

Universidad de Sevilla

Doc-Course 2026

Topics in Inverse Problems

Otared Kavian

October 2026

3

Foreword

We gather here, in chapter 3, some notes about inverse spectral problems, corresponding to three lectures given during the week of October 5 to October 9, 2026 in « Doc-Course 2026 », a series of introductory and advanced doctoral courses on « PDEs : Theoretical and Numerical Analysis and Control », organized by the IMUS (Instituto de Matemáticas de la Universidad de Sevilla).

For the students' convenience, in order to have a minimal self-contained lecture notes, and allow them to get familiar with some of the tools used in the study of partial differential equations, and in particular the study of inverse problems, we added chapters 1 and 2 which have been essentially published in « Three Courses on Partial Differential Equations », pp. 125–162, IRMA Lect. Math. Theor. Phys., n° 4, de Gruyter, Berlin, 2003.

It is an immense pleasure for me to thank my friends, and nevertheless collegues, at the Universidad de Sevilla, in particular Enrique Fernández Cara, Anna Doubova and Diego Araújo de Souza, for having given me the opportunity to give these lectures.

Otared Kavian
Sevilla, October 12, 2026

3

Content

1. Introduction and Motivations 3

1.1 Examples and motivations 3

1.2 Some basic facts about linear elliptic equations 8

2. Identification of conductivities 23

2.1 A.P. Calderón's approach 23

2.2 Identification on the boundary 30

2.3 Identification of potentials in Schrödinger operators 39

2.4 Density of products of solutions 45

3. Inverse spectral problems 53

3.1 Sturm-Liouville problems 55

3.2 Higher Dimensions and Boundary Spectral Data 57

3.3 The case of Dirichlet Boundary Spectral Data 59

3.4 The case of Neumann Boundary Spectral Data 66

Bibliography 77

1

Introduction and Motivations

The terms *Inverse Problems* and *Parameter Identification* can be considered synonymous, although the latter is probably less absconse to the non specialist. Both refer to situations in which a certain quantity, represented by a *parameter* or a *coefficient*, is to be determined via an indirect measurement, that is via the measurement or determination of a function, parameter or coefficient, which depends on the desired parameter in a more or less complicated way.

1.1 Examples and motivations

A simple, elementary, example of an *inverse problem* is when one determines the height of an inaccessible tower via the measurement of various angles and sides of appropriate triangulations (elementary Euclidian geometry establishes relations between the length of the sides and angles of a triangle).

Another example, mathematically more involved, is when one knocks at different heights on a closed container, which is supposed to contain a liquid, in order to determine to what extent it is full or empty. In this case the frequencies of the resulting sound depend on whether at the height where one has knocked there is liquid or air, and in a way or another by hearing the main frequencies of the resulting sound one is able to guess the height of the liquid. This is related to inverse spectral problems, where one wishes to recover some informations about the coefficients of certain elliptic operators through informations on the eigenvalues and some other measurements on the boundary of the eigenfunctions.

We give below a few more examples in order to illustrate the kind of problems we shall be dealing with in these lecture notes.

1.1.1 Example. Following the presentation of A. Kirsch [18], a good example of an inverse problem which can be described, and solved, is the Abel's problem

(1823), in which one desires to determine the *shape* of the slope of a hill by letting a ball roll from a certain height h downhill until a certain point considered to be the origin, and then measure the time $T(h)$ spent by the ball on its trajectory.

In order to write a mathematical model of this problem, let

$$\Gamma := \left\{ (\psi(y), y) \ ; \ 0 \le y \le H \right\}$$

describe the *shape* of the slope, where we wish to determine the function ψ for all heights between zero and H. If the ball of mass m rolls, with initial velocity zero, from the point $P_1 = (\psi(h), h)$ at height $h \in (0, H)$ to the origin $P_0 = (0, 0)$ in the measured time $T(h)$, the principle of conservation of energy yields that the sum of the kinetic energy and the potential energy at each point $P = (\psi(y), y)$, for $y = y(t) \in (0, h)$ is constant and equals the potential energy at P_1 (here g is the acceleration of the gravity supposed to be constant) :

$$\frac{1}{2} m v^2 + m g y = m g h.$$

Denoting by s the curvilinear abscissa on the curve Γ defined by

$$ds = \sqrt{1 + (\psi'(y))^2}\, dy$$

this yields that

$$v(t) = \sqrt{2g(h - y(t))} = \frac{ds}{dt},$$

and finally

$$T(h) = \int_{\Gamma(P_0, P_1)} \frac{ds}{v} = \int_0^h \frac{\sqrt{1 + (\psi'(y))^2}}{\sqrt{2g(h - y)}}\, dy.$$

Therefore the problem can be stated as follows : the function $h \mapsto T(h)$ being known, determine a function ψ defined on $(0, h)$ such that

$$\forall h \in (0, H), \qquad \int_0^h \frac{\sqrt{1 + (\psi'(y))^2}}{\sqrt{h - y}}\, dy = \sqrt{2g}\, T(h). \tag{1.1.1}$$

