Introduction to option pricing models in quantitative finance

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1 *1. Dynamic hedging and Black-Scholes models*

- Motivation and introduction
- Dynamic hedging for European options
- Black-Scholes for European vanilla call options
- Black-Scholes equation for European vanilla put options
- Examples of European vanilla options pricing
- Hedging and pricing of American vanilla options

Example of financial structured product

Let us consider a structured deposit indexed to a basket of 6 assets: Telefónica, Inditex, Banco Santander, Mapfre, Repsol and Dia. The financial institution sells it from 15 of October until 25 of November. The return is governed by the following conditions:

- An initial fixed coupon equivalent to $2,5\%$ equivalent annual yield from subscription is paid at 30-11-2026
- A coupon at maturity date (01-12-2028) is paid depending on the different assets behavior. More precisely, if at maturity date:
  - All asset prices are equal or greater to their initial value at 1 of December 2026, a 20% coupon is paid
  - Otherwise, if five asset prices are equal or greater to their initial value at 1 of December 2026, a 10% coupon is paid
  - Otherwise, if four asset prices are equal or greater to their initial value at 1 of December 2026, a 5% coupon is paid
  - Otherwise, a 2% coupon is paid

Moreover, the institution indicates that the maximum equivalent annual yield is $9,517\%$ and the minimum one is $0,93\%$

Some financial markets

- **Commodities markets:** Goods and commodities are negotiated (oil, gold, silver, cotton, electricity,...)
- **Foreign exchange markets:** Currency exchange rates are negotiated (pound-euro, dollar-yen,...)
- **Bond markets:** Bonds issued by state, by private companies, ... are traded
- **Stock exchange:** Assets of different companies, usually they are the largest and most popular financial markets
- **Futures and options markets:** Mainly the financial products of derivatives (options, interest rate derivatives,...) type are negotiated. They recently become increasingly large and popular (CBOE-U.S.A., MEFF-Madrid,...)

Option contract

- An option is a contract which gives its owner the **right** (not the obligation) to buy (call option) or sell (put option) a certain amount of an (underlying) good (asset, commodity, etc) at a certain prescribed price (*strike or exercise price*) before or at a prescribed date (expiry or maturity date).
- In order to get the rights involved in the option contract, its owner has paid a certain amount of money (option price or prime).

Example:

In the MEFF we buy today a call option that gives the right to buy 1 Telefónica asset by 3 euros, we can exercise this right until 1 of February 2027

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Advantages from operating with options

- **Naive example :**
  Today (3 of November of 2026) we pay 1 euro to buy a call option on an asset of certain ZZZZZ company:

  Exercise price: 20 euros
  Expiry date: 1 of February 2027
- We can figure out two possibilities for the asset price at the expiry date (01/02/2027):
  - 22 euros (We exercise) $\Rightarrow$ Profit $= 22 - 20 - 1 = 1$ euro
  - 18 euros (We do not exercise) $\Rightarrow$ Profit $= -1$ euro

Question:

What is the market price of the option contract?

$$\Uparrow$$

Supply vs. Demand Law

What is the fair price of the option contract?

$$\Downarrow$$

Option pricing $\Rightarrow$ Numerical and mathematical problems

Mathematical study of a real problem

Motivation of the problem: process arising from Physics, Engineering, Medicine, Economy, ...

$$\Downarrow$$

Comprehension and analysis of the process

$$\Downarrow$$

MATHEMATICAL MODEL

Steps in the study of a mathematical model

- **Theoretical study of the model:** existence, uniqueness and properties of the solution
- **If possible, calculus of the exact solution**
- **Otherwise, computation of the approximated solution: Numerical methods**
  - Theoretical analysis of the numerical approximation
  - Computer implementation of the numerical method
  - Computation output: Set of numbers (numerical solution), postprocessing (graphs of the numerical solution, isolines, gradients,...)
- **Verification of numerical results against real experiments or data $\Rightarrow$ parameters calibration and model revision**

Some problems requiring numerical methods

1. Nonlinear equations
   Computation of a bond yield
2. Linear and nonlinear systems of equations
   Solution of discretized Black-Scholes models for options
3. Numerical interpolation, derivation and integration
   Computation of the Greeks of an option
4. Linear/nonlinear ordinary and partial differential equations
   Black-Scholes models with one or several stochastic factors
5. Numerical optimization
   Calibration, portfolio optimization,...

All this kind of problems arise in financial pricing

Hypotheses in Black-Scholes modeling:

- The asset (or underlying) price, S_t, follows a geometric Brownian motion (GBM).
- The risk free interest rate, r, and the asset volatility, σ, are assumed to be deterministic functions of time.
- Fiscal and transaction costs are not taken into account (frictionless markets).
- No arbitrage assumption: all risk free portfolios have the same return rate, which is the risk-free interest rate.
- Continuous market: a non necessarily integer number of assets can be bought or sold in continuous time.
- Initially we assume that assets do not pay dividends.
- Market with long and short positions: shortselling is allowed.

