Reconstruction algorithm of some parameter in the wave equation.

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joint works with M. de Buhan, S. Ervedoza, A. Osses, J. Valein and E. Crépeau.



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Coefficient inverse problem in the wave equation

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$, it writes for instance,

$$\begin{cases} \partial_{tt} y(t,x) - \Delta_x y(t,x) + p^*(x) y(t,x) = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

or with variable speed

$$\begin{cases} \partial_{tt} y - \nabla \cdot (a^*(x) \nabla y) = f, & \text{in } (0,T) \times \Omega, \\ y = g, & \text{on } (0,T) \times \partial\Omega, \\ (y(0), \partial_t y(0)) = (y^0, 0), & \text{in } \Omega, \end{cases}$$

- Given data : Source terms f, g ; initial data : (y^0, y^1) ;
- Unknown : the potential $p^* = p^*(x)$ or the speed $a^* = a^*(x)$;
- Additional measurement : the flux $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

Motivation

- The determination in Ω of p^* or a^* from an additional measurement are inverse problems for which *uniqueness* and *stability* are well-known and proved using Carleman estimates.
- Classical reconstruction : from the measurement $d^* = \partial_\nu y[p^*]$, calculate

$$\min J(p^k) = \|\partial_\nu y[p^k] - d^*\|$$

But J is not convex and may have several local minima, so that the solution will depend on the initialization p_0.
Algorithms not guaranteed to converge to the global minimum.

C-bRec Algorithm

Initialization : μ^* is given, take any $q^0 \in L^\infty_m(\Omega)$.

Iteration : Given q^k,

1 - Compute $w[q^k]$ the solution of $\begin{cases} \partial_t^2 w - \Delta w + q^k w = f, \\ w = f_\partial, \\ w(0) = w^0, \quad \partial_t w(0) = w^1, \end{cases}$

and set $\mu^k = \eta(\varphi)\partial_t \left(\partial_\nu w[q^k] - \mu^*\right)$ on $\Gamma_0 \times (0,T)$.

2 - Introduce the functional $J_s[q^k, 0, \mu^k] = J_s^k$

$$J_s^k(z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|\partial_t^2 z - \Delta z + q^k z|^2 + \frac{s}{2}\int_0^T \int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \mu^k|^2 + \frac{s^3}{2}\iint_{\mathcal{O}} e^{2s\varphi}|z|^2.$$

Let $\widetilde{Z}^k$ be the unique minimizer of J_s^k on the appropriate trajectories.

3 - Set $\tilde{q}^{k+1} = q^k + \dfrac{\partial_t \widetilde{Z}^k(0)}{w_0}$, in Ω, where w_0 is the initial condition.

4 - Finally, set $q^{k+1} = T_m(\tilde{q}^{k+1})$, with $T_m(q) = \begin{cases} q, & \text{if } |q| \le m, \\ \textit{sign}(q)m, & \text{if } |q| \ge m, \end{cases}$

where m is the a priori bound.

In theory, C-bRec works. But in practice ?

Two remarks :

- Discretizing the wave equation brings numerical artefacts ;
- Minimizing a strictly convex coercive quadratic functional based on a Carleman estimate means dealing with $e^{2s\phi}$ for large s.

- Goal : propose a numerically efficient algorithm.

Ideas : We actually need an algorithm constructed with at least

- a regularization term in the cost functional,
- some improvements in the preconditioning,
- some splitting of the observations.

Natural idea for reconstruction

Given a continuous measurement $\mathscr{M}[p^*] = \partial_\nu y[p^*]|_{(0,T)\times\partial\Omega}$

- Discretize the wave equation

$$\begin{cases} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h \simeq f, \\ y_h|_{(0,T)\times\partial\Omega} = g_h \simeq g, \\ (y_h, \partial_t y_h)(t=0) = (y_h^0, y_h^1) \simeq (y^0, y^1). \end{cases}$$

- Solve the following discrete inverse problem : Find a potential p_h so that the corresponding discrete solution $y_h[p_h]$ approximates at best the measurement :

$$\partial_h y_h[p_h]|_{(0,T)\times\partial\Omega}(t,x) \simeq \mathscr{M}[p^*](t,x)$$

$$\text{i.e. } p_h = \text{Argmin}_{q_h} \|\partial_h y_h[q_h] - \mathscr{M}[p^*]\|$$

Question : Do we get $p_h \simeq p^*$?

Convergence of the discrete inverse problems

Remarks :

- Natural question for all inverse problems in infinite dimensions : Finding a source term, a conductivity...
- Depends *a priori* on the numerical scheme employed.

