind the proofs of the inverse problem stability by compactness uniqueness arguments as in [PY96, PY97, Yam99] or by Carleman estimates given in [IY01a, IY01b, IY03, Bau10].

The construction of this cut-off function could be a real challenge, for instance when working on a network of equations, as one can read in [BdBCV25].

Properties of the cost functional J_s - Let us now give some insight about the chosen cost functional J_s presented in (34) and used in the algorithm.

Lemma 2 *Let us assume that the hypotheses of Theorem 6 are satisfied. Let φ be a weight function defined as in* (11)*. Then for all $s > 0$ large enough, $q \in L^\infty(\Omega)$, $h \in L^2(0,T;L^2(\Omega))$ and $\mu \in L^2(0,T)$, the functional $J_s[q,h,\mu]$ that generalizes* (34)

$$J_s[q,h,\mu](z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 z - \Delta z + qz - h|^2 \, dxdt + \frac{s}{2}\int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu|^2 \, d\sigma dt + \frac{s^3}{2}\iint_{\mathcal{O}} e^{2s\varphi} |z|^2 \, dxdt, \tag{40}$$

is continuous, strictly convex and coercive on $\mathcal{T}$ defined by

$$\mathcal{T} = \Big\{ v \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)), \partial_t^2 v - \Delta v \in L^2((0,T)\times\Omega), v(0,\cdot) = 0 \text{ in } \Omega \Big\}.$$

and equipped with an appropriate weighed norm.
Thenceforth, the functional $J_s[q,h,\mu]$ admits a unique minimizer on the set $\mathcal{T}$.

Proof. Let us begin by defining the following weighed seminorm for any $z \in \mathcal{T}$:

$$\|z\|^2_{obs,s} = 2J_s[q,0,0](z). \tag{41}$$

This is actually a well defined norm thanks to the Carleman estimate (16) applied to the extension of $z \in \mathcal{T}$ to negative times as an odd function. Indeed, since φ is a bounded function over the bounded domain $(0,T)\times\Omega$, as soon as s is large enough, one has

$$s^3\|z\|^2_{L^2(-T,T;L^2(\Omega))} \leq CJ_s[q,0,0](z)$$

bringing $z=0$ if $J_s[q,0,0](z)=0$.

Besides, the linearity of the properties defining $\mathcal{T}$ allow it to be a closed set.
On the other hand, the functional $J_s[q,h,\mu]$ is continuous and differentiable on $\mathcal{T}$ since one can calculate its gradient as follows, for any $z, Z \in \mathcal{T}$:

$$\begin{aligned}\langle \nabla J_s[q,h,\mu](z), Z\rangle_{\mathcal{T}} = &\int_0^T\int_\Omega e^{2s\varphi}(\partial_t^2 z - \Delta z + qz - h)(\partial_t^2 Z - \Delta Z + qZ)\,dxdt \\ &+ s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu z - \mu)\partial_\nu Z\,d\sigma dt + s^3\iint_{\mathcal{O}} e^{2s\varphi} zZ\,dxdt. \end{aligned}\tag{42}$$

Now taking $y, z \in \mathcal{T}$ and $Z = y - z$ one obtains

$$\langle \nabla J_s[q,h,\mu](y) - \nabla J_s[q,h,\mu](z), y-z\rangle_{\mathcal{T}} = 2J_s[q,0,0](y-z).$$

Therefore, it shows the coercivity condition

$$\langle \nabla J_s[q,h,\mu](y) - \nabla J_s[q,h,\mu](z), y-z\rangle_{\mathcal{T}} \geq \|y-z\|^2_{obs,s}$$

yielding the strong convexity of $J_s[q,h,\mu]$ from basic properties of differentiable convex functions (see the book [All07] for instance if needed).

As a consequence, the functional $J_s[q,h,\mu]$ admits a unique minimizer on the set $\mathcal{T}$ and allows to define properly the C-bRec Algorithm 1. □

5.3.2 Proof of the convergence of the C-bRec algorithm

In order to show estimate (36) that characterizes the convergence of Algorithm 1, instead of considering only functionals of the form $J_s[q,0,\tilde{\mu}]$, we will rely on the more general functionals $J_s[q,h,\tilde{\mu}]$ given by (40) that admit a unique minimizer for each $\tilde{\mu} \in L^2((0,T)\times\Gamma_0)$ and $h \in L^2((0,T)\times\Omega)$.

We then observe that $\tilde{z}^k := \eta(\varphi)z^k$, where z^k satisfies (38), is the minimizer of $J_s[q^k, f^k, \tilde{\mu}^k]$ with

$$f^k = \eta(\varphi)(q^\star - q^k)\partial_t w[q^\star] + [\partial_t^2 - \Delta, \eta(\varphi)]z^k, \tag{43}$$

since it solves Equation (39) and $\partial_\nu \tilde{z}^k = \tilde{\mu}^k = \eta(\varphi)\partial_t\left(\partial_\nu w[q^k] - \partial_\nu w[q^\star]\right)$ on $(0,T)\times\Gamma_0$.

We shall then compare $\widetilde{Z}^k$ and $\tilde{z}^k$, the minimizers of the functionals $J_s[q^k,0,\tilde{\mu}^k]$ and $J_s[q^k,f^k,\tilde{\mu}^k]$ respectively, especially at the time $t=0$ corresponding to the set in which the information on $(q^\star - q^k)$ is encoded. The result is stated as follows:

Lemma 3 *Assume the geometric and time conditions* (H1)-(H2) *on* Γ_0 *and* T*, that* β *is chosen as in* (14)*, and let* $\mu \in L^2((0,T)\times\Gamma_0)$ *and* g^a, $g^b \in L^2((0,T)\times\Omega)$*. Assume also that* $q \in L^\infty_{\leq m}(\Omega)$*.*
Let Z^j *be the unique minimizer of the functional* $J_s[q, g^j, \mu]$ *on* $\mathcal{T}$ *for* $j \in \{a, b\}$*. Then there exist* $s_0(m) > 0$ *and* $M = M(m) > 0$ *such that for* $s \geq s_0(m)$ *we have:*

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t Z^a(0) - \partial_t Z^b(0)|^2\,dx \leq M\int_0^T\int_\Omega e^{2s\varphi}|g^a - g^b|^2\,dxdt. \tag{44}$$

where φ *and* $s_0(m)$ *are chosen so that Theorem* 5 *holds.*

We postpone the proof of Lemma 3 to the end of the section and first show how it can be used for the proof of Theorem 6.

Let us define the integral

$$I = s^{1/2}\int_\Omega e^{2s\varphi(0,x)}|q^{k+1}(x) - q^\star(x)|^2 dx \tag{45}$$

that we will estimate successively using Algorithm 1 and Carleman estimate (16).

First notice that the projection operator T_m (of Step 4 of the algorithm) is 1-Lipschitzian, and is also such that $q^\star = T_m(q^\star)$, since $q^\star \in L^\infty_m(\Omega)$, and $q^{k+1} = T_m(\tilde{q}^{k+1})$ by construction. Hence, we have

$$I = s^{1/2}\int_\Omega e^{2s\varphi(0)}|T_m(\tilde{q}^{k+1}) - T_m(q^\star)|^2 dx \leq s^{1/2}\int_\Omega e^{2s\varphi(0)}|\tilde{q}^{k+1} - q^\star|^2 dx.$$

Moreover, the initial data w_0 satisfies the positivity assumption H4, so that

$$I \leq s^{1/2}\int_\Omega e^{2s\varphi(0)}|\tilde{q}^{k+1} - q^\star|^2\frac{|w_0|^2}{r^2}dx = \frac{s^{1/2}}{r^2}\int_\Omega e^{2s\varphi(0)}\left|(\tilde{q}^{k+1} - q^\star)w_0\right|^2 dx.$$

Let us now split the gap between $\tilde{q}^{k+1}$ and $q^\star$ using the previous value q^k of the process, and take advantage of the optimization information of Step 2 and 3 of the algorithm.