Dynamic hedging (Δ-hedging) (I)

- The asset price, S_t, follows satisfies the SDE (GBM):

$$dS_t = \mu S_t \, dt \; + \; \sigma S_t \, dX_t$$

- The option price, v_t, depends on S_t and t in the form $v_t = V(t, S_t)$.
- By applying Ito lemma we get:

$$dv_t = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) (t, S_t) \, dt + \sigma S \frac{\partial V}{\partial S} (t, S_t) \, dX_t$$

- By combining processes S_t and v_t we can build the deterministic process π_t in the form:

$$\pi_t = v_t - \Delta S_t$$

- In practice the process, π_t, in $[t, t + dt]$ can be obtained, for example, by selling Δ shares and buying 1 call option at time t.

Dynamic hedging (Δ-hedging) (II)

- By using stochastic calculus:

$$\pi_t = v_t - \Delta S_t \Rightarrow d\pi_t = dv_t - \Delta dS_t \Rightarrow$$

$$d\pi_t = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S\right)(t, S_t)\, dt + \sigma S_t \left(\frac{\partial V}{\partial S} - \Delta\right)(t, S_t)\, dX_t$$

- Thus, the portfolio π_t becomes risk free $\Leftrightarrow \Delta = \dfrac{\partial V}{\partial S}$. In this case:

$$d\pi_t = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)(t; S_t)\, dt$$

- Thus, as π_t is risk free, using the arbitrage free hypothesis we get

$$d\pi_t = r\pi_t dt = r\left(v_t - S_t \frac{\partial V}{\partial S}(t, S_t)\right) dt$$

Dynamic hedging (Δ-hedging) (III)

- So, identifying previous expressions we get:

$$\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right) = r\left(V - S\frac{\partial V}{\partial S}\right)$$

- Finally we obtain the well known **Black-Scholes (BS) equation**:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- The asset drift, μ, does not affect the option value
- Black-Scholes equation is a degenerated parabolic PDE at $S = 0$
- In order to define a well posed problem a final condition has to be imposed, i.e. $V(T, S)$.

European vanilla options

- In Spanish future markets: Options on IBEX-35 Index.
- The involved rights can be only exercised at maturity date.
- The objective is to determine the price of the prime to be paid at any instant before maturity.

BS equation for European vanilla call options

Let $D = [0, T] \times [0, \infty)$
Find the function $C = C(t, S)$ such that:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \text{ in } D$$

$$C(T, S) = \max(S - E, 0), \qquad S > 0$$

r : risk free interest rate $\quad \sigma$: asset volatility
E : exercise price $\quad T$: maturity date

BS formula for European vanilla call options (I)

Exact solution of Black-Scholes equation:

$$C(t,S) = S\ N(d_1)\ -\ E\ \exp(r(t-T))N(d_2)$$

$$\begin{aligned} d_1 &= \frac{\log(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\log(S/E) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-y^2/2)\, dy \end{aligned}$$

r : risk free interest rate $\quad$ σ : asset volatility
E : exercise (strike) price $\quad$ T : maturity date

BS formula for European vanilla call options (I)

Change of variables:

$$S = E\,e^x,\ -\infty < x < +\infty;\ \ t = T - \frac{2\tau}{\sigma^2},\ 0 \le \tau \le \frac{\sigma^2 T}{2}$$

First change of unknown:

$$C(t,S) \ = \ E\,v(\tau,x)$$

If $k = 2r/\sigma^2$ then v is the solution of problem:

Find $v = v(\tau, x)$ such that:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - k\,v$$
$$v(0,x) = \max\left(e^x - 1, 0\right)$$

(Linear convection-diffusion-reaction equation)

BS formula for European vanilla call options (II)

Second change of variable: Find a solution u in the form

$$v(\tau, x) = e^{(\alpha x + \beta \tau)}\, u(\tau, x),$$

of the pure diffusion equation. After some computations, the choice

$$\alpha = -\frac{k-1}{2},\ \beta = -\frac{(k+1)^2}{4} \Rightarrow v(\tau, x) = e^{-\frac{x}{2}(k-1) - \frac{\tau}{4}(k+1)^2}\, u(\tau, x)$$

implies that u verifies the following Cauchy problem for the heat equation

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(0, x) = \max\left(e^{\frac{x}{2}(k+1)} - e^{\frac{x}{2}(k-1)}, 0\right)$$

BS formula for European vanilla call options(III)

The solution of the Cauchy problem for the heat equation is:

$$u(\tau,x)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{+\infty}u(0,s)\,e^{-\frac{1}{4\tau}(x-s)^2}\,ds\;=$$

$$\Big(\text{using the change of variable } x'=(s-x)/\sqrt{2\tau}\Big)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}u(0,x+x'\sqrt{2\tau})\,e^{-\frac{x'^2}{2}}\,dx'\;=$$

$$\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{+\infty}e^{\frac{(k+1)}{2}(x+x'\sqrt{2\tau})-\frac{x'^2}{2}}\,dx'\;-$$

$$\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{+\infty}e^{\frac{(k-1)}{2}(x+x'\sqrt{2\tau})-\frac{x'^2}{2}}\,dx'\;=I_1-I_2.$$

Next, by using in the exponents the identity:

$$b^2-(a-b)^2=-a^2+2ab,\ \text{con } a=x',\ b=(k+1)\sqrt{2\tau}/2$$

we can get:

BS formula for European vanilla call options(IV)

$$I_1 = \frac{e^{\frac{x}{2}(k+1)}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{+\infty} e^{\frac{(k+1)^2}{4}\tau - \frac{1}{2}(x' - (k+1)\sqrt{2\tau}/2)^2}\, dx' =$$

$$\frac{e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{+\infty} e^{-\frac{1}{2}(x' - (k+1)\sqrt{2\tau}/2)^2}\, dx' =$$

$$\frac{e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}}{\sqrt{2\pi}} \int_{-d_1}^{+\infty} e^{-\frac{\rho^2}{2}}\, d\rho\ ,$$

with $\rho = x' - (k+1)\sqrt{2\tau}/2$ y $d_1 = x/\sqrt{2\tau} + (k+1)\sqrt{2\tau}/2$. Finally,

$$I_1 = \frac{e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{\rho^2}{2}}\, d\rho = e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}\ N(d_1)$$

Analogously, we have

$$I_2 = e^{\frac{x}{2}(k-1)+\frac{(k-1)^2}{4}\tau}\ N(d_2), \quad d_2 = x/\sqrt{2\tau} + (k-1)\sqrt{2\tau}/2$$

BS formula for European vanilla call options (V)

Therefore, we have: $v(\tau,x) = e^{-\frac{x}{2}(k-1)-\frac{\tau}{4}(k+1)^2}\,(I_1 - I_2) \;=$

$$e^{-\frac{x}{2}(k-1))-\frac{\tau}{4}(k+1)^2}\left(e^{\frac{x}{2}(k+1)+\frac{(k+1)^2}{4}\tau}\,N(d_1) - e^{\frac{x}{2}(k-1)+\frac{(k+1)^2}{4}\tau}\,N(d_2)\right) \;=$$

$$e^x\,N(d_1) \;-\; N(d_2).$$

Finally,

$$\begin{aligned} C(t,S) &= E\,v\left(\frac{\sigma^2}{2}(T-t), log(S/E)\right) \\ &= E\frac{S}{E}\,N(d_1) \;-\; Ee^{-k\frac{\sigma^2}{2}(T-t)}\,N(d_2) \\ &= S\,N(d_1) \;-\; E\,e^{-r(T-t)}\,N(d_2)\,, \quad \text{with} \end{aligned}$$

$$d_1 = x/\sqrt{2\tau} + (k+1)\sqrt{2\tau}/2 = \frac{\log(S/E) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = x/\sqrt{2\tau} + (k-1)\sqrt{2\tau}/2 = \frac{\log(S/E) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

BS formula for European vanilla call options (VI)

- Notice that boundary conditions at $S = 0$ and as $S \to \infty$ are not required
- Also, when using numerical methods, conditions at $C(t, 0)$ are not required
- Concerning the behavior when $S \to \infty$, when using the numerical methods a domain truncation technique will be used and appropriate boundary conditions are required at the new artificial boundary of the bounded domain

BS equation for European vanilla put options

Let $D = [0, T] \times [0, \infty)$, find the function $P = P(t, S)$ such that:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \text{ in } D$$

$$P(T, S) = \max(E - S, 0), \qquad S > 0$$

Exact solution (Black-Scholes formula for European vanilla put options):

$$P(t, S) = E\ \exp(-r(T-t))N(-d_2)\ -\ S\ N(-d_1)$$

Greeks or sensitivities (I)

Greeks measure the sensitivities of the options with respect to the variations in financial variables or parameters. They are very useful for quants and traders.

- Δ: measures the sensitivity with respect to the asset value. In case of call and put European vanilla options, they are respectively given by:

$$\Delta_C = \frac{\partial C}{\partial S} = N(d_1),\ \Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1$$

- Γ: measures the sensitivity of Δ with respect to the asset value:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$

- Θ: measures the sensitivity with respect to time:

$$\Theta = \frac{\partial V}{\partial t}$$

- Sensitivities with respect to other parameters, such as volatility (Vega), interest rate (Rho), etc can also be defined.

Computation of the Greeks

- Notice that when exact formulas for the option price are available then Greeks can be obtained by taking the appropriate derivatives, although sometimes results easier to use numerical derivation.
- For example, to approximate the value of Vega we can use:

$$\frac{\partial V}{\partial \sigma}(t,S) \approx \frac{V_{\sigma+d\sigma}(t,S) - V_{\sigma}(t,S)}{d\sigma},$$