Main difficulty :

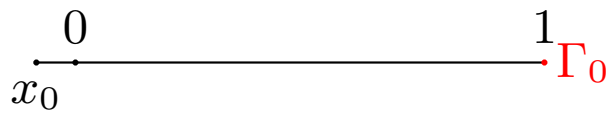
- Different dynamics for the continuous wave equation versus its discrete approximations, cf Ervedoza - Zuazua '11 :
  $\rightsquigarrow$ Numerical artefacts : High-frequency spurious waves.

Convergence results for the inverse problem :

- Penalization of high-frequencies with a regularization term in the discrete Carleman estimates.
- 1D (LB & Ervedoza '13) and 2D (LB & Ervedoza & Osses '15)

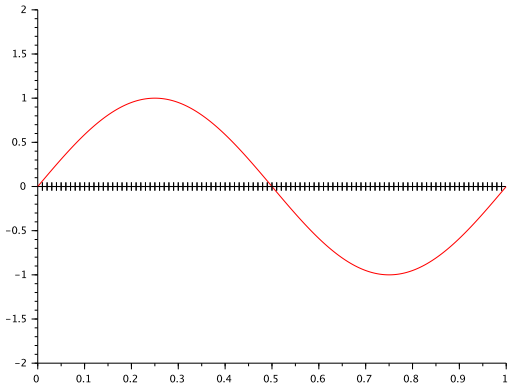
Discretization of the problem

- $\Omega = [0,1]$, $x_0 = -0.3$, $\Gamma_0 = \{x = 1\}$, $\beta = 0.99$, $T = 1.3$, $s = 100$, $f = 0$, $g = 2$, $u_0(x) = 2 + \sin(x\pi)$ and $u_1 = 0$.

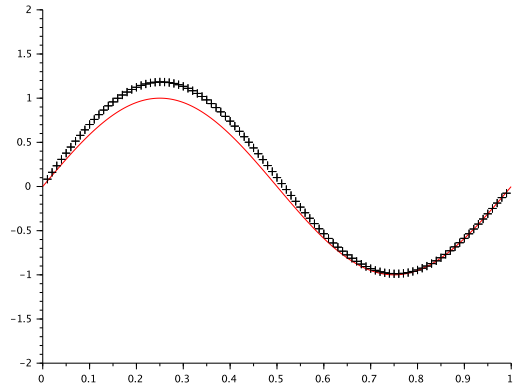


- To avoid the inverse crime, we use $\neq$ schemes and $\neq$ meshes in the direct and inverse problems.
- Discretization with the finite-difference method : $N + 1 = \frac{1}{h}$, $(\Delta_h y_h)_j = \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}, \quad \forall j \in \{1, \cdots, N\}$
- Addition of a regularization term $s \int_0^T \int_0^1 e^{2s\varphi} |h\partial_h^+ \partial_t z_h|^2 \, dt$ to the cost functional - from discrete Carleman estimates - to have uniformity with respect to h. Constraint : sh small enough.
- Other approach : use high order finite elements to guarantee a conformal approximation (Cîndea-FernándezCara-Münch '13).

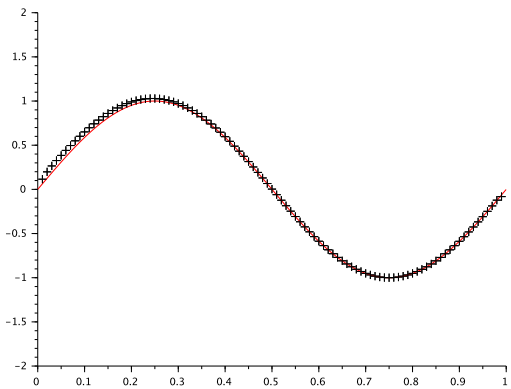
Illustration of the convergence of the algorithm