This is a particular case of the so-called Abel's integral equations : setting

$$\varphi(y) := \sqrt{1 + (\psi'(y))^2} \quad \text{and} \quad f(h) := \sqrt{2g}\, T(h)$$

we have to find a positive function φ defined on $(0, H)$ such that for all $h \in (0, H)$ one has :

$$\int_0^h \frac{\varphi(y)\,dy}{\sqrt{h-y}} = f(h).$$

This is solved using Abel's inversion scheme, namely

$$\begin{aligned}\int_0^s \frac{f(h)\,dh}{\sqrt{s-h}} &= \int_0^s \int_0^h \frac{\varphi(y)\,dy\,dh}{\sqrt{h-y}\,\sqrt{s-h}} \\ &= \int_0^s \varphi(y) \left(\int_y^s \frac{dh}{\sqrt{h-y}\,\sqrt{s-h}} \right) dy = \pi \int_0^s \varphi(y)\,dy,\end{aligned}$$

and finally one obtains

$$\varphi(y) = \frac{1}{\pi} \frac{d}{dy} \int_0^y \frac{f(h)\,dh}{\sqrt{y-h}}.$$

In particular when $f(h) \equiv f_0$ is constant, that is when the time $T(h)$ is independent of the height h at which the ball has been dropped on the hill, one has $\varphi(y) = f_0/(\pi\sqrt{y})$ and the curve Γ is a cycloid : this is the reason for which cycloids are called *tautochrone*[1].

1.1.2 Example. Another physical example in which determination of coefficients of an elliptic operator is of importance, is the following. Consider a domain Ω representing an inhomogeneous conducting body, with electric conductivity $a(x)$ at $x \in \Omega$ (the *electric conductivity* is the inverse of the electric resistance). If one assumes that this body undergoes a known electric potential $\varphi(\sigma)$ at $\sigma \in \partial\Omega$, the boundary of Ω, then the electric potential $u(x)$ in Ω satisfies the elliptic equation :

$$\begin{cases} -\operatorname{div}(a(x)\,\nabla u(x)) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.1.2}$$

In this setting, the (local) intensity of the current is represented by

$$a(\sigma)\frac{\partial u(\sigma)}{\partial \boldsymbol{n}(\sigma)},$$

and the total electric power necessary to maintain the potential φ on the boundary $\partial\Omega$ will be

$$Q_a(\varphi) := \int_\Omega a(x)\,|\,\nabla u(x)\,|^2 dx.$$

[1] From the Greek *tauto* meaning *same*, and *khrônos* meaning *time*.

Now the question is the following : what can be deduced about the conductivity $a(x)$ of the body, if one knows the physical data φ, $a\,\partial u/\partial \boldsymbol{n}$ and $Q_{\boldsymbol{a}}(\varphi)$? This question arises for instance when one wishes to obtain some informations about the material constituents of the body, without *destructive testings*.

Note that, as a matter of fact, in the above problem the electric power $Q_{\boldsymbol{a}}(\varphi)$ is a redundant information, as it is entirely determined by the potential φ and the current $a\,\partial u/\partial \boldsymbol{n}$ on the boundary : indeed if one multiplies equation (1.1.2) by u, after an integration by parts (i.e. using Green's formula) one gets :

$$\int_{\Omega} a(x)\,|\,\nabla u(x)\,|^2 dx = \int_{\partial\Omega} a(\sigma)\,\frac{\partial u(\sigma)}{\partial \boldsymbol{n}(\sigma)}\,\varphi(\sigma)\,d\sigma.$$

Therefore we can restrict our attention to the problem in which a known potential φ and the induced current density $a\,\partial u/\partial \boldsymbol{n}$ on the boundary are given, and we wish to give some informations on the function a. By inspection of a few physical examples, one is convinced that knowledge of a single set of data, that is one given pair of potential and current, in general would not be enough information to determine $a(x)$ in Ω. Therefore, we may give a new version of the above question, which is mathematically more precise : can one determine uniquely the conductivity $a(x)$ if for all φ (in a certain functional space) one knows the current density $a\,\partial u/\partial \boldsymbol{n}$, where u satisfies (1.1.2) ? Indeed, if any such space of potentials φ exists, one may also ask what is the minimal space which permits this determination (note that since the mapping $\varphi \mapsto a\,\partial u/\partial \mathbf{n}$ is linear, we may restrict our attention to a "basis" $(\varphi_j)_{j\in J}$ of that space).

There exist numerous practical situations in which an identification problem may be reduced to the determination of the coefficients of an elliptic operator. For instance the case of a homogeneous body filling a domain Ω in which some small inhomogeneities located in $\omega \subset \Omega$ are included, can be modelled by saying that the conductivity a satisfies $a(x) \coloneqq a_0$, a known positive constant, when x is in the homogeneous part $\Omega \setminus \omega$ of the medium, and $a(x) \coloneqq a_1$, an unknown positive constant, when $x \in \omega$. In this case the mathematical problem would be the determination of the function a, which amounts to the determination of both the unknown positive constant a_1 *and* the position and shape of the subdomain ω containing the inhomogeneities.