where $d\sigma$ denote a small volatility increment

A European call option pricing

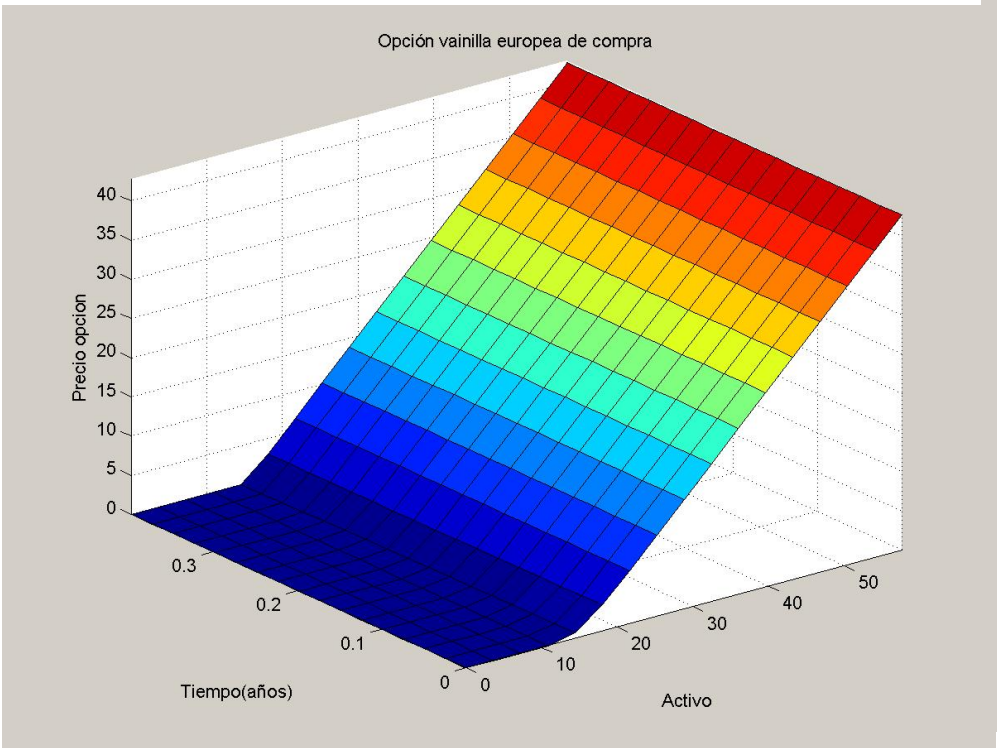
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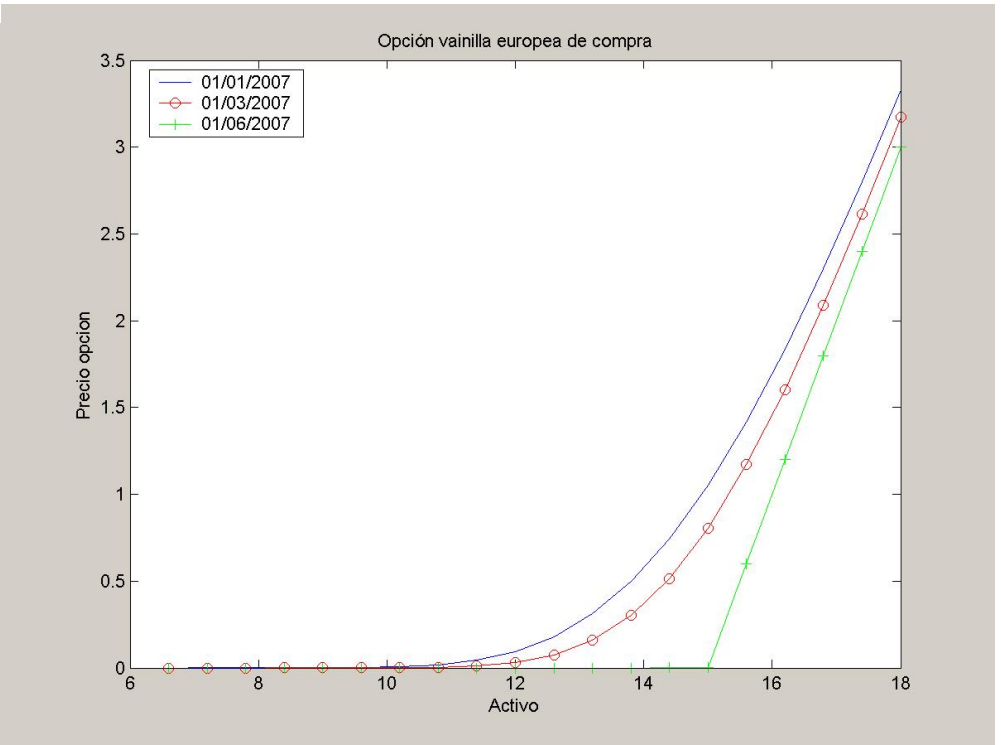
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Expiry date:	$21-03-2027$
Exercise price:	$15,00$
Interest rate:	3%
Asset volatility:	25%
Pricing date:	$03-11-2026$
Spot asset price:	$17,00$

Pricing and Greeks:

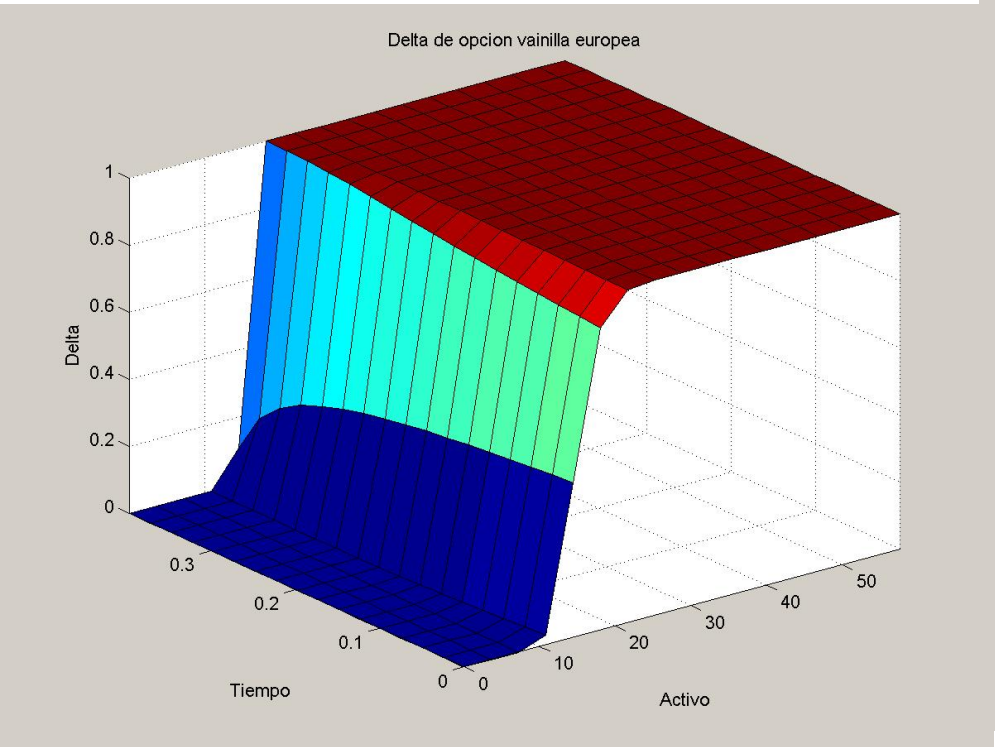
Option:	$2,3277$
Delta:	$0,8515$
Gamma:	$0,0988$
Theta:	$-1,2568$
Vega:	$2,1710$
Rho:	$3,6944$