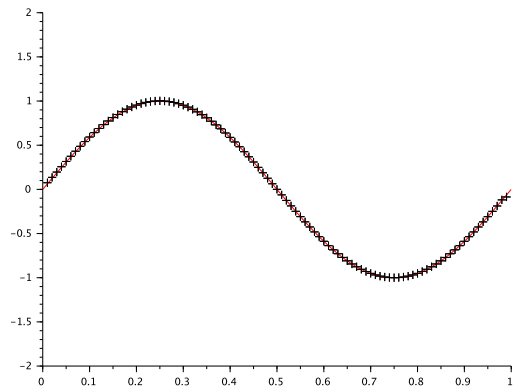
(a) q^0



(b) q^1

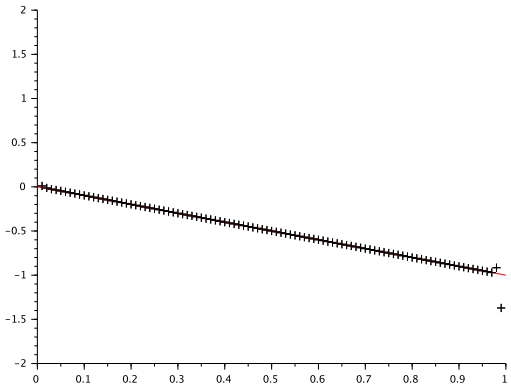


(c) q^2

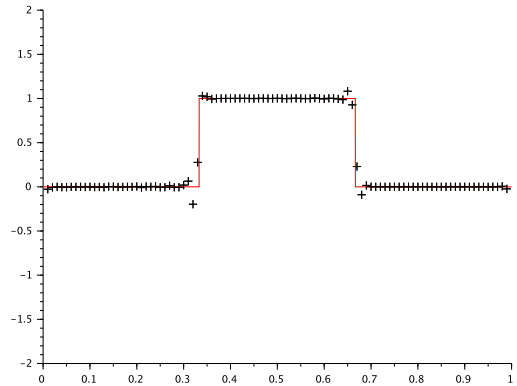


(d) q^3

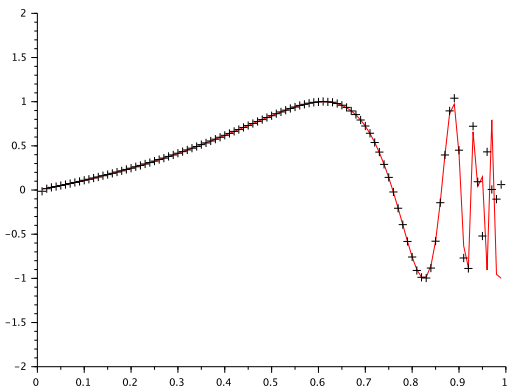
Other 1D simulations



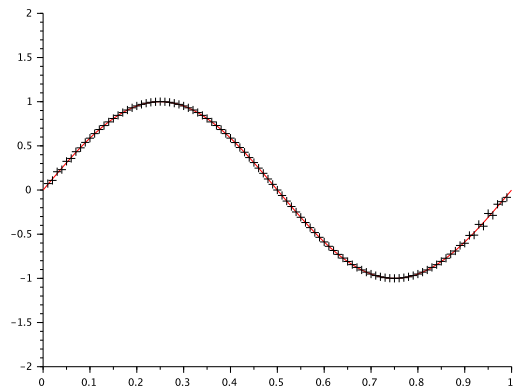
(a) $p^* = -x$



(b) $p^* = \text{gate}(x)$

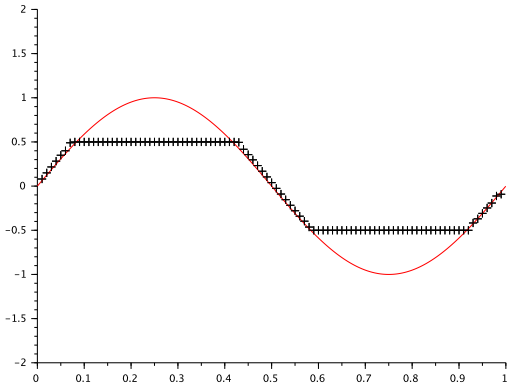


(c) $p^*(x) = \sin(\frac{x}{1-x})$

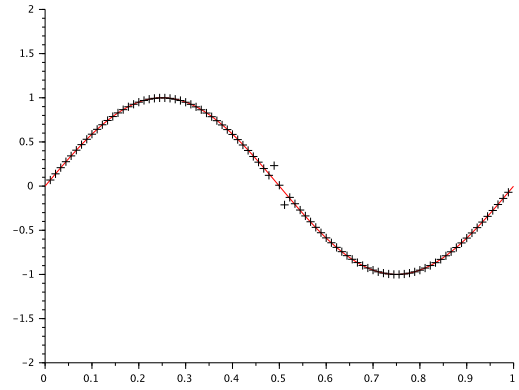


(d) $p^*(x) = \sin(2\pi x)$, with $q^0 = 10$

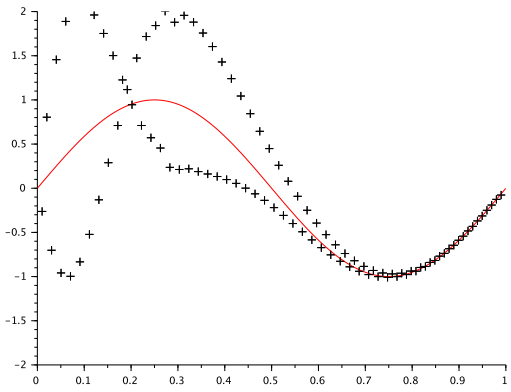
Wrong choices of the parameters



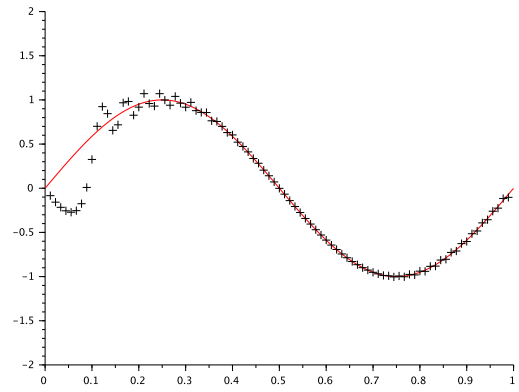
(a) Wrong choice of m