These questions, aside being interesting in their own right, both from a mathematical point of view and in many physical applications, are of deep importance in many other problems (inverse scattering problems, inverse spectral problems,

thermography, tomography, etc.). Later on we shall give more details in this respect.

In the following sections we address the problem of determining the coefficient a, as well as some related questions regarding elliptic operators. □

1.1.3 Remark. When Ω is a bounded interval of $\mathbb{R}$, say $\Omega = (0, \ell)$, the above problem has a quite simple answer : it is not possible to determine the conductivity $a(x)$ by measuring potentials and currents at the endpoints of the interval. From a physical point of view this is due to the fact that different conductivities in a conducting rod of length ℓ, represented by the interval $(0, \ell)$, may lead to the same observed potential and current at the endpoints. The only information one may get is the average resistance, that is an equivalent homogeneous resistance one may put between the endpoints of the interval in order to obtain a given potential and current passing through the conductor. Indeed for $j = 0$ and $j = 1$ consider u_j solution of

$$-(a(x)\,u_j'(x))' = 0, \qquad u_j(0) = 1 - j, \qquad u_j(\ell) = j.$$

Then it is clear that if we denote by $M(a)$ the *harmonic mean value* of a, i.e.

$$M(a) := \left(\int_0^\ell \frac{ds}{a(s)} \right)^{-1},$$

then we have

$$u_0(x) = 1 - M(a) \int_0^x a(s)^{-1}\,ds, \qquad u_1(x) = M(a) \int_0^x a(s)^{-1}\,ds.$$

Now if one solves $-(a(x)\,u'(x))' = 0$ with $u(0) = \alpha$ and $u(\ell) = \beta$, one has $u(x) = \alpha u_0(x) + \beta u_1(x)$, and the intensity of the current at the endpoints will be given by

$$a(0)\,u'(0) = a(\ell)\,u'(\ell) = (\beta - \alpha)\,M(a).$$

From this it is clear that the only information we might obtain is the value of $M(a)$, provided one applies a potential at the endpoints such that $\beta - \alpha \neq 0$.

From a heuristic point of view, one can explain why it is not possible to determine the function $a(x)$ on $[0, \ell]$: indeed, the space of data at the extremities of the interval is a two dimensional linear space (one value at the origin, another one at ℓ), that is there are two degrees of freedom, while the set

of functions a is an infinite dimensional manifold (any positive function a such that $M(a) = a(0)\,u'(0)/(\beta - \alpha)$ solves the inverse problem). □

1.2 Some basic facts about linear elliptic equations

In this section we gather a few results concerning solutions of elliptic equations which should be well known before studying the related inverse problems.

In what follows we assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, and that $N \geq 2$; we shall denote the usual norm of $L^2(\Omega)$ by $\|\cdot\|_2$. For more details on any of the results sketched in this section the reader may refer to one of the classics on elliptic partial differential equations or D. Gilbarg-N.S. Trudinger [13], J.L. Lions [24], G. Stampacchia [29] or O.A. Ladyženskaja & N.N. Ural'ceva [21].

Thanks to the trace inequality (here we denote $\gamma_0(u) := u_{|\partial\Omega}$) :

$$\exists\, c > 0,\ \forall\, u \in C^1(\overline{\Omega}),\quad \int_{\partial\Omega} |\gamma_0(u)(\sigma)|^2 d\sigma \leq c(\|\nabla u\|_2^2 + \|u\|_2^2),$$

since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, the trace operator γ_0 may be extended to $H^1(\Omega)$ and by definition $H^{1/2}(\partial\Omega) := \gamma_0(H^1(\Omega))$. The Sobolev space $H^1(\Omega)$ will be equipped with the scalar product

$$(u|v)_{H^1} := \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_{\partial\Omega} u(\sigma)\,v(\sigma)\,d\sigma, \tag{1.2.1}$$

and, the associated norm $\|u\|_{H^1} := (u|u)_{H^1}^{1/2}$ is equivalent to

$$u \mapsto (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}$$

the usual norm of $H^1(\Omega)$.

In this setting $H_0^1(\Omega)$ is the kernel of γ_0 and the norm on $H_0^1(\Omega)$ will be $u \mapsto \|\nabla u\|_2$. The orthogonal space of $H_0^1(\Omega)$ in $H^1(\Omega)$, with respect to the scalar product (1.2.1), is the space of harmonic functions on Ω :

$$H_0(\Delta) := \left\{ v \in H^1(\Omega)\ ;\ \Delta v = 0 \ \text{ in } H^{-1}(\Omega) \right\}. \tag{1.2.2}$$

Since by definition $H^1(\Omega) = H_0(\Delta) \oplus H_0^1(\Omega)$, any $u \in H^1(\Omega)$ has the unique decomposition $u = \psi + u_0$ where $\psi \in H_0(\Delta)$ and $u_0 \in H_0^1(\Omega)$. This decomposition is characterized by the fact that ψ and u_0 satisfy respectively the equations