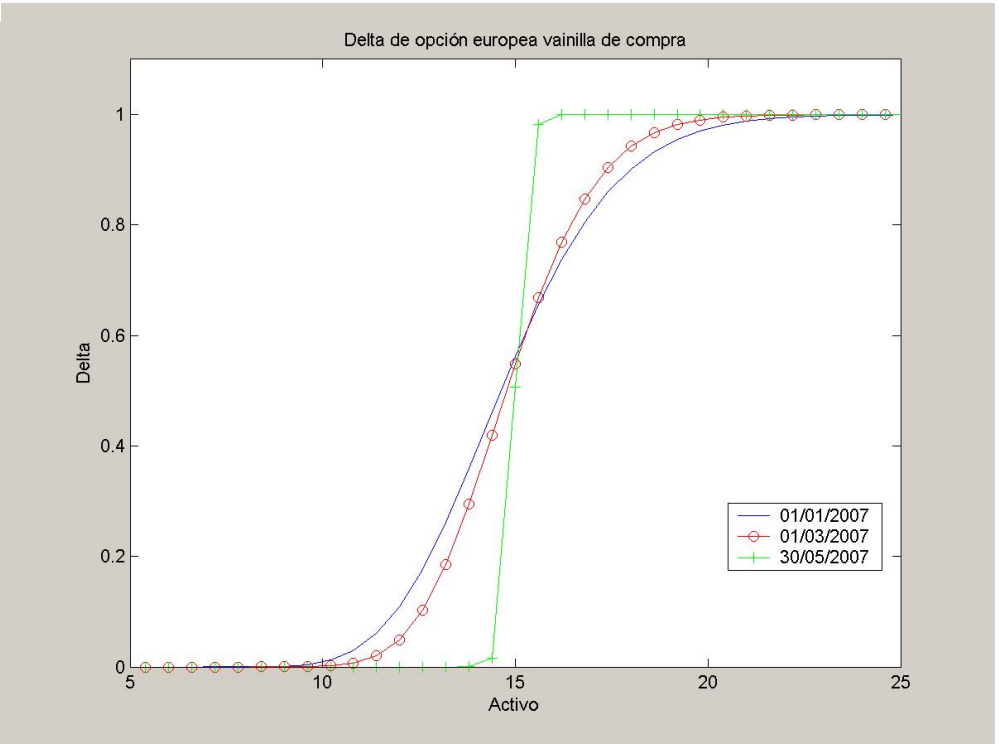
A European call option pricing





Delta of a European call option pricing





A European put option pricing

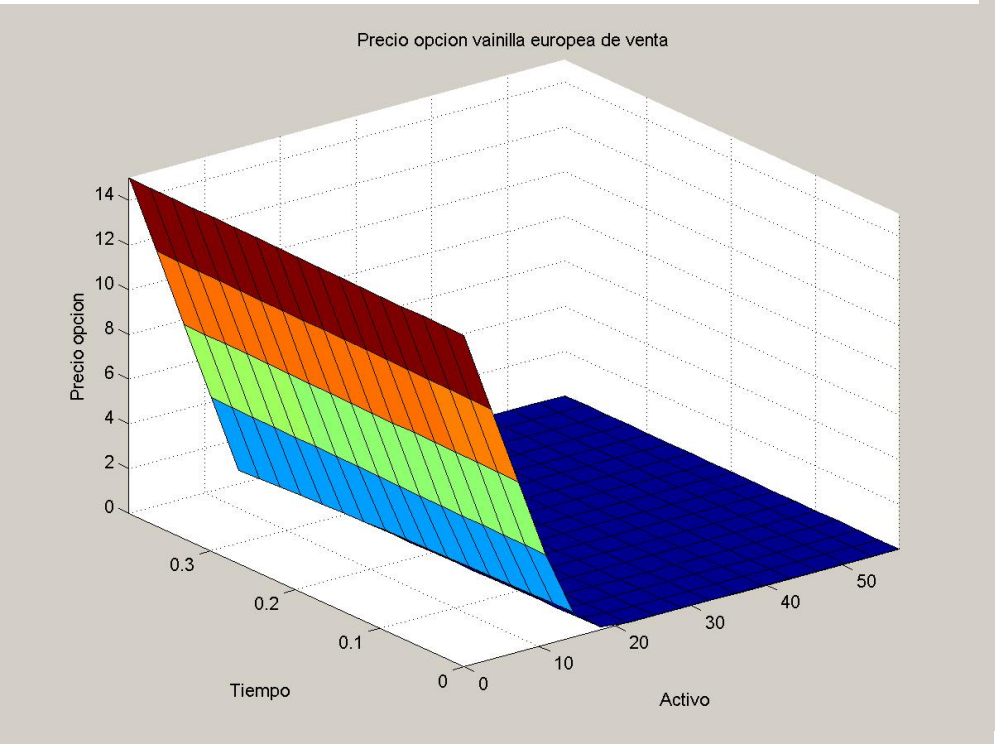
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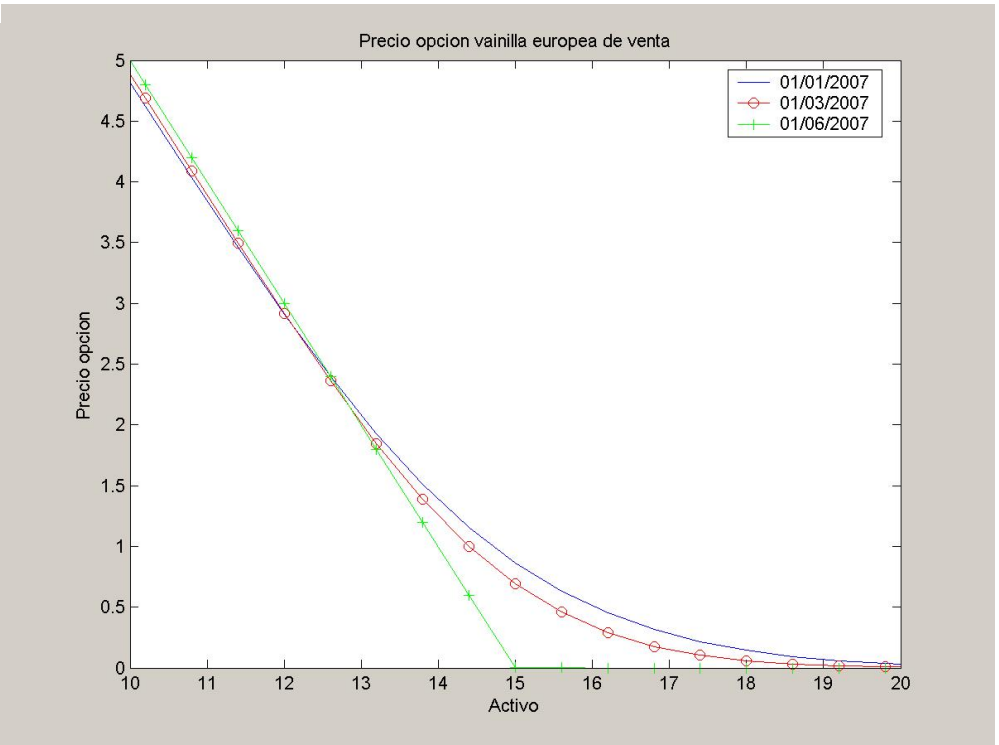
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Expiry date:	$21-03-2027$
Exercise price:	$15,00$
Interest rate:	3%
Asset volatility:	25%
Pricing date:	$03-11-2026$
Spot asset price:	$17,00$

Pricing and Greeks:

Option:	$0,1915$
Delta:	$-0,1485$
Gamma:	$0,0988$
Theta:	$-0,8109$
Vega:	$2,1710$
Rho:	$-0,8529$

A European put option pricing





American vanilla options

- Example in Spanish futures market (MEFF): Options on assets.
- They can be exercised at any time before expiry date or at expiry date.
- The objective is to determine the value of the option at any time before expiry date.
- The pricing in the frame of Black-Scholes models requires the use of numerical methods as no analytical formulas are available.

Dynamic hedging for American options

If $p_t = P(t, S_t)$ denotes the American put option price, then the no arbitrage hypothesis implies that

- $p_t \geq \max(E - S_t, 0)$
- $r\pi_t dt \geq \left(\dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P}{\partial S^2} \right)(t, S_t)\, dt$

Note that in American options case, the inequality

$$r\pi_t\, dt > \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \right)(t, S_t)\, dt$$

does not lead to an arbitrage opportunity, unlike the European case. This is related to the fact that the option buyer can decide to exercise it in the time interval $[t, t + dt]$, which implies risk in the strategy and avoids arbitrage.