Since $\widetilde{Z}^k$ is the unique minimizer of $J_s[q^k, 0, \tilde{\mu}^k]$ and $\tilde{z}^k$ the one of $J_s[q^k, f^k, \tilde{\mu}^k]$ on $\mathcal{T}$, one can apply Lemma 3 to $Z^a = \widetilde{Z}^k$ and $Z^b = \tilde{z}^k$ and get that for $s \geq s_0(m)$:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t\widetilde{Z}^k(0) - \partial_t\tilde{z}^k(0)|^2\,dx \leq M\int_0^T\int_\Omega e^{2s\varphi}\left|f^k\right|^2 dxdt.$$

Recalling that $\partial_t\tilde{z}^k(0,\cdot) = (q^\star - q^k)w_0$ from (39) and $\partial_t\widetilde{Z}^k(0,\cdot) = (\tilde{q}^{k+1} - q^k)w_0$ from (35), it becomes

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}\left|\left(\tilde{q}^{k+1} - q^\star\right)w_0\right|^2 dx \leq M\int_0^T\int_\Omega e^{2s\varphi}\left|f^k\right|^2 dxdt$$

so that

$$I \leq \frac{M}{r^2}\int_0^T\int_\Omega e^{2s\varphi}\left|f^k\right|^2 dxdt. \tag{46}$$

The next step is to get an estimate of f^k defined by (43). There, we will use the fact that η varies only when its argument is between 0 and d_0^2, so that $\eta(\varphi)$ is contant as soon as $\varphi < 0$ or $\varphi > d_0^2$, and consequentely $[\partial_t^2 - \Delta, \eta(\varphi)]\, z$ is supported in a region where $0 < \varphi < d_0^2$ since

$$\left[\partial_t^2 - \Delta, \eta(\varphi)\right] z = (\partial_t^2 - \Delta)(\eta(\varphi))z + 2\partial_t(\eta(\varphi))\partial_t z + 2\nabla(\eta(\varphi)) \cdot \nabla z.$$

Using also that $\varphi(t,\cdot) \leq \varphi(0,\cdot)$ in Ω, we thus obtain

$$\begin{aligned}
\int_0^T \int_\Omega e^{2s\varphi} \left|f^k\right|^2 dxdt &\leq M \int_0^T \int_\Omega e^{2s\varphi} |\eta(\varphi)(q^\star - q^k)\partial_t w[q^\star]|^2 \, dxdt \\
&\quad + M \int_0^T \int_\Omega e^{2s\varphi} \left|\left[\partial_t^2 - \Delta, \eta(\varphi)\right] z^k\right|^2 dxdt \\
&\leq M \left\|w[q^\star]\right\|^2_{H^1(0,T;L^\infty(\Omega))} \int_\Omega e^{2s\varphi(0)} |q^k - q^\star|^2 \, dx \\
&\quad + M e^{2sd_0^2} \int_0^T \int_\Omega \left(|\nabla z^k|^2 + |\partial_t z^k|^2 + |z^k|^2\right) dxdt.
\end{aligned}$$

From the *a priori* energy estimates (see Theorem 1) for z^k solution of (38) we have

$$\begin{aligned}
\|z^k\|_{L^\infty(0,T;H_0^1(\Omega))} &+ \|\partial_t z^k\|_{L^\infty(0,T;L^2(\Omega))} \leq M \left(\|z_1^k\|_{L^2(\Omega)} + \|g^k\|_{L^1(0,T;L^2(\Omega))}\right) \\
&\leq M \|q^\star - q^k\|_{L^2(\Omega)} \left(\|w_0\|_{L^\infty(\Omega)} + \|\partial_t w[q^\star]\|_{L^1(0,T;L^\infty(\Omega))}\right) \\
&\leq M \left\|w[q^\star]\right\|_{H^1(0,T;L^\infty(\Omega))} \|q^\star - q^k\|_{L^2(\Omega)},
\end{aligned}$$

so that combining the above estimates, using K from assumption (H3), we get

$$I \leq \frac{M}{r^2} K^2 \int_\Omega e^{2s\varphi(0)} |q^k - q^\star|^2 \, dx + \frac{M}{r^2} K^2 e^{2sd_0^2} \|q^\star - q^k\|^2_{L^2(\Omega)}.$$

Using finally $\varphi(0,\cdot) \geq d_0^2$ in Ω, we deduce

$$I = s^{1/2} \int_\Omega e^{2s\varphi(0)} |q^{k+1} - q^\star|^2 dx \leq \frac{M}{r^2} K^2 \int_\Omega e^{2s\varphi(0)} |q^k - q^\star|^2 \, dx,$$

completing the proof of Theorem 6. Indeed, by simple iteration on k, we conclude to the desired estimate (36) and the convergence result, when k tends to infinity, as soon as s is large enough to have $\frac{C}{s^{-1/2}} = \frac{MK^2}{r^2 s^{1/2}} < 1$. □

It only remains to prove the former lemma.

Proof of Lemma 3 - Since Z^j is the unique minimizer of the functional $J_s[q, g^j, \mu]$ on $\mathcal{T}$, let us write the Euler Lagrange equations satisfied by Z^j, for $j \in \{a, b\}$. For all $z \in \mathcal{T}$, we have

$$\begin{aligned}
\int_0^T \int_\Omega e^{2s\varphi} (\partial_t^2 Z^j - \Delta Z^j + q Z^j - g^j)(\partial_t^2 z - \Delta z + qz) \, dxdt \\
+ s \int_0^T \int_{\Gamma_0} e^{2s\varphi} (\partial_\nu Z^j - \mu) \partial_\nu z \, d\sigma dt + s^3 \iint_{\mathcal{O}} e^{2s\varphi} Z^j z \, dxdt = 0.
\end{aligned}$$

Applying this identity for $j = a$ and $j = b$ to $z = Z = Z^a - Z^b$ and subtracting the two identities, we obtain:

$$\int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 Z - \Delta Z + qZ|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu Z|^2 \, d\sigma dt + s^3 \iint_{\mathcal{O}} e^{2s\varphi} |Z|^2 \, dxdt$$
$$= \int_0^T \int_\Omega e^{2s\varphi} (g^a - g^b)(\partial_t^2 Z - \Delta Z + qZ) \, dxdt.$$

This implies, using Young's inequality,

$$\frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 Z - \Delta Z + qZ|^2 \, dxdt + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu Z|^2 \, d\sigma dt$$
$$+ s^3 \iint_{\mathcal{O}} e^{2s\varphi} |Z|^2 \, dxdt \leq \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |g^a - g^b|^2 \, dxdt. \quad (47)$$

Since the left hand side of (47) is precisely the right hand side of the Carleman estimate (16), applying Theorem 5 to Z, we immediately deduce (44). Indeed, $\partial_t Z(0) = \partial_t Z^a(0) - \partial_t Z^b(0)$ so that it brings

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t Z^a(0) - \partial_t Z^b(0)|^2 \, dx \leq \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |\partial_t^2 Z - \Delta Z + qZ|^2 \, dxdt$$
$$+ s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu Z|^2 \, d\sigma dt + s^3 \iint_{\mathcal{O}} e^{2s\varphi} |Z|^2 \, dxdt \leq \frac{1}{2} \int_0^T \int_\Omega e^{2s\varphi} |g^a - g^b|^2 \, dxdt.$$

□

6 Numerics and further applications

Let us now give some more numerical context § 6.1 in order to introduce properly several numerical simulations that illustrate the C-bRec algorithm for the wave equation, both in the setting presented above § 6.2 and in some new related settings showing even more complexity § 6.3.

6.1 Numerical tractability

The fact that the cost functional J_s in (34) contains weight functions with only one exponential makes the problem less difficult to implement than the one with two parameters/exponentials as in [BdBE13]. However, it is still numerically challenging to use such functionals, especially as the convergence of Algorithm 1 gets better for large parameter s. We propose in [BdBE17] two ideas recalled below to make it numerically tractable.