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \psi - u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta u_0 = -\Delta u & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular since $\|\psi\|_{H^1} \le \|u\|_{H^1}$, we may state the following result :

1.2.1 Lemma. *For any $\varphi \in H^{1/2}(\partial\Omega)$ there exists a unique $\psi \in H^1(\Omega)$ such that*

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \psi = \varphi & \text{on } \partial\Omega. \end{cases}$$

Moreover for some positive constants c_0, c_1 depending on Ω, we have

$$c_0\|\varphi\|_{H^{1/2}(\partial\Omega)} \le \|\psi\|_{H^1} \le c_1\|\varphi\|_{H^{1/2}(\partial\Omega)}.$$

We shall denote by $\mathcal{A}_{\mathrm{ad}}$ the set of admissible coefficients

(1.2.3)
$$\mathcal{A}_{\mathrm{ad}} := \{a \in L^\infty(\Omega)\ ;\ \exists \varepsilon_0 > 0 \text{ such that } a \ge \varepsilon_0 \text{ a.e. on } \Omega\}.$$

For $q \in L^\infty(\Omega)$ and $a \in \mathcal{A}_{\mathrm{ad}}$ we denote by $L_{a,q}$ the elliptic operator

(1.2.4)
$$L_{a,q}u := -\mathrm{div}(a\nabla u) + qu,$$

which, with the domain

(1.2.5)
$$D(L_{a,q}) := \left\{u \in H_0^1(\Omega)\ ;\ -\mathrm{div}(a\nabla u) + qu \in L^2(\Omega)\right\},$$

is a self-adjoint operator acting on $L^2(\Omega)$ (see for instance [17], chapitre 1, § 9). The operator $(L_{a,q}, D(L_{a,q}))$ has a compact resolvent, more precisely there exists $\lambda_* \in \mathbb{R}$ such that for all $\lambda > \lambda_*$ the resolvent operator

$$(\lambda I + L_{a,q})^{-1} : L^2(\Omega) \longrightarrow L^2(\Omega)$$

is compact and, as a matter of fact, the image $R((\lambda I + L_{a,q})^{-1})$ is precisely $D(L_{a,q})$ (for instance one may take $\lambda_* \in \mathbb{R}$ such that $q + \lambda_* \ge 1$ a.e. in Ω). As a consequence, $L_{a,q}$ has a nondecreasing sequence of eigenvalues $(\lambda_k)_{k\ge1}$, each being of finite multiplicity, and eigenfunctions $(\varphi_k)_{k\ge1}$:

$$L_{a,q}\varphi_k = \lambda_k\varphi_k, \qquad \varphi_k \in D(L_{a,q}), \quad \int_\Omega \varphi_k(x)\,\varphi_j(x)\,dx = \delta_k^j,$$

where δ_k^j stands for the Kronecker symbol. The first eigenvalue λ_1 is simple, $\lambda_1 < \lambda_2$, and one may choose the corresponding eigenfunction φ_1 to be positive in Ω, that is so that $\varphi_1 > 0$ in Ω and $\varphi_1 = 0$ on $\partial\Omega$. Also each eigenfunction φ_k is Lipschitz continuous in Ω, that is it belongs to $W^{1,\infty}(\Omega)$.

We shall denote by $N(L_{a,q})$ the kernel of the operator $(L_{a,q}, D(L_{a,q}))$ and

$$H_1 := \bigoplus_{\lambda_k<0} \mathbb{R}\varphi_k \qquad \text{and} \qquad H_2 := \bigoplus_{\lambda_k>0} \mathbb{R}\varphi_k,$$

so that the space $L^2(\Omega)$ has the orthogonal decomposition

$$L^2(\Omega) = H_1 \oplus N(L_{a,q}) \oplus H_2.$$

(Note that $N(L_{a,q}) \neq \{0\}$ only if zero is an eigenvalue). Any $u \in L^2(\Omega)$ has a unique decomposition $u = u_0 + u_1 + u_2$ with $u_j \in H_j$ for $j = 1, 2$ and $u_0 \in N(L_{a,q})$, which are orthogonal projections of u on the corresponding subspaces. In particular, since H_1 and $N(L_{a,q})$ are finite dimensional spaces spanned by eigenfunctions of $L_{a,q}$, one sees that $H_1 \oplus N(L_{a,q}) \subset D(L_{a,q})$ and so the projections u_0 and u_1 belong to $D(L_{a,q})$. Therefore when $u \in H_0^1(\Omega)$ the above decomposition yields that we have $u_2 \in H_0^1(\Omega)$.