Black-Scholes for American vanilla put options

Let $D = [0,T] \times [0,\infty)$, find the function $P = P(t,S)$ such that:

$$L(P) = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0 \text{ in } D$$

$$P(t,S) \geq \max(E-S,0)\,, \qquad \text{in } D$$

$$L(P)\cdot(P - \max(E-S,0)) = 0, \qquad \text{in } D$$

$$P(T,S) = \max(E-S,0), \qquad S > 0$$

$$P(t,S) \approx 0, \qquad t \in [0,T],\ S \to \infty$$

$$P(t,0) = E\,\exp(-r(T-t)), \qquad t \in [0,T]$$

American vanilla put options pricing

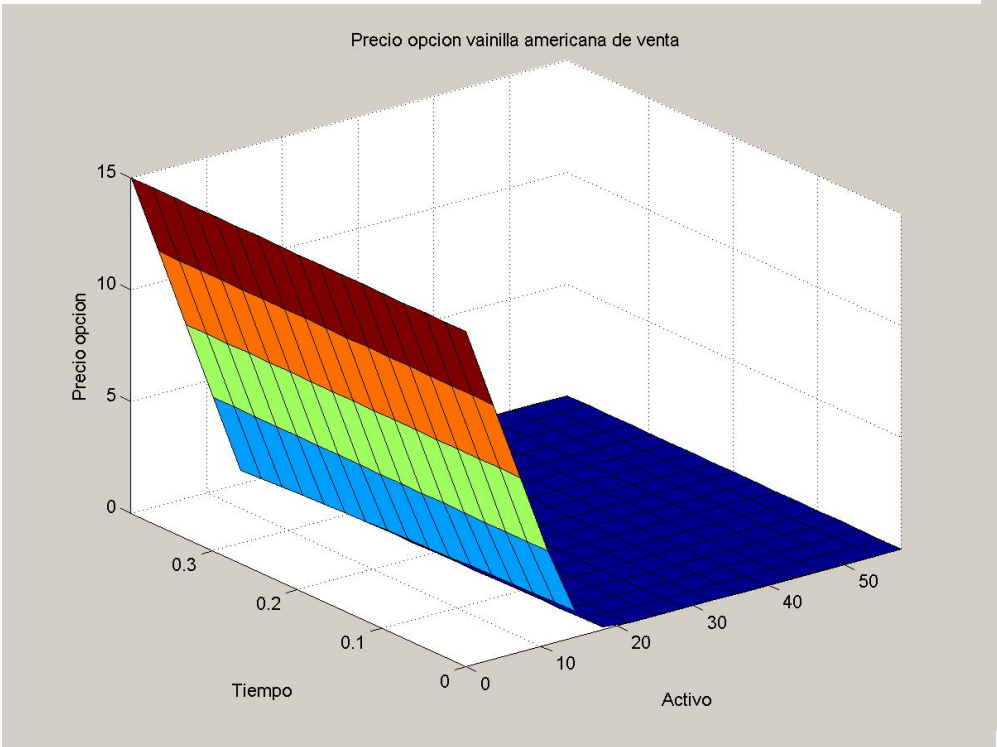
Option data:

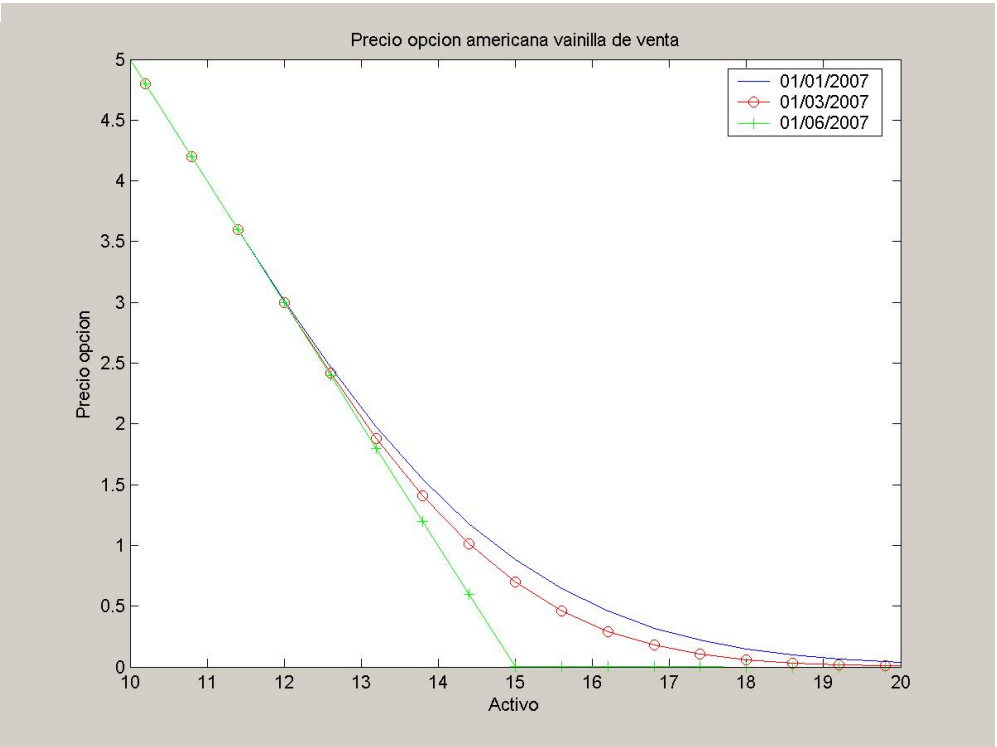
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Exercise price:	$15,00$
Interest rate:	3%
Asset volatility:	25%
Pricing date:	$03-11-2026$
Spot asset price:	$17,00$