(b) y^0 vanishes at $x = 0.5$

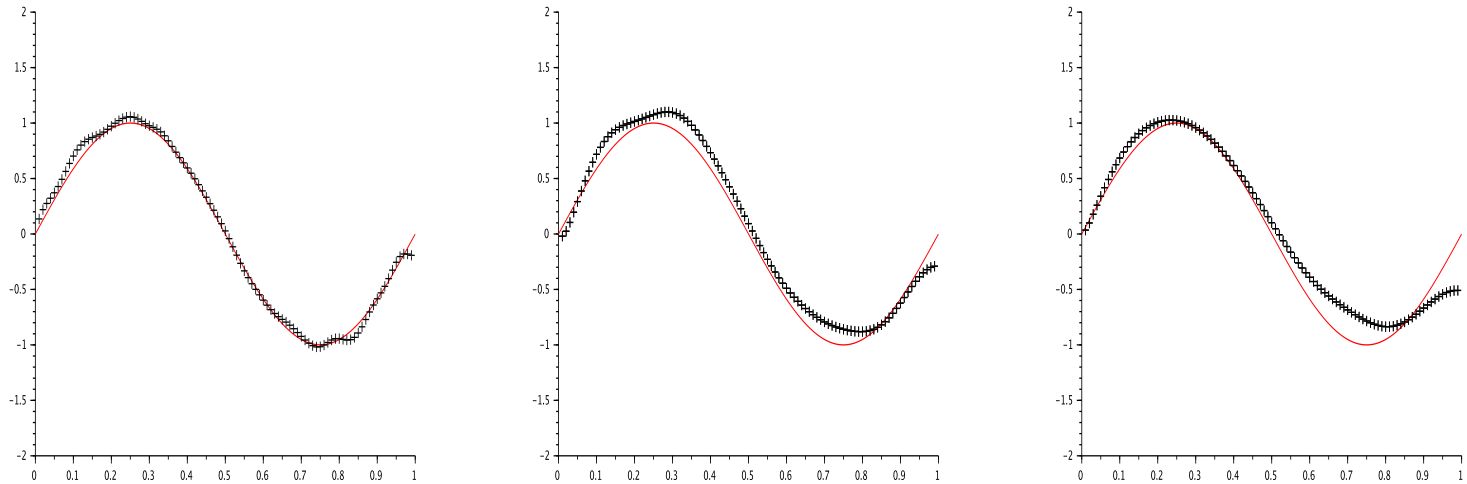


(c) No viscous term or sh too large



(d) $T = 0.9 < 1$

With noise on the measurement of the flux



$s = 10$ and the noise is multiplicative : 1%, 5%, 10%.

Taking s too large seems to amplify the effects of the noise...

Recovery of the main coefficient

Wave equation with variable speed :

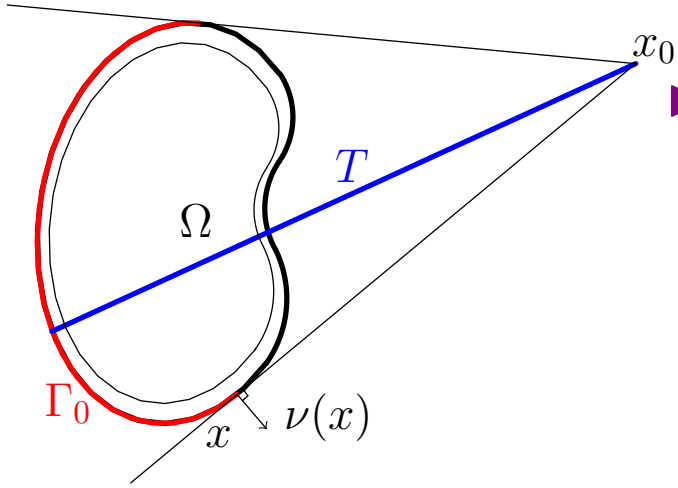
$$\begin{cases} \partial_{tt} y - \nabla \cdot (a^*(x) \nabla y) = f, & \text{in } (0,T) \times \Omega, \\ y = g, & \text{on } (0,T) \times \partial\Omega, \\ y(0) = y^0, \quad \partial_t y(0) = y^1, & \text{in } \Omega, \end{cases}$$