Under the assumption that $N(L_{a,q}) = \{0\}$, for any given $f \in H^{-1}(\Omega)$, one can solve the problem

$$\begin{cases} -\operatorname{div}(a\,\nabla u) + qu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2.6}$$

by a variational method. Namely, for $v \in H_0^1(\Omega)$ setting

$$J(v) := \frac{1}{2}\int_\Omega a(x)\,|\nabla v(x)|^2 dx + \frac{1}{2}\int_\Omega q(x)\,|v(x)|^2 dx - \int_\Omega f(x)\,v(x)\,dx,$$

there exists a unique $u \in H_0^1(\Omega)$ such that $u = u_1 + u_2$, with $u_j \in H_j$ and

$$J(u) = \min_{v_2 \in H_2 \cap H_0^1(\Omega)} \max_{v_1 \in H_1} J(v_1 + v_2), \tag{1.2.7}$$

in other words $(u_1, u_2) \in H_1 \times (H_2 \cap H_0^1(\Omega))$ is the unique saddle point of the concave-convex function $(v_1, v_2) \mapsto J(v_1 + v_2)$ (see for instance [17], chapitre 3, § 4). Moreover if we denote by $B := B_{a,q} : H^{-1}(\Omega) \longrightarrow H_0^1(\Omega)$ the mapping $f \mapsto u$, we have

$$\|u\|_{H_0^1(\Omega)} = \|Bf\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha_0}\,\|f\|_{H^{-1}}, \qquad \text{where } \alpha_0 := \min\{|\lambda_k|\ ;\ k \geq 1\}.$$

In the particular case of an operator $L_{a,q}$ which is coercive, that is $\lambda_1 > 0$, the functional J is strictly convex on $H_0^1(\Omega)$ and the above resolution of equation (1.2.6) is equivalent to using Lax-Milgram theorem (see for instance J.L. Lions [24], J.L. Lions & E. Magenes [25], K. Yosida [31]).

When zero is an eigenvalue of $(L_{a,q}, D(L_{a,q}))$, that is $N(L_{a,q}) \neq \{0\}$, since $L_{a,q}$ is self-adjoint and $R(L_{a,q}) = N(L_{a,q})^{\perp} = H_1 \oplus H_2$, equation (1.2.6) has a solution if and only if f is orthogonal to $N(L_{a,q})$ (this is a consequence of the Fredholm alternative ; here $R(L_{a,q})$ denotes the range of the operator $L_{a,q}$). In this case, provided $f \in H_1 \oplus H_2 = R(L_{a,q})$, one may prove easily that (1.2.6) has a unique solution $u \in H_1 \oplus (H_2 \cap H_0^1(\Omega))$, and u is characterized by the equation (1.2.7). This means that $u \in N(L_{a,q})^{\perp}$ and that

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha_0} \|f\|_{H^{-1}}, \qquad \text{where } \alpha_0 := \min\{|\lambda_k| \,;\, k \geq 1,\ \lambda_k \neq 0\}.$$

(However note that $u + \varphi_0$ is a solution of (1.2.6) for any $\varphi_0 \in N(L_{a,q})$).

In order to solve more general non homogeneous boundary value problems, for instance equations such as (1.2.6) where the second condition is replaced with $\alpha u + \partial u/\partial n = \varphi$ on $\partial\Omega$, the following result concerning existence of a *normal trace* is extremely useful.

1.2.2 J.L. Lions Lemma. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. There exists a constant $c_1 > 0$ such that if $F \in (L^2(\Omega))^N$ satisfies $\operatorname{div}(F) \in L^2(\Omega)$, then $F \cdot n$ can be defined as an element of $H^{-1/2}(\partial\Omega)$ and :*

$$\|F \cdot n\|_{H^{-1/2}(\partial\Omega)} \leq c_1(\|F\|_2 + \|\operatorname{div}(F)\|_2).$$

Proof. Assume that $F \in W^{1,\infty}(\Omega)$. Then, using an integration by parts, for any $\psi \in H^1(\Omega)$ we have

$$\int_{\partial\Omega} \psi(\sigma)\, F(\sigma) \cdot n(\sigma)\, d\sigma = \int_{\Omega} F(x) \cdot \nabla\psi(x)\, dx + \int_{\Omega} \operatorname{div}(F(x))\, \psi(x)\, dx.$$

Therefore, denoting the left hand side of the above identity as $\langle F \cdot n, \gamma_0(\psi)\rangle$, with the brackets $\langle \cdot, \cdot \rangle$ meaning the duality between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ and $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ being the trace operator, we have :

$$|\langle F \cdot n, \gamma_0(\psi)\rangle| \leq \|F\|_2 \|\nabla\psi\|_2 + \|\operatorname{div}(F)\|_2 \|\psi\|_2.$$

For a given $\varphi \in H^{1/2}(\partial\Omega)$ we may choose $\psi \in H^1(\Omega)$ as in Lemma 1.2.1 such that $\psi = \varphi$ on $\partial\Omega$, which satisfies $\|\psi\|_{H^1} \leq c_1 \|\varphi\|_{H^{1/2}}$, for a constant c_1 depending only on Ω. Hence

$$|\langle F\cdot \boldsymbol{n}, \varphi\rangle| \leq c_1(\|F\|_2 + \|\mathrm{div}(F)\|_2)\|\varphi\|_{H^{1/2}}$$