Pricing and Greeks:

Option:	$0,1978$
Delta:	$-0,1534$
Gamma:	$0,1000$
Theta:	$-0,8305$
Vega:	$2,7161$
Rho:	$-0,6108$

American vanilla put options price





Free boundary problem for American vanilla put options

- Besides the option price at each time for each asset price, it is important to identify if it is better to maintain or exercise the American option.
- Mathematically, for each $t \in [0,T]$ two sets can be distinguished:
  - Exercise region (coincidence set): The option price coincides with the exercise price

$$\Omega^0(t) = \{S \in [0,\infty)/P(t,S) = max(E-S,0)\}$$

  - Non exercise region (non coincidence set):

$$\Omega^+(t) = \{S \in [0,\infty)/P(t,S) > max(E-S,0)\}$$

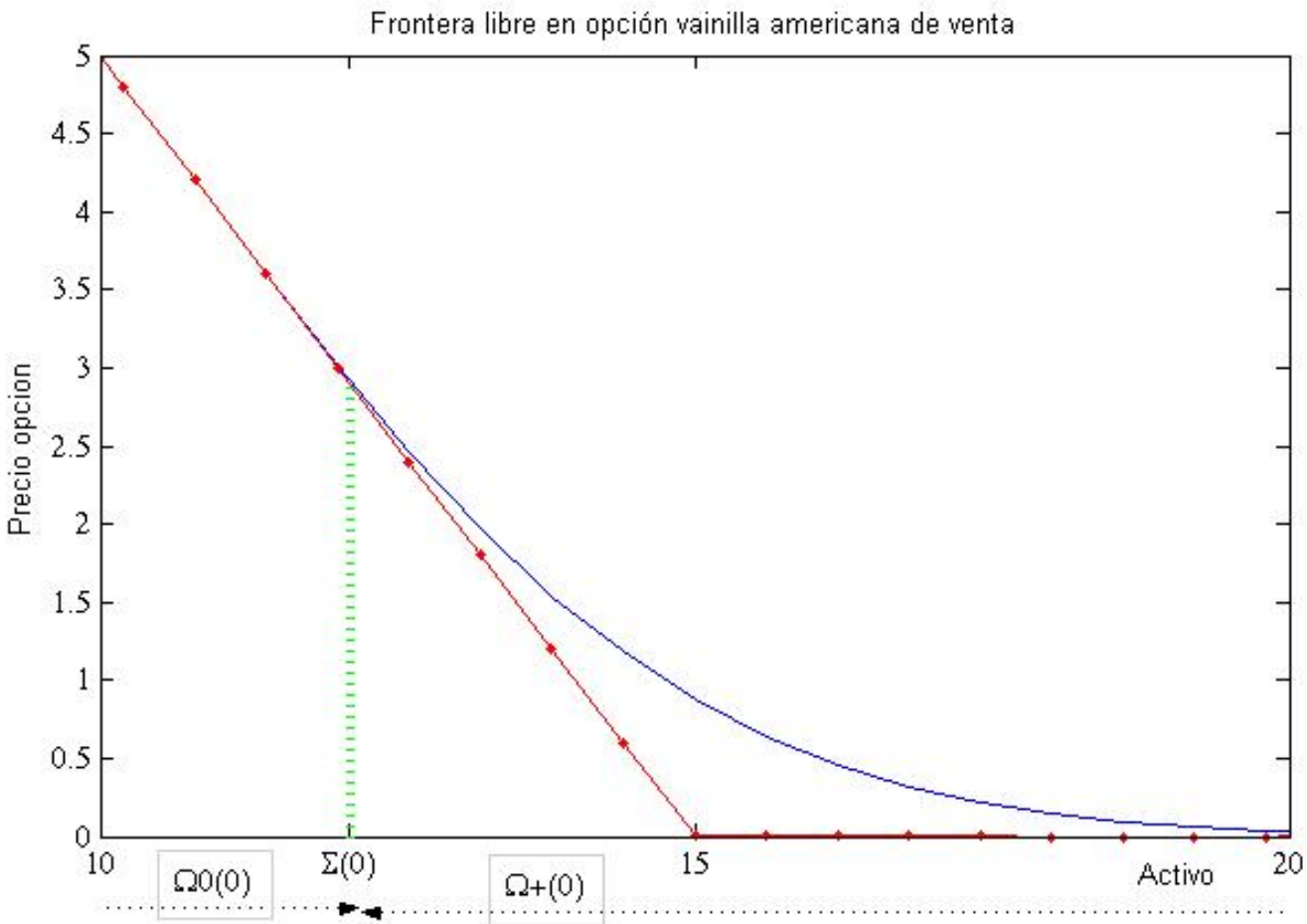
  - At each t, there exists an asset price separating both connexe regions (free boundary):

$$\Sigma(t) = \partial\Omega^+(t) \cap \partial\Omega^0(t)$$

- Moreover, the optimal exercise boundary (moving boundary) is defined by:

$$\Sigma = \{(t,\Sigma(t))/t \in [0,T]\}$$

Free boundary for an American put option



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