- Given data : Source terms (f, g), initial data : (y^0, y^1),
  boundary values $a^* = \mathbf{a}$ and $\partial_\nu a^* = \mathbf{a}_\nu$ on $\partial\Omega$.
- Unknown : the speed $a^* = a^*(x) > 0$, inside Ω.
- Additional measurement : the flux $\partial_\nu y^*(t,x)$ on $(0,T) \times \partial\Omega$.

Goal : Find the variable speed $a^* = a^*(x)$.

$\rightsquigarrow$ Application in medical imaging ?

Setting and assumptions



- Geometric and time conditions :

  $\exists\, x_0 \notin \overline{\Omega},$ such that

  $$\Gamma_0 \supset \{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) \geq 0\},$$
  $$T > \frac{\sup_{x\in\Omega} |x - x_0|}{\sqrt{\alpha_0 \rho_0}}.$$

- Regularity assumption $y^* = y[a^*] \in H^2(0, T; W^{2,\infty}(\Omega))$.
- Initial conditions : $|\nabla y^0(x) \cdot (x - x_0)| \geq r_0 > 0$ and $y^1 = 0$ in Ω.
- $\mathcal{V}_{\mathbf{a},\mathbf{a}_\nu} = \{a \in C^1(\overline{\Omega}) \cap H^2(\Omega), \|\nabla a\|_{L^\infty(\Omega)} \leq m,\ 0 < \alpha_0 \leq a \leq \alpha_1,$
  $\nabla a \cdot (x - x_0) \leq 2(1 - \rho)a$ in $\Omega,\ a = \mathbf{a}, \partial_\nu a = \mathbf{a}_\nu$ on $\partial\Omega\}$.

Theorem (Inverse problem stability)

There exists a positive constant $M = M(\Omega, T, x_0, r_0, \mathbf{a}, \mathbf{a}_\nu, \alpha_0, \alpha_1)$
such that for all $a, a^* \in \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$ *:*

$$\|a - a^*\|_{H_0^1(\Omega)} \leq M \|\partial_\nu y - \partial_\nu y^*\|_{H^2(0,T;L^2(\Gamma_0))}.$$

Idea

The speed reconstruction algorithm is based on the fact that if $y[a^*]$, $y[a^k]$, are the solution of the wave equation, then

$$z^k = \partial_t^2 \left(y[a^k] - y[a^*] \right)$$

solves

$$\begin{cases} \partial_{tt} z^k - \nabla \cdot (a^k \nabla z^k) = g^k, & \text{in } (0,T) \times \Omega, \\ z^k = 0, & \text{on } (0,T) \times \partial\Omega, \\ z^k(0,\cdot) = z_0^k, \quad \partial_t z^k(0,\cdot) = 0, & \text{in } \Omega, \end{cases}$$

where

$$g^k = \nabla \cdot ((a^k - a^*) \nabla \partial_t^2 y[a^*]), \qquad z_0^k = \nabla \cdot ((a^k - a^*) \nabla w_0),$$

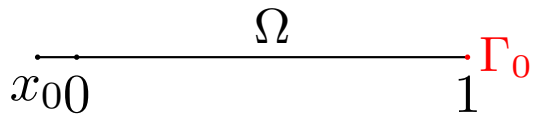
and for both operators (wave and first order) we can prove Carleman estimates.

$\rightsquigarrow$ Holder stability results (Imanuvilov Yamamoto '03)

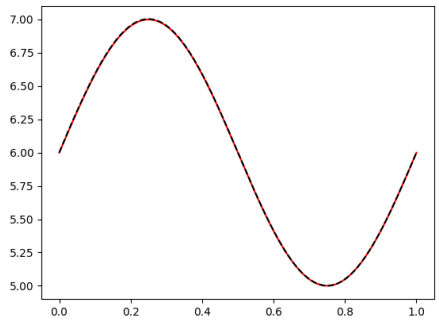
$\rightsquigarrow$ Lipschitz stability results (Klibanov Yamamoto '06)

$\rightsquigarrow$ Γ_0 small, Logarithmic stability (Bellassoued Yamamoto '06)