This means that

$$\|F\cdot \boldsymbol{n}\|_{H^{-1/2}(\partial\Omega)} \leq c_1(\|F\|_2 + \|\mathrm{div}(F)\|_2)\,. \tag{1.2.8}$$

When F is no longer in $W^{1,\infty}(\Omega)$, using a density argument (see the classical books on Sobolev spaces, for instance R.A. Adams [1] or V.G. Maz'ya [27]) one can find a sequence $(F_k)_k$ of $W^{1,\infty}(\Omega)$ such that

$$F_k \to F \ \text{ in } \ (L^2(\Omega))^N, \quad \mathrm{div}(F_k) \to \mathrm{div}(F) \ \text{ in } \ L^2(\Omega)\,.$$

Inequality (1.2.8) shows that $(F_k\cdot \boldsymbol{n})_k$ is a Cauchy sequence in $H^{-1/2}(\partial\Omega)$ whose limit, which is independent of the particular choice of the sequence $(F_k)_k$, will be denoted by $F\cdot \boldsymbol{n}$. The lemma is proved. □

One of the consequences of this lemma is that whenever $u\in H^1(\Omega)$ satisfies

$$-\mathrm{div}(a\,\nabla u) + qu = 0 \ \text{ in } \ \Omega,$$

since $a\,\nabla u \in (L^2(\Omega))^N$ and $\mathrm{div}(a\,\nabla u)\in L^2(\Omega)$, we have

$$a\frac{\partial u}{\partial \boldsymbol{n}} \in H^{-1/2}(\partial\Omega)\,.$$

Hence, if we denote by $\varphi = u_{|\partial\Omega}$ the trace of u on the boundary $\partial\Omega$, we can consider the mapping

$$\varphi \mapsto a\frac{\partial u}{\partial \boldsymbol{n}}$$

as a linear continuous operator from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. This operator, denoted by $\Lambda_{a,q}$, is the *Poincaré-Steklov operator* which we will study in the next sections.

An important result in the study of inverse problems for partial differential equations is the following unique continuation principle. Let Ω be domain of $\mathbb{R}^N$ and $u\in C^2(\Omega)$ a function satisfying $\Delta u = 0$ in Ω. Then knowing that such a function is analytique in Ω, if $\omega\neq\emptyset$ is an open set contained in Ω and $u\equiv 0$ in ω, then we conclude that $u\equiv 0$. In the case of the Laplace operator, we know even more : if u vanishes at order infinity at $x_0\in\Omega$, that is if for any multi-index $\alpha\in\mathbb{N}^N$ we have $\partial^\alpha u(x_0)$, then $u\equiv 0$ in Ω. This property, called the *unique continuation principle* has a wide variety of flavors and it is very important to know for what kind of differential operators it holds.

In our study of inverse problems we shall use the following version of unique continuation which can be found for instance in V. Adolfsson & L. Escauriaza [2], L.V. Hörmander [15], Chapter 17, Theorem 17.2.6 (this latter book is a classical reference for this kind of results).

1.2.3 Theorem. *Let Ω be a domain of class C^2. The coefficients $a \in \mathcal{A}_{\mathbf{ad}} \cap W^{1,\infty}(\Omega)$ and $q \in L^\infty(\Omega)$ being given, consider a part $\Gamma_0 \subset \partial\Omega$ which has non empty relative interior with respect to $\partial\Omega$. If $u \in H^1(\Omega)$ satisfies*

$$\begin{cases} -\operatorname{div}(a\nabla u) + qu = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_0. \end{cases}$$

Then $u \equiv 0$ in Ω.

We consider next, for a given $\varphi \in H^{1/2}(\partial\Omega)$, the following equation

$$\begin{cases} -\operatorname{div}(a\nabla u) + qu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{1.2.9}$$

which is closely related to equation (1.2.6). Choosing a $\psi \in H^1(\Omega)$ such that $\psi = \varphi$ on $\partial\Omega$, for instance ψ given by Lemma 1.2.1, the above equation is reduced to (1.2.6), upon setting $u =: v + \psi$ and seeking $v \in H_0^1(\Omega)$ which solves

$$-\operatorname{div}(a\nabla v) + qv = \operatorname{div}(a\nabla\psi) - q\psi \text{ in } H^{-1}(\Omega), \quad v \in H_0^1(\Omega). \tag{1.2.10}$$

This equation has a unique solution when $N(L_{a,q}) = \{0\}$, for any given $\psi \in H^1(\Omega)$, and if we choose ψ given by Lemma 1.2.1, since we have $\|\psi\|_{H^1(\Omega)} \leq c\|\varphi\|_{H^{1/2}}$, we deduce that