Preconditioning - When considering the functional $J_s[q, 0, \tilde{\mu}]$, one easily sees that exponentials factors in (34) could be removed if considering the unknown $ze^{s\varphi}$ instead of z. Such transformation corresponds to a preconditioning of the functional J_s. Indeed, that way, exponential factors do not appear anymore when computing its gradient. Nevertheless, there are

still exponentials factors appearing in the measurements. We therefore also develop a progressive algorithm in the resolution of the minimization process. The idea is to consider intervals in which the weight function φ does not significantly change, allowing to preserve numerical accuracy despite the possible large values of s.

Indeed, more precisely, minimizing $J_s[q,0,\tilde{\mu}]$ in (34) is equivalent to minimize, on the same set $\mathcal{T}$, the functional $\widetilde{J}_s[q,0,\tilde{\mu}]$ defined by

$$\widetilde{J}_s[q,0,\tilde{\mu}](y) = \frac{1}{2}\int_0^T \int_\Omega |\mathscr{L}_{s,q} y|^2 \, dxdt + \frac{s}{2}\int_0^T \int_{\Gamma_0} |\partial_\nu y - \tilde{\mu} e^{s\varphi}|^2 \, d\sigma dt + \frac{s^3}{2}\iint_{\mathcal{O}} y^2 \, dxdt$$

where $\mathscr{L}_{s,q} = e^{s\varphi}(\partial_t^2 - \Delta + q)e^{-s\varphi}$.

In the same objective, from the smooth cut-off function η equal to 1 for $\tau \geq d_0^2$ defined in (31), we introduce N_s cut-off functions $\{\eta_j\}_{1\leq j\leq N_s}$ (these ones are not all necessarily smooth) such that

$$\forall \tau \in \mathbb{R}, \quad \sum_{j=1}^{N_s} \eta_j(\tau) = \eta(\tau), \tag{48}$$

as illustrated in Figure 4. Then, the target flux is written as

$$\tilde{\mu} = \eta(\varphi)\mu = \sum_{j=1}^{N_s} \tilde{\mu}_j = \sum_{j=1}^{N_s} \eta_j(\varphi)\mu,$$

and if for each $j \in \{1, \cdots, N_s\}$, we denote by Y_j the minimizer of $\widetilde{J}_q[q,0,\tilde{\mu}_j]$ on $\mathcal{T}$, then the minimizer Y of $\widetilde{J}_q[q,0,\tilde{\mu}]$ is simply given by $Y = \sum_{j=1}^{N_s} Y_j$. The interest of this approach is that the target flux $\tilde{\mu}_j$ involves exponential terms in φ on the support of $\eta_j(\varphi(t,x))$, that can be chosen according to a prescribed bound on the ratio (30) between the maximum and the minimum of $e^{s\varphi}$ over this support. Every details cans be read in [BdBE17].

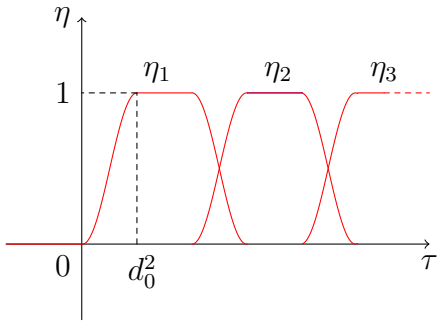


Figure 4: Example of cut-off functions η_j for $1 \leq j \leq 3$.

Processing and discretizing the cost functional - When implementing Algorithm 1 numerically, one has to discretize the wave equation under consideration, and to adapt the cost functional J_s to the discrete setting. It is quite well-known that most numerical schemes

for the waves exhibit some pathologies at high-frequency [Tre82, Zua05], namely the occurence of discrete rays propagating at velocity 0 or the blow up of observability estimates. The adaptation of the cost functional to the discrete setting thus has to take advantage of any available penalization term of the high frequencies.

Following some ideas developed in the context of the observability of discrete waves (see [Zua05]), we use in [BdBE17] a naive discrete version of $J_s[q, 0, \tilde{\mu}]$ that indeed penalize the high-frequencies. One can read there the $d = 1$-dimensional case that takes advantage of the discrete Carleman estimates obtained in [BE13] (for a multi-dimensional setting see [BEO15]). In such estimates for a semi-discrete operator (where the space semi-discretization is using the finite-difference method on a uniform mesh), a new observation term appears, that writes

$$Msh^2 \int_{-T}^{T} \int_{[0,L)_h} |\partial_t \partial_h^+ y_h|^2 \, dt, \tag{49}$$

where h is the mesh size, and for $f_h = (f_j)_{j\in\{0,\dots,N+1\}}$,

$$\int_{[0,L)_h} f_h = h \sum_{j=0}^{N} f_j \quad \text{and} \quad (\partial_h^+ f_h)_j = \frac{f_{j+1} - f_j}{h}.$$

This is a high-frequency term. Indeed, as $h\partial_h^+$ is of the order of $h|\xi|$ for frequencies ξ, this term can be absorbed for large parameter s by the left hand-side of the discrete Carleman estimate for frequencies $\xi = o(1/h)$. However, for frequencies of the order of the mesh-size h, this term cannot be absorbed anymore. This is coherent with the lack of uniform observability for discrete waves, and the various comments from [BE13].

Moreover, one should also be warned that the aforementioned discrete Carleman estimates also bring into play the constraint that the product sh will need to be chosen small enough. One can also read detailed explanations of this condition in [BdBE17, Section 4.2].

6.2 Numerical simulations of Algorithm 1 in 1-D

Setting - Even if we have already given a glimpse of the challenges when implementing numerically the C-bRec Algorithm, let us now give some details about the setting for some one-dimensional simulations of our reconstruction approach. The article [BdBE17] contains a full description of the numerical implementation in a 1-D space semi-discrete framework, that we do not wish to present extensively here. It also contains a discretized algorithm, as well as the proof of its convergence, for those who wish to get into these details.

In this subsection, we present the discretization schemes and the technical solutions we have developed to implement numerically the algorithm. In order to simplify the presentation, from now on, we focus on the one-dimensional case $\Omega = (0, L_x)$ and $\Gamma_0 = \{x = L_x\}$. Note that in this case, our problem can probably be solved in an easier way using the specific features of the 1-d wave equation (see e.g. [Isa06, Section 8.1]), but our goal is to propose an approach that is generalizable to the multi-dimensional setting.

Generation of the data and noise - We will work with synthetic data. To discretize the wave equation (1) in $(0, T) \times (0, L_x)$, we use a finite differences scheme in time and a $\theta-$scheme

in space domains. The space and time steps are denoted by h and τ respectively. We introduce N_x and N_t such that $L_x = (N_x+1)h$ and $T = (N_t+1)\tau$, and we define, for $0 \le j \le N_x+1$ and $0 \le n \le N_t+1$, w_j^n a numerical approximation of the solution $w(t^n, x_j)$ with $t^n = n\tau$ and $x_j = jh$. It satisfies the following system:

$$\begin{cases} \dfrac{1}{\tau^2}\left(w_j^{n+1} - 2w_j^n + w_j^{n-1}\right) - \dfrac{\theta}{2}(\Delta_h w^{n+1})_j - (1-\theta)(\Delta_h w^n)_j - \dfrac{\theta}{2}(\Delta_h w^{n-1})_j \\ \qquad\qquad - q^\star(x_j) w_j^n = f(t^n, x_j), \quad 1 \le j \le N_x,\ 1 \le n \le N_t, \\ w_j^1 = w_0(x_j) + \tau w_1(x_j) + \dfrac{\tau^2}{2}\left((\Delta_h w_0)(x_j) - q^\star(x_j) w_0(x_j) + f(0, x_j)\right) \quad \text{and}, \\ w_j^0 = w_0(x_j) \qquad\qquad 1 \le j \le N_x \\ w_0^n = f_\partial(t^n, 0) \quad \text{and} \quad w_{N_x+1}^n = f_\partial(t^n, L_x), \qquad 0 \le n \le N_t+1, \end{cases} \tag{50}$$

where

$$(\Delta_h w)_j = \frac{w_{j+1} - 2w_j - w_{j-1}}{h^2}.$$

Then, we compute the discrete counterpart of the continuous measurement $\mathscr{M} = \partial_\nu w|_{(0,T)\times\Gamma_0}$ given in (9) as follows:

$$\mathscr{M}^n = \frac{w_{N_x+1}^n - w_{N_x}^n}{h}, \qquad 0 \le n \le N_t+1.$$

On these measured data, we may add a Gaussian noise:

$$\mathscr{M}^n \longleftarrow \mathscr{M}^n + \alpha(\max_\ell \mathscr{M}^\ell)\mathcal{N}(0,1), \quad 0 \le n \le N_t+1 \tag{51}$$

where $\mathcal{N}(0,1)$ satisfies a centered normal law with deviation 1 and α is the level of noise. Note that the chosen model of noise is additive. It is a way to model experimental inaccuracies in the measurements. In the presence of noise, the first step of the inverse problem is to regularize the data in a way that depends on the nature of the noise and the physical phenomena. In our approach, the entry data for the inverse problem is directly the time derivative of the flux $\mathscr{M}$, see e.g. formula (32). Since the classical finite difference formulae generates instabilities when computing this discrete time derivative, we need to propose some regularization in order to guarantee the smoothness of the result even in the presence of noise (see [BdBE17] for details).

Avoidance of 'inverse crime' - We use neither the same schemes nor the same meshes for the direct and the inverse problems in order to prevent any *inverse crime* to occur. Indeed, we solve (1) thanks to an implicit scheme (meaning (50) with $\theta = 0$) and we use an explicit scheme for equation (33) (meaning (50) with $\theta = 1$), with different mesh size in time and space for each. This implies that even in the absence of additional noise, the reconstruction has to deal with numerical errors. Moreover, for the explicit scheme used in (50) to remain stable, one has to impose the CFL condition $\tau \le h$ on the time and space steps. Thus, we set $N_t = \lfloor \frac{T(N_x+1)}{L_x} \rfloor$ where $\lfloor\cdot\rfloor$ stands for the integer part.

Stopping criterion - Finally, the iterative loop is stopped when one of the three following criteria is satisfied:

$$\frac{\|\tilde{\mu}^n\|_2}{\|\mathscr{M}\|_2} \le \max(\epsilon, \alpha) \quad \text{or} \quad \frac{\|q^{k+1} - q^k\|_\infty}{\alpha} \le \epsilon, \tag{52}$$

where ϵ is a fixed tolerance and α is the level of noise in (51), or when the maximal number of iterations is reached.

Numerical results - To illustrate the properties of the reconstruction algorithm and its efficiency let us now present some simulations. All of them are executed with PYTHON and the source codes are available on request. Table 1 gathers the numerical values used for all the following examples, bearing in mind that T respects (H2) and N_t follows the CFL condition with respect to N_x.

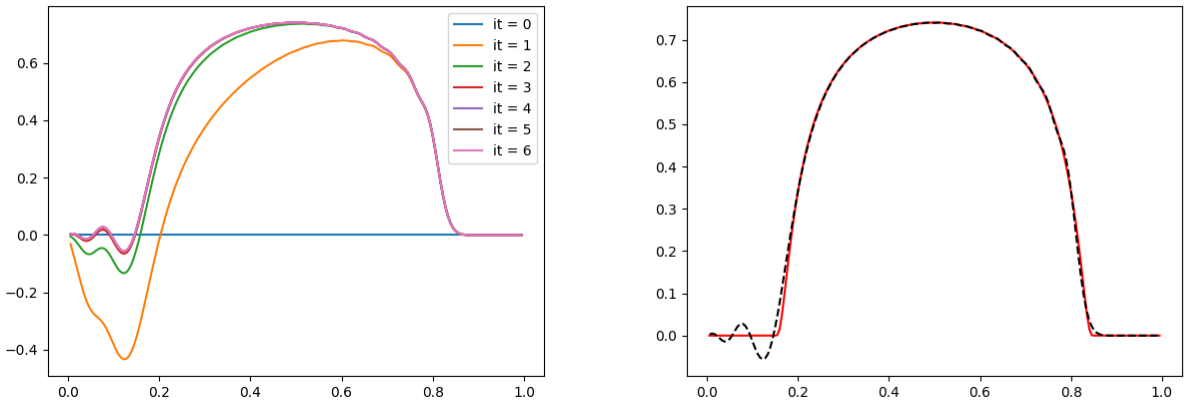
L_x	N_x	w_0	f_∂	f
1	$200L_x$	x	$x\cos(\pi t)$	0
s in (36)	x_0 in (H1)	d_0^2 in (31)	Γ_0 in (H1)	ϵ in (52)
10	-0.3	x_0^2	$\{x = L_x\}$	$1 \cdot 10^{-3}$

Table 1: Numerical values of the variables used for all the numerical examples in the 1-D setting according to equation (50).

For all the forthcoming illustrations, we have chosen to recover a coefficient that takes the following "hat shape"

$$q^*(x) = e^{\frac{-0.3}{1-8|x-0.4|^2}} \mathbb{1}_{1>8|x-0.4|^2}. \tag{53}$$

In the Figures 5 and 6 showing reconstruction results, the exact coefficient that we want to recover is plotted by a red line, whereas the numerical coefficient recovered by the algorithm is represented by a dotted black line.

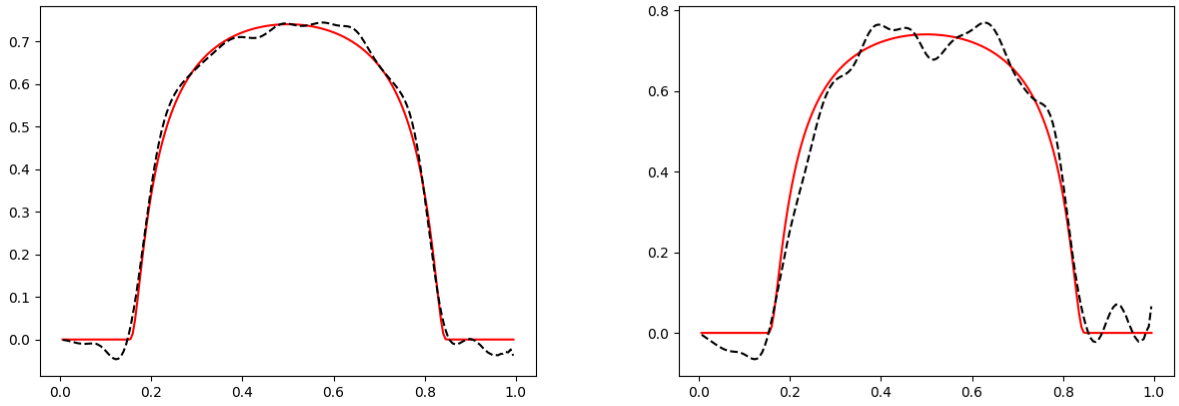


(a) Iterative steps

(b) Exact/recovered (dashed) potential

Figure 5: Simulation without noise for the recovery of the potential $q^\star$.

If you want to see more examples of potential q^* to reconstruct, and more convergence informations (number of iterations, running time, convergence error), have a look at the final section of [BdBE17]. Along with 2-D simulations, one can also find there various illustrations of the of the effects of the preconditioning, the progressive process, the regularization of the cost function and of the violation of some of the assumptions on the reconstruction of the potential.



(a) $\alpha = 1\%$ of noise

(b) $\alpha = 3\%$ of noise

Figure 6: Simulation with noise - Exact (red) and recovered (dashed) potential $q^\star$.

6.3 Some other related settings where C-bRec was succesfully applied

Using the same approach as the one described in this document to construct a Carleman-based reconstruction algorithm, some others settings were tackled more recently. The idea remains to use the knowledge of the proof of the lipschitz stability of the corresponding inverse problem, proof that uses Carleman estimates, and build the cost functional of the algorithm from those. Although the process could now seem quite well controlled, every case has proven to raise new challenges to overcome (see [BdBEO21] and [BdBCV25]). We would like to give below a glimpse of two of them, both for the wave equation. But we can also point out at [BdBS21] for the reconstruction of a source term in a reaction-diffusion equation.

6.3.1 Reconstruction of a potential on a network of string equations

In the context of a network of vibrating strings, modelled by interconnected linear partial differential wave equations, the article [BdBCV25] is interested in the reconstruction of a zeroth order term of each one-dimensional wave equation involved, using some appropriate external boundary measurements. More precisely, the inverse problem under consideration is set on a tree shaped network where each edge behaves according to the wave equation with potential, external nodes have Dirichlet boundary conditions and internal nodes follow the Kirchoff law as presented below.

The main goal is the reconstruction of the potential everywhere on the network, from the Neumann boundary measurements at all but one external vertices as presented in the generic Figure 7 or in the examples of Figure 8 or 11.

Leveraging from the Lipschitz stability of this inverse problem, already proved in [BCV11], we provided an efficient reconstruction algorithm based on the use of a specific new global Carleman estimate. The proof of the main tool and of the convergence of the algorithm are given; along with a detailed description of the numerical illustrations presented at the end of the article.

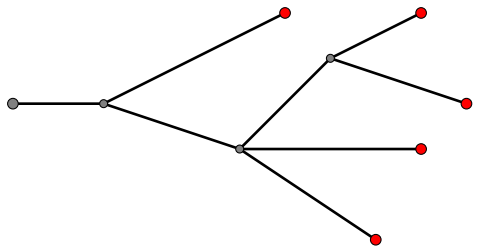


Figure 7: A tree-shaped network $\mathcal{R}$, with unobserved root node and observed leaf nodes •.

Here, in order to be able to give some other numerical simulations, let us give quickly the setting. Denoting $\mathcal{J}_{ext} = \{j \in \mathcal{J}, N_j = 0\}$ the set of the branches having a final leaf node and $\mathcal{J}_{int} = \{j \in \mathcal{J}, N_j > 0\}$ the other branches, we define a function f on the network $\mathcal{R}$ through its restriction f_j to each of the branches $j \in \mathcal{J}$ and we introduce the following notations:

$$\int_{\mathcal{R}} f(x)dx := \sum_{j\in\mathcal{J}} \int_0^{\ell_j} f_j(x)dx, \qquad [f]_j := f_j(\ell_j) - \sum_{i=1}^{N_j} f_{j_i}(0), \quad \forall j \in \mathcal{J}_{int}.$$

On each branch $j \in \mathcal{J}$ of the network, we consider the one-dimensional wave equation system

$$\begin{cases} \partial_{tt} u_j(t,x) - \partial_{xx} u_j(t,x) + q_j^\star(x) u_j(t,x) = g_j(t,x), & \forall (t,x) \in (0,T)\times(0,\ell_j), \\ u_j(0,x) = u_j^0(x), \qquad \partial_t u_j(0,x) = u_j^1(x), & \forall x \in (0,\ell_j), \end{cases} \tag{54}$$

where the time-independant potential $q_j^\star$ is an unknown coefficient.
In order to complete system (54), we then equip each branch $j \in \mathcal{J}$ with boundary conditions: Dirichlet boundary conditions at external nodes (root and leaves), and continuity condition and Kirchhoff law at any internal node. So that for all $t \in (0,T)$:

$$\begin{cases} \text{for } j = 1, & u_1(t,0) = h_1(t), \\ \text{if } j \in \mathcal{J}_{ext}, & u_j(t,\ell_j) = h_j(t), \\ \text{if } j \in \mathcal{J}_{int}, & u_j(t,\ell_j) = u_{j_i}(t,0), \ \forall i \in [[1,N_j]], \\ & [\partial_x u]_j(t) = 0, \end{cases} \tag{55}$$

assuming additionally some compatibility conditions between Dirichlet boundary and initial data: $u_1^0(0) = h_1(0)$ and $u_j^0(\ell_j) = h_j(0)$ for all $j \in \mathcal{J}_{ext}$.

For the following numerical simulations, we have chosen again to recover coefficients $q_j^\star$ that all takes the "hat shape" of (53). The exact coefficient that we want to recover is plotted by a red line, whereas the numerical coefficient recovered by the algorithm is represented by a dotted black line. The PYTHON source codes are available on request and similar numerical values to the one gathered in Table 1 are used here as well.

Besides the necessity of proving a new global Carleman estimate with one parameter for the whole network, the main technical difficulty in this setting was probably the construction of the appropriate cutoff function for the new and unwanted terms of the Carleman inequality.

In a setting with 3 branches as is Figure 8, one gets the recovery of $q_1^\star$, $q_{11}^\star$ and $q_{12}^\star$ as in Figures 9 (without noise) and Figures 10 (with 3% of noise). A very visual appreciation of the result allows to see that the potential of branch 1, that is the furthest from the measurements, is less good than the others.

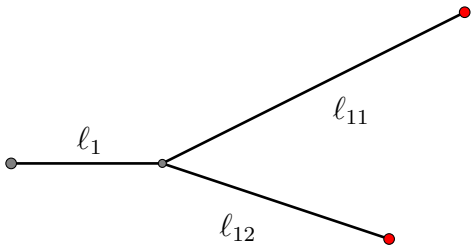


Figure 8: First setting - a 3 branches network, with observations at ●.

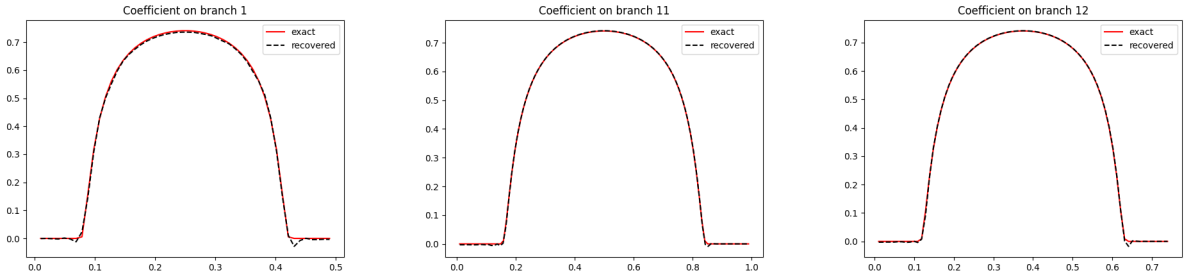


Figure 9: Simulation without noise.

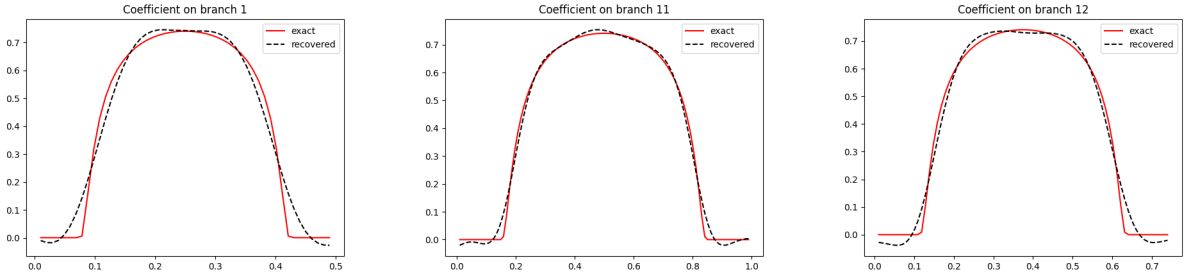


Figure 10: Simulation with $\alpha = 3\%$ of noise.

In a setting with 9 branches as is Figure 11 one gets, without noise, Figures 12 and with 3% of noise, Figure13.

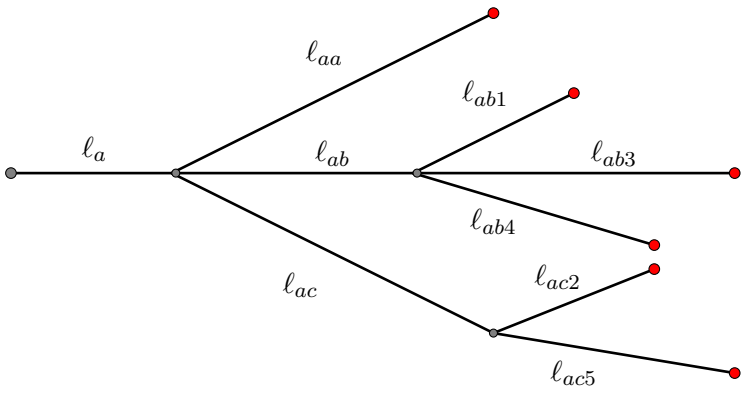


Figure 11: Second setting - a 9 branches network, with observation at ●.

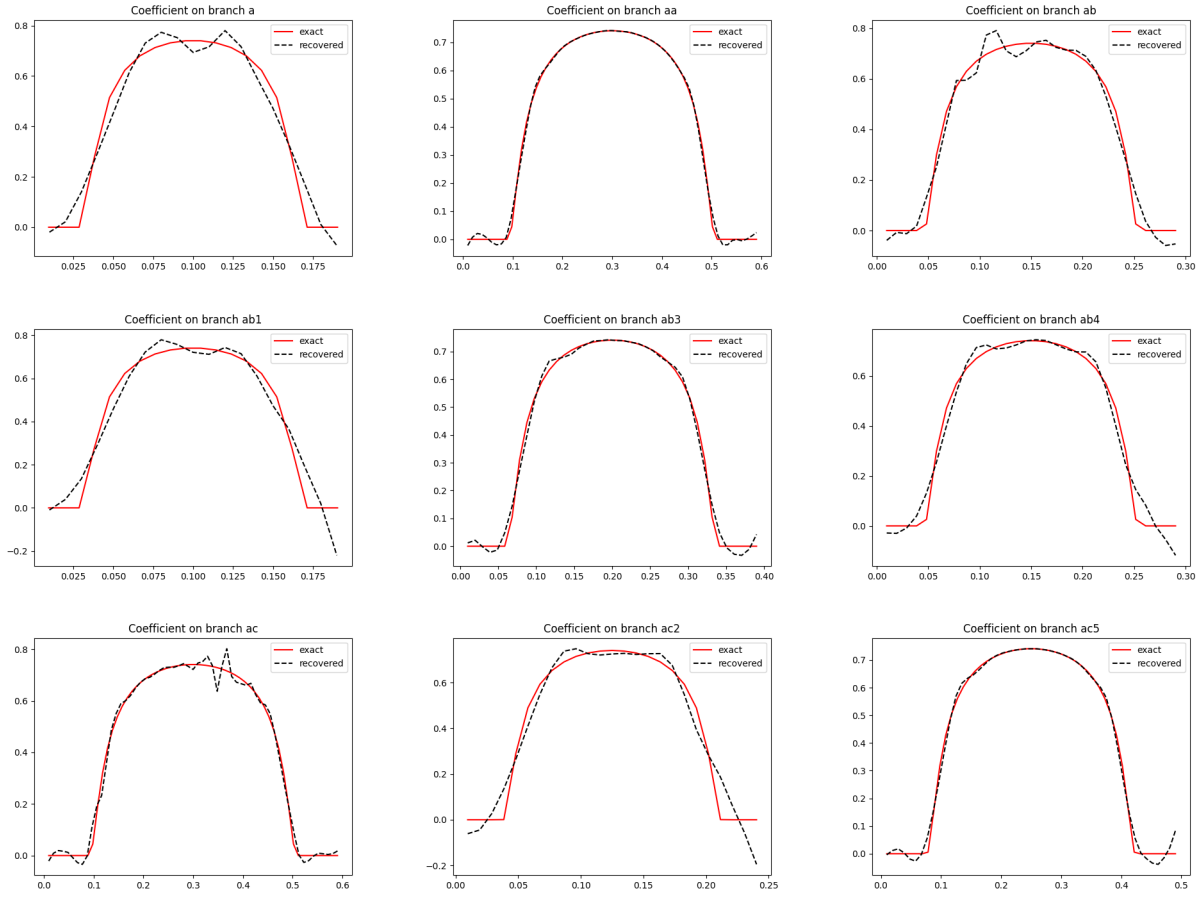


Figure 12: Simulation without noise.

If you want to see more examples and convergence informations (number of iterations, running time, convergence error), have a look at the final section of [BdBCV25].

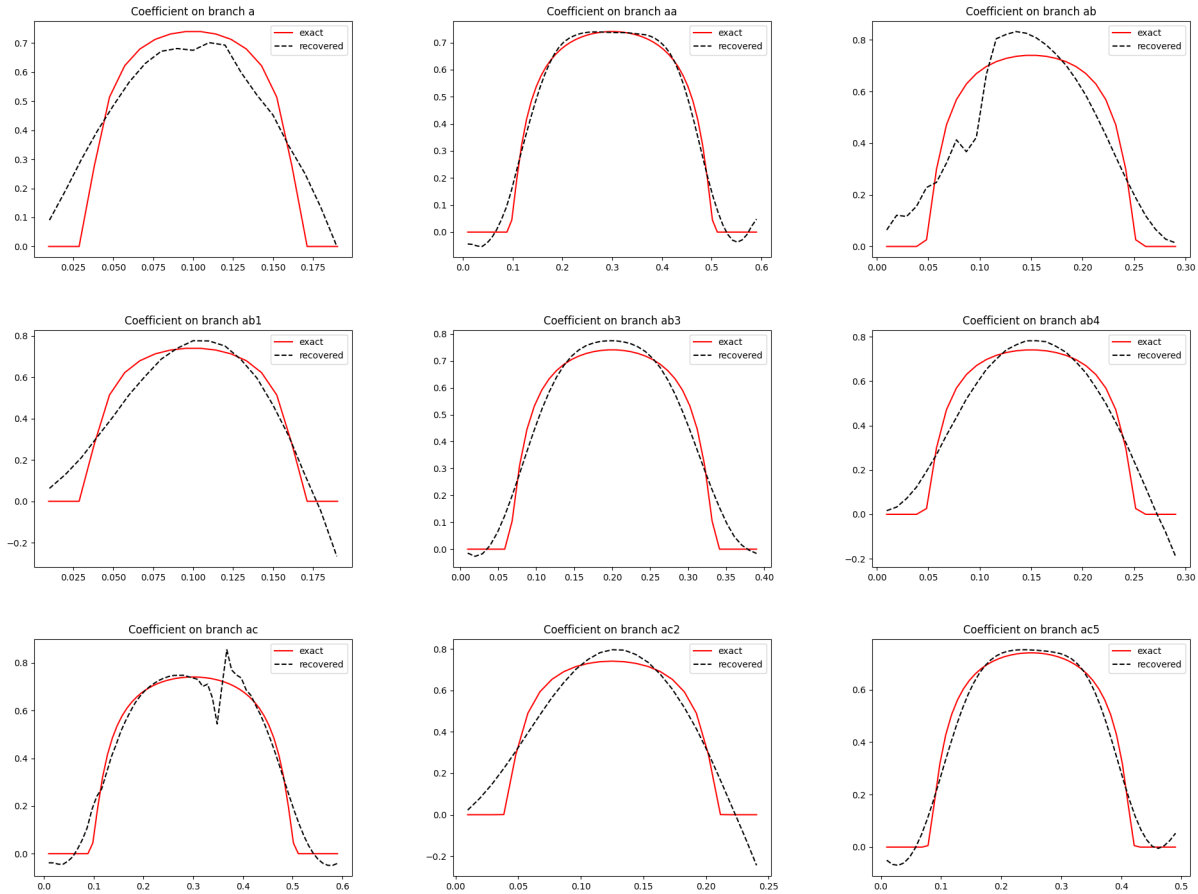


Figure 13: Simulation with $\alpha = 3\%$ of noise.

6.3.2 Reconstruction of the speed in a wave equation

When considering the wave equation trying to recover the source term of the initial data is easier than looking for the potential, but it is even more difficult to aim at reconstructing the speed of propagation if is is unknown. This means considering the following system :

$$\begin{cases} \partial_t^2 w - \nabla \cdot (a_* \nabla w) = f, & \text{in } (0,T) \times \Omega, \\ w = f_\partial, & \text{on } (0,T) \times \partial\Omega, \\ w(0) = w_0, \quad \partial_t w(0) = 0, & \text{in } \Omega, \end{cases} \tag{56}$$

where the speed $a_* = a_*(x)$ is unknown.

In [BdBEO21] we presented another globally convergent numerical C-bRec algorithm based on global Carleman estimates to reconstruct the speed of propagation of waves in a bounded domain with Dirichlet boundary conditions from a single measurement of the boundary flux of the solutions in a finite time interval. Although adapted from the same core idea to take advantage of Carleman estimates to build the C-bRec algorithm, this was quite a new challenge. The two main reasons for that is, first, the fact that the proof of this inverse problem's Lipschitz stability relies on two Carleman estimates (see e.g. [IY03]) and second, the fact that the admissible set of speed of propagation (given below) is quite demanding.

On a related topic, one can read in [KLZ22] an other type of convergent numerical method for the recovery of the inverse of the speed in a wave-like equation from boundary observations. The authors use a Carleman estimate to prove Lipschitz stability and construct a Tikhonov-like cost functional (using the Carleman weight function) which is globally strictly convex.

We will only give here some key elements to be able to show a numerical illustration and refer the interested reader to [BdBEO21] for all the details, specifically for the adapted Algorithm with two cost functional to minimize. The inverse problem is the following:

> Given the distributed and boundary source terms f and f_∂ and the initial position and velocity data $(w_0, 0)$, considering the solution of the wave partial differential equation system (56), can we determine the unknown speed coefficient $a_* = a_*(x)$, assumed to depend only on $x \in \Omega$, from the additional knowledge of the flux of the solution through a part Γ_0 of the boundary $\partial\Omega$, namely
>
> $$a_* \partial_\nu w = a_* \nabla w \cdot \nu, \quad \text{on } (0,T) \times \Gamma_0 ? \tag{57}$$

The speed of propagation a_* is supposed to belong to the following admissible set that limits the speed fluctuations with respect to a given exterior point x_0 and allows to avoid the possibility of trapped rays in the geometry of Ω because of the variations of a_*. One can indeed prove uniqueness and stability of this inverse problem in the following class (see [IY03] and [KY06]):

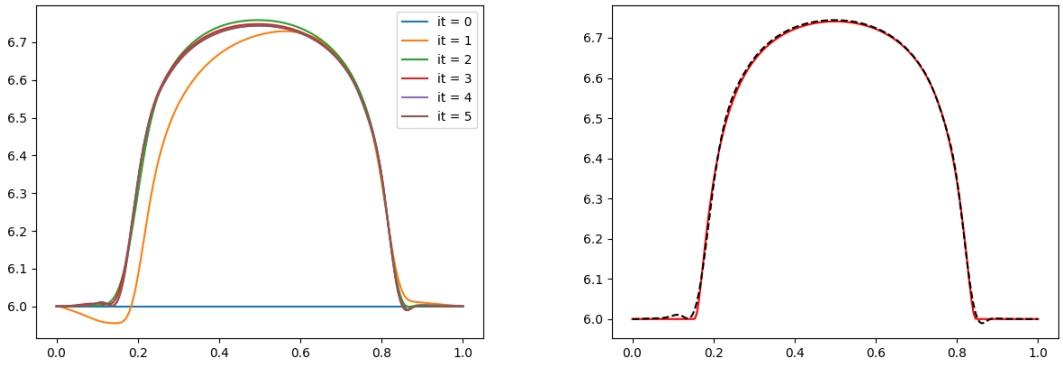
$$\begin{aligned}\mathcal{V} := \big\{ a \in C^1(\overline{\Omega}) \cap H^2(\Omega),\ 0 < \alpha_0 \leq \inf_\Omega a \leq \sup_\Omega a \leq \alpha_1, \\ \|\nabla a\|_{L^\infty(\Omega)} \leq m,\ \nabla a \cdot (x - x_0) \leq 2(1-\rho) a \text{ in } \Omega,\ a = \mathbf{a} \text{ and } \partial_\nu a = \mathbf{a}_\nu \text{ on } \partial\Omega \big\},\end{aligned} \tag{58}$$

a class of coefficients where we also managed to apply the C-bRec approach in [BdBEO21].

In order to recover a_*, we require a single experiment with null initial velocity $w_1 = 0$ and initial deformation having some monotonicity properties in the direction of $x - x_0$. We perform here numerical simulations in the same discrete setting as above in order to illustrate feasibility of our approach in this new challenging setting. As proved theoretically in [BdBEO21], we verify that the numerical reconstruction is achieved for any admissible initial guess, even in the presence of small random disturbances on the measurements, as presented in Figure 14 and Figure 15, for

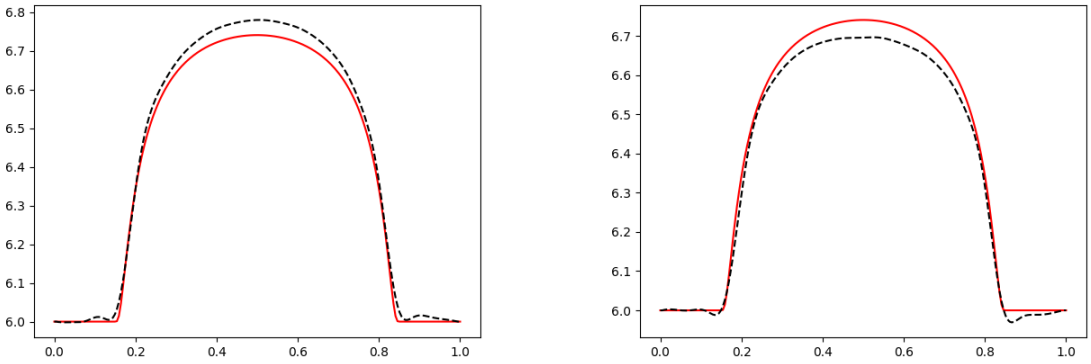
$$a^*(x) = 6 + e^{\frac{-0.3}{1-8|x-0.4|^2}} \mathbb{1}_{1>8|x-0.4|^2}, \quad a^* \in \mathcal{V}.$$

One can see in in [BdBEO21] simulations for several others syntetised a^*, with various levels of noise, wrong choices of the setting's parameters T, $a^* \notin \mathcal{V}$ and even 2-D space simulations.



(a) Iterative steps (b) Exact/recovered (dashed) potential

Figure 14: Simulation without noise for the recovery of the speed $a^\star$.



(a) $\alpha = 1\%$ of noise (b) $\alpha = 3\%$ of noise

Figure 15: Simulation with noise - Exact (red) and recovered (dashed) speed.

References

[All07] G. Allaire, *Numerical analysis and optimization*, Numerical Mathematics and Scientific Computation, Oxford Univ. Press, Oxford, 2007.

[BLR92] C. Bardos, G. Lebeau, and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control and Optim. **30** (1992), no. 5, 1024–1065.

[Bau10] L. Baudouin, *Lipschitz stability in an inverse problem for the wave equation*, 2010, hal-00598876.

[BCV11] L. Baudouin, Emmanuelle Crépeau and J. Valein, *Global carleman estimate on a network for the wave equation and application to an inverse problem.*, Mathematical

control and related fields, **1** (2011), no. 3.

[BdBCV25] L. Baudouin, M. de Buhan, Emmanuelle Crépeau and J. Valein, *Carleman-Based Reconstruction Algorithm on a wave Network*, Math. Control Signals Systems, **37** (2025), no. 3, 621–660.

[BdBE13] L. Baudouin, M. de Buhan and S. Ervedoza, *Global carleman estimates for waves and applications*, Comm. Partial Differential Equations **38** (2013), no. 5, 823–859.

[BdBE17] L. Baudouin, M. de Buhan and S. Ervedoza, *Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation*, SIAM J. Numer. Anal. **55** (2017), no. 4, 1578–1613.

[BdBEO21] L. Baudouin, M. de Buhan and S. Ervedoza, Osses, A, *Carleman-based reconstruction algorithm for waves.* , SIAM J. Numer. Anal. **59** (2021), no. 2, 998–1039.

[BE13] L. Baudouin and S. Ervedoza, *Convergence of an Inverse Problem for a 1-D Discrete Wave Equation*, SIAM J. Control Optim. **51** (2013), no. 1, 556–598.

[BEO15] L. Baudouin, S. Ervedoza, and A. Osses, *Stability of an inverse problem for the discrete wave equation and convergence results*, Journal de Mathématiques Pures et Appliquées (9) **103** (2015), no. 6, 1475–1522.

[BK12] L. Beilina and M. V. Klibanov, *Approximate global convergence and adaptivity for coefficient inverse problems*, Springer, 2012.

[BdBS21] M. Boulakia, M. de Buhan and, E. Schwindt, *Numerical reconstruction based on carleman estimates of a source term in a reaction-diffusion equation*, ESAIM: Control, Optimisation and Calculus of Variations **27** (2021), no. 1, 1-34.

[BdBDIM26] M. Boulakia, M. de Buhan, T. Delaunay, S. Imperiale and P. Moireau, *Solving inverse source wave problem – From Carleman estimates to observer design*, Mathematical control and related fields textbf16 (2026), 40-81.

[BK81] A. L. Bukhgeĭm and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.

[Car39] T. Carleman, *Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat. Astr. Fys. **26** (1939), no. 17.

[CM25] C. Castro and S. Micu, *A mixed finite elements approximation of inverse source problems for the wave equation with variable coefficients using observability*, Numer. Math. **157** (2025), no. 5, 1847–1895

[CFCM13] N. Cîndea, E. Fernández-Cara, and A. Münch, *Numerical controllability of the wave equation through primal methods and Carleman estimates*, ESAIM: Control, Optimisation and Calculus of Variations **19** (2013), no. 4, 1076-1108.

[DL90] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 1 & 5*, translated from the French by Ian N. Sneddon, Springer, Berlin, 1990.

[FYZ07] X. Fu, J. Yong, and X. Zhang, *Exact controllability for multidimensional semilinear hyperbolic equations*, SIAM J. Control Optim. **46** (2007), no. 5, 1578–1614 (electronic).

[Had02] J. Hadamard, *Sur les problèmes aux dérivées partielles et leur signification physique*, Princeton University Bulletin, (1902), 49–52.

[Ho86] L. F. Ho, *Observabilité frontière de l'équation des ondes*, C. R. Acad. Sci. Paris Sér. I Math. **302** (1986), no. 12, 443–446.

[Hor63] L. V. Hörmander, *Linear partial differential operators*, Springer, Berlin-Gottingen-Heidelberg, (1963), Academic Press, Inc., Publishers, New York.

[Ima02] O. Y. Imanuvilov. *On Carleman estimates for hyperbolic equations*, Asymptot. Anal., 32 (2002), no. 3-4, 185–220.

[IY01a] O. Y. Imanuvilov and M. Yamamoto, *Global Lipschitz stability in an inverse hyperbolic problem by interior observations*, Inverse Problems **17** (2001), no. 4, 717–728, Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).

[IY01b] O. Y. Imanuvilov and M. Yamamoto, *Global uniqueness and stability in determining coefficients of wave equations*, Comm. Partial Differential Equations **26** (2001), no. 7-8, 1409–1425.

[IY03] O. Y. Imanuvilov and M. Yamamoto, *Determination of a coefficient in an acoustic equation with a single measurement*, Inverse Problems **19** (2003), no. 1, 157–171.

[Isa06] V. Isakov, *Inverse problems for partial differential equations*, second ed., Applied Mathematical Sciences, vol. 127, Springer, New York, 2006.

[Kli97] M. V. Klibanov, *Global convexity in a three-dimensional inverse acoustic problem*, SIAM J. Math. Anal. **28** (1997), no. 6, 1371–1388.

[Kli13] M. V. Klibanov, *Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems*, J. Inverse Ill-Posed Probl. **21** (2013), no. 4, 477–560.

[Kli92] M. V. Klibanov, *Inverse problems and Carleman estimates*, Inverse Problems **9** (1992), no. 4, 575-596.

[KLZ22] M. V. Klibanov, J. Li and W. Zhang, *A globally convergent numerical method for a 3D coefficient inverse problem for a wave-like equation,* SIAM J. Sci. Comput. **44** (2022), no. 5, 3341– 3365;

[KY06] M. V. Klibanov and M. Yamamoto, *Lipschitz stability of an inverse problem for an acoustic equation*, Appl. Anal. **85** (2006), no. 5, 515-538.

[Kom94] V. Komornik, *Exact controllability and stabilization*, RAM: Research in Applied Mathematics, Masson, Paris, 1994 Wiley, Chichester, 1994

[LLT86] I. Lasiecka, J.-L. Lions, and R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures Appl. (9) **65** (1986), no. 2, 149–192.

[LTZ00] I. Lasiecka, R. Triggiani, and X. Zhang. *Nonconservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot*, volume 268 of *Contemp. Math.*, pages 227–325. Amer. Math. Soc., Providence, RI, 2000.

[LRS86] M.M. Lavrentiev, V.G. Romanov, and S.P. Shishatskiĭ, *Ill-posed problems of mathematical physics and analysis*, Translations of Mathematical Monographs, American Mathematical Society, Providence, R.I., 1986.

[Lio88] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués. tome 1. Contrôlabilité exacte*, vol. RMA 8, Masson, 1988.

[LM72] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[PY96] J.-P. Puel and M. Yamamoto, *On a global estimate in a linear inverse hyperbolic problem*, Inverse Problems **12** (1996), no. 6, 995–1002.

[PY97] J.-P. Puel and M. Yamamoto, *Generic well-posedness in a multidimensional hyperbolic inverse problem*, J. Inverse Ill-Posed Probl. **5** (1997), no. 1, 55–83.

[Tat96] D. Tataru. Carleman estimates and unique continuation for solutions to boundary value problems. *J. Math. Pures Appl. (9)*, 75(4):367–408, 1996.

[Tre82] L. N. Trefethen, *Group velocity in finite difference schemes*, SIAM Rev. **24** (1982), no. 2, 113–136.

[TW09] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basel Textbooks. Birkhäuser Verlag, Basel, 2009.

[Yam99] M. Yamamoto, *Uniqueness and stability in multidimensional hyperbolic inverse problems*, J. Math. Pures Appl. (9) **78** (1999), no. 1, 65–98.

[Zha00] X. Zhang, *Explicit observability estimate for the wave equation with potential and its application*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **456** (2000), no. 1997, 1101–1115.

[Zua24] E. Zuazua, *Exact controllability and stabilization of the wave equation.*, Translated from the Spanish by Darlis Bracho Tudares, Springer Cham, Unitext Vol. 162, 2024.

[Zua05] E. Zuazua, *Propagation, observation, and control of waves approximated by finite difference methods*, SIAM Rev. **47** (2005), no. 2, 197–243 (electronic).