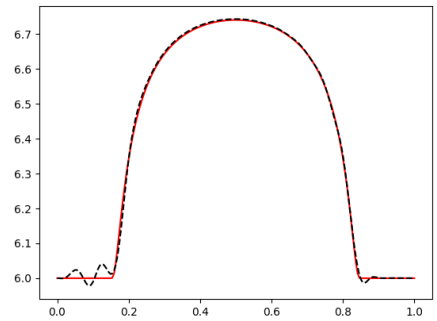
1D Numerical results



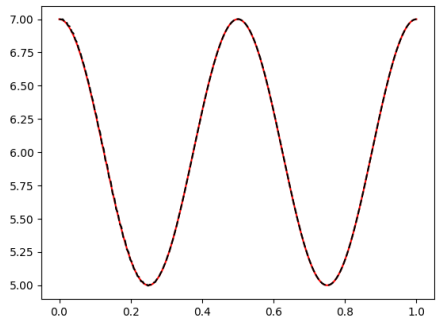
- $f = 0$, $T = 5$, $s = 100$, $y_0(x) = x - 1$,
- Finite differences scheme in space and time, avoiding inverse crime : $\neq$ meshes & scheme for direct and inverse problems ;



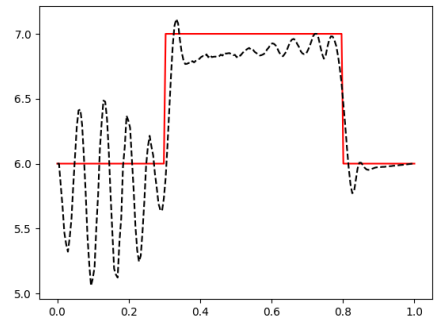
(a) $a = 6 + \sin(2\pi x)$



(b) $a \in \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$



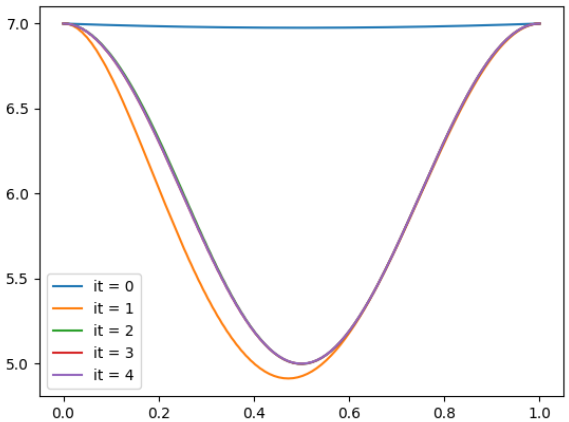
(c) $a \notin \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$



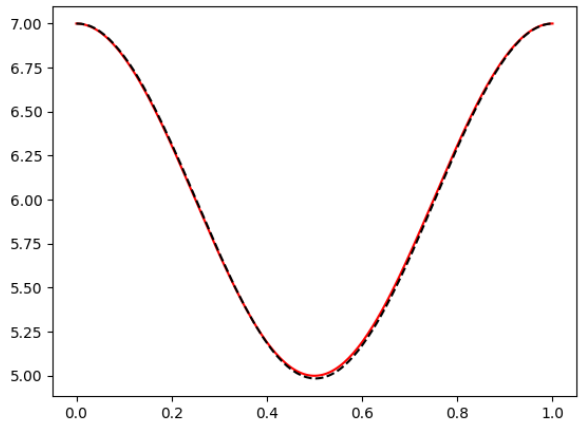
(d) $a \notin \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$

Numerical results with noisy data

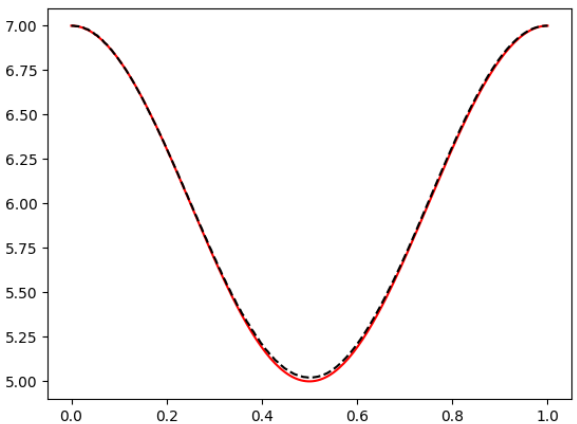
- $\partial_t^2 \mu = (1 + \alpha \mathcal{N}(0, 0.5))\partial_t^2 \mu, \quad \alpha \geq 0, \qquad a = 6 + \cos(2\pi x)$