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha_0}\,\|\operatorname{div}(a\nabla\psi) - q\psi\|_{H^{-1}} \leq \frac{c}{\alpha_0}\,\|\varphi\|_{H^{1/2}},$$

where we use the fact that $\|q\psi\|_{H^{-1}} \leq c\|q\psi\|_2 \leq c\|\psi\|_{H^1(\Omega)} \leq c\|\varphi\|_{H^{1/2}}$, and that

$$\begin{aligned} \|\operatorname{div}(a\nabla\psi)\|_{H^{-1}} &= \sup\Big\{ -\int_\Omega a\nabla\psi\cdot\nabla w\,dx\ ;\ w \in H_0^1(\Omega),\ \|w\|_{H_0^1(\Omega)} = 1 \Big\} \\ &\leq \|a\|_\infty \|\psi\|_{H^1(\Omega)} \leq c\|\varphi\|_{H^{1/2}}. \end{aligned}$$

When $N(L_{a,q}) \neq \{0\}$, according to our above analysis, equation (1.2.10) has a solution if and only if the right hand side is orthogonal to the kernel $N(L_{a,q})$, that is if for all $\varphi_0 \in N(L_{a,q})$ one has :

$$\int_{\partial\Omega} a(\sigma)\frac{\partial\varphi_0(\sigma)}{\partial n(\sigma)}\,\psi(\sigma)\,d\sigma = \int_{\Omega} a\,\nabla\psi.\nabla\varphi_0 dx + \int_{\Omega} q\psi\varphi_0 dx = 0.$$

Since the kernel $N(L_{a,q})$ is finite dimensional, this means that equation (1.2.9) has a unique solution if and only if $\varphi \in H^{1/2}(\partial\Omega)$ is orthogonal to the finite dimensional space

$$\text{span}\Big\{a\frac{\partial\varphi_0}{\partial n}\ ;\ \varphi_0 \in N(L_{a,q})\Big\}.$$

Actually this space and $N(L_{a,q})$ have the same dimension, at least when the coefficient a and the boundary $\partial\Omega$ are smooth enough so that, by the unique continuation theorem 1.2.3 one may assert that the conditions

$$-\text{div}(a\,\nabla w) + qw = 0, \quad \text{in } \Omega, \qquad \text{and } w = a\frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega$$

imply $w \equiv 0$.

To sum up the above remarks, we give the following :

1.2.4 Corollary. *Let $\Omega \subset \mathbb{R}^N$ a bounded Lipschitz domain, $a \in \mathcal{A}_{ad}$ and $q \in L^\infty(\Omega)$. Then there exist two positive constants c_1 and c_2 such that for any $\varphi \in H^{1/2}(\partial\Omega)$ satisfying*

(1.2.11)
$$\forall \psi \in N(L_{a,q}), \qquad \int_{\partial\Omega} a(\sigma)\frac{\partial\psi(\sigma)}{\partial n(\sigma)}\,\varphi(\sigma)\,d\sigma = 0,$$

equation (1.2.9) has a unique solution $u \in H^1(\Omega) \cap N(L^{\perp}_{a,q})$ which satisfies :

(1.2.12)
$$c_1\,\|\varphi\|_{H^{1/2}} \leq \|u\|_{H^1} \leq c_2\,\|\varphi\|_{H^{1/2}}.$$

1.2.5 Remark. Note that when $N(L_{a,q}) = \{0\}$, the orthogonality condition (1.2.11) is satisfied for all $\varphi \in H^{1/2}(\partial\Omega)$. □

We recall here the following local regularity result concerning solutions of elliptic boundary value problems (see for instance D. Gilbarg & N.S. Trudinger [13], G. Stampacchia [29]).

1.2.6 Proposition. *Let Ω be a Lipschitz domain, $\sigma_* \in \partial\Omega$ and $R_* > 0$ be such that $\partial\Omega \cap B(\sigma_*, R_*)$ is of class $C^{1,1}$, while $a \in \mathcal{A}_{ad} \cap W^{1,\infty}(B(\sigma_*, R_*) \cap \Omega)$.*

Assume that φ has its support in $B(\sigma_*, R_*) \cap \partial\Omega$ and $\varphi \in H^{3/2}(B(\sigma_*, R_*) \cap \partial\Omega)$. If $u \in H^1(\Omega)$ satisfies

$$\begin{cases} -\operatorname{div}(a\,\nabla u) + qu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$, then

$$u \in H^2(\Omega \cap B(\sigma_*, R_*)), \qquad a\frac{\partial u}{\partial \boldsymbol{n}} \in H^{1/2}(B(\sigma_*, R_*) \cap \partial\Omega).$$

Also there exists a positive constant c_1 depending only on Ω and R_* and on the coefficients a, q, such that

$$\|u\|_{H^2(\Omega \cap B(\sigma_*, R_*))} \le c_1(\|f\|_2 + \|\varphi\|_{H^{3/2}(\Omega \cap B(\sigma_*, R_*))}).$$

This regularity result may be used to prove existence of solutions to equations such as (1.2.9) when φ is not anymore in $H^{1/2}(\partial\Omega)$, for instance when $\varphi \in H^{-1/2}(\partial\Omega)$. Indeed in this case one has to interpret the equation not in $H^{-1}(\Omega)$, but rather in the distributions sense, because the solution u would not be anymore in $H^1(\Omega)$ and so the term $\operatorname{div}(a\,\nabla u)$ would make sense only in $\mathcal{D}'(\Omega)$.

For simplicity and clarity of exposition we shall consider only the case of two coefficients a, q such that $N(L_{a,q}) = \{0\}$, and we shall assume global regularity assumptions on Ω and a. However one may prove analoguous local results when zero is an eigenvalue of $(L_{a,q}, D(L_{a,q}))$ or when Ω and a have only local regularity properties. In the particular case $a \equiv 1$ and $q \equiv 0$, a different proof of the following lemma may be found in J.L. Lions [24], chapitre VII, § 5.

1.2.7 Lemma. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain, $a \in \mathcal{A}_{ad} \cap W^{1,\infty}(\Omega)$ and $q \in L^\infty(\Omega)$ be such that $N(L_{a,q}) = \{0\}$. Then for a given $\varphi \in H^{-1/2}(\partial\Omega)$, there exists a unique $u \in L^2(\Omega)$ such that $\operatorname{div}(a\,\nabla u) \in L^2(\Omega)$ and*

$$\begin{cases} -\operatorname{div}(a\,\nabla u) + qu = 0 & \text{in } L^2(\Omega) \\ u = \varphi & \text{in } H^{-1/2}(\partial\Omega) \end{cases}$$

Moreover there exist two positive constants c_1, c_2 depending only on Ω and the coefficients a, q such that

$$\|\operatorname{div}(a\,\nabla u)\|_2 + \|u\|_2 \le c_1\|\varphi\|_{H^{-1/2}(\partial\Omega)}, \tag{1.2.13}$$

and $a\,\partial u/\partial \boldsymbol{n} \in H^{-3/2}(\partial\Omega)$ satisfies

$$\left\| a \frac{\partial u}{\partial \boldsymbol{n}} \right\|_{H^{-3/2}(\partial\Omega)} \leq c_2 \|\varphi\|_{H^{-1/2}(\partial\Omega)}. \tag{1.2.14}$$

Proof. Assume first that $\varphi \in H^{1/2}(\partial\Omega)$. Then there exists a unique $u \in H^1(\Omega)$ such that $u = \varphi$ on the boundary $\partial\Omega$ and

$$-\operatorname{div}(a\,\nabla u) + qu = 0. \tag{1.2.15}$$

Now let $v \in H_0^1(\Omega)$ solve

$$-\operatorname{div}(a\,\nabla v) + qv = u \ \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega. \tag{1.2.16}$$

According to Proposition 1.2.6 we know that $v \in H^2(\Omega)$ and that

$$\left\| a \frac{\partial v}{\partial \boldsymbol{n}} \right\|_{H^{1/2}(\partial\Omega)} \leq c \|v\|_{H^2(\Omega)} \leq c \|u\|_2. \tag{1.2.17}$$

Upon multiplying equation (1.2.16) by u and integrating by parts we see that since :

$$\int_\Omega (a\,\nabla v \cdot \nabla u + quv)\,dx = 0,$$

(because $v \in H_0^1(\Omega)$ and u satisfies the homogeneous equation (1.2.15) in Ω), we have :

$$\begin{aligned}
\int_\Omega u(x)^2\,dx &= -\int_{\partial\Omega} a \frac{\partial v(\sigma)}{\partial \mathbf{n}(\sigma)} u(\sigma)\,d\sigma + \int_\Omega (a\,\nabla v \cdot \nabla u + quv)\,dx \\
&= -\int_{\partial\Omega} a \frac{\partial v(\sigma)}{\partial \boldsymbol{n}(\sigma)} \varphi(\sigma)\,d\sigma \\
&\leq \|\varphi\|_{H^{-1/2}(\partial\Omega)} \left\| a \frac{\partial v}{\partial \boldsymbol{n}} \right\|_{H^{1/2}(\partial\Omega)} \\
&\leq c \|\varphi\|_{H^{-1/2}(\partial\Omega)} \|u\|_2,
\end{aligned}$$

where in the last step we have used the estimate (1.2.17) . Therefore for any $\varphi \in H^{1/2}(\partial\Omega)$, if $u \in H^1(\Omega)$ solves (1.2.15) and $u = \varphi$ on $\partial\Omega$, we have $\|u\|_2 \leq c\|\varphi\|_{H^{-1/2}}$, and since $\operatorname{div}(a\,\nabla u) = qu$, the following estimate holds :

$$\|\operatorname{div}(a\,\nabla u)\|_2 + \|u\|_2 \leq c \|\varphi\|_{H^{-1/2}(\partial\Omega)}. \tag{1.2.18}$$

Also, in order to see that the *normal derivative* $a\,\partial u/\partial \boldsymbol{n}$ has a meaning in $H^{-3/2}(\partial\Omega)$ we proceed as follows. Let $\psi \in H^{3/2}(\partial\Omega)$ be given and consider $w \in H^2(\Omega)$ solution to

$$\begin{cases} -\operatorname{div}(a\,\nabla w) + qw = 0 & \text{