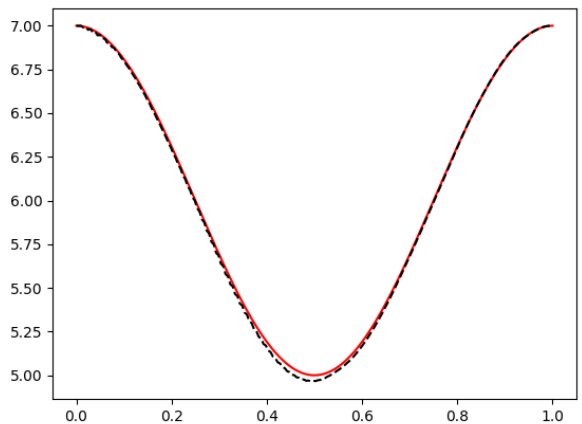
(a) $\alpha = 0\%$ - iterations



(b) $\alpha = 5\%$

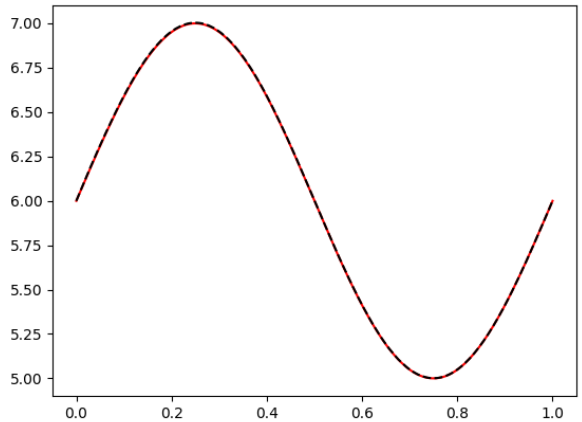


(c) $\alpha = 10\%$

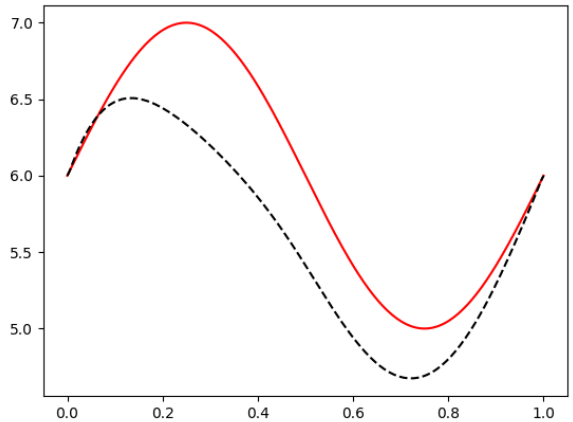


(d) $\alpha = 20\%$

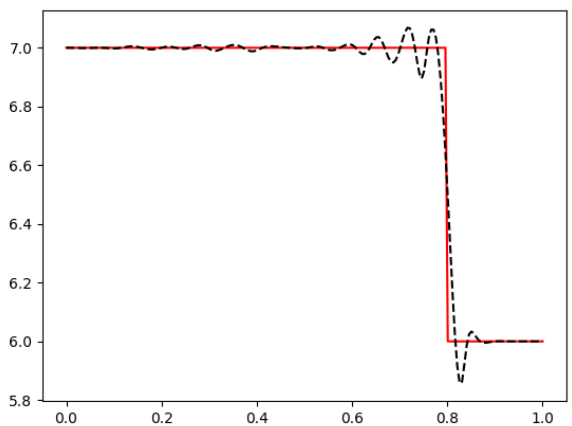
Other numerics



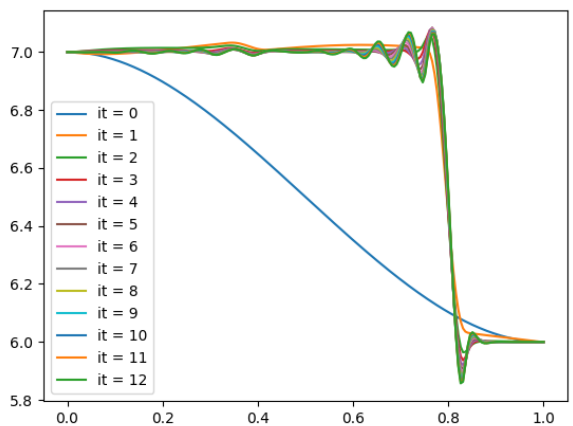
(a) $T = L/\sqrt{\alpha_0}$



(b) $T = L/2\sqrt{\alpha_0}$

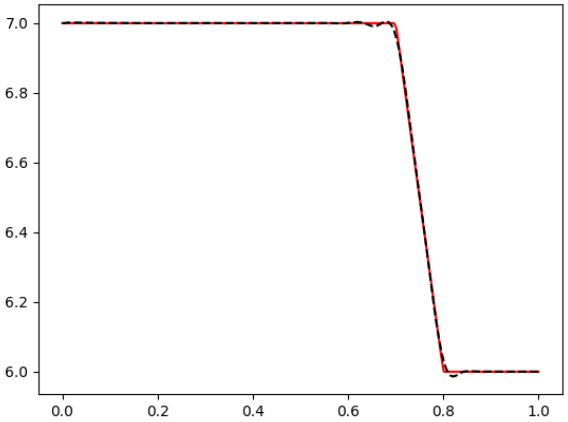


(c) $a \notin \mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$

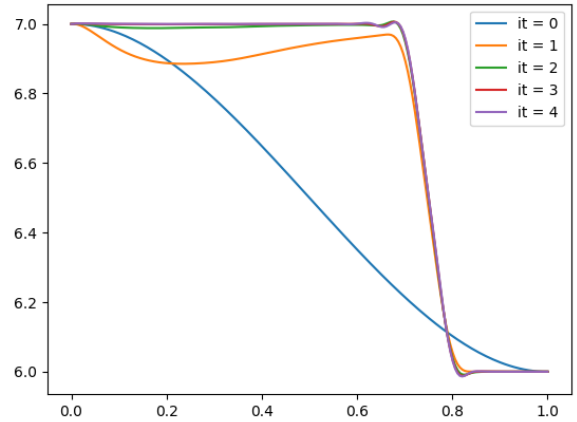


(d) iterations

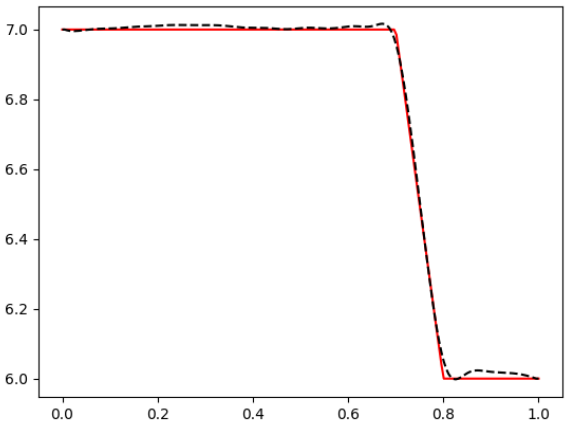
Other numerics



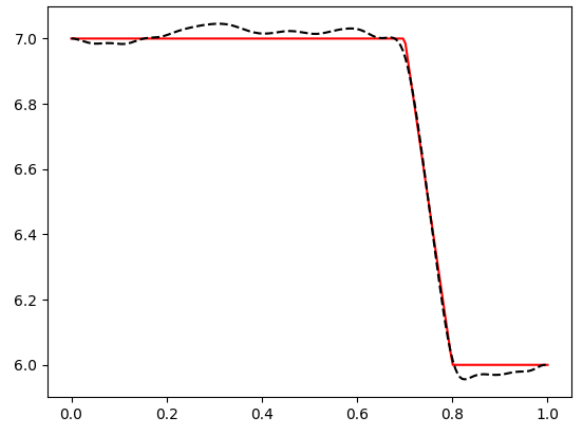
(a) $\alpha = 0\%$



(b) iterations



(c) $\alpha = 5\%$



(d) $\alpha = 10\%$

Conclusion

Flaws

- Projection operator $T_{\mathbf{a},\mathbf{a}_\nu}$ on the admissible set $\mathcal{V}_{\mathbf{a},\mathbf{a}_\nu}$...
- Constraints on the initial condition y^0
- Strongly linked to the Bukhgeim-Klibanov method to prove Lipschitz stability of the inverse problem

Hopes

- Challenges to work with other equations ? Or in domains with interface of discontinuity.
- Make our approach more educational ?
- Could we design a real imaging system using this strategy ?

$\star\star\star$

Thank you for your attention.

$\star$

Related articles

- *Carleman-based reconstruction algorithm on a wave network*,
  L. B., M. de Buhan, E. Crépeau & J. Valein, MCSS 2025.
- *Carleman-based Reconstruction algorithm*,
  L. B., M. de Buhan, S. Ervedoza & A. Osses, SINUM 2021.
- *Convergent algorithm based on Carleman estimates for the recovery of a potential in the wave equation*,
  L. B., M. de Buhan & S. Ervedoza, SINUM 2017.
- *Stability of an inverse problem for the discrete wave equation and convergence results*,
  L. B., S. Ervedoza & A. Osses, JMPA 2015.
- *Global Carleman estimates for waves and applications*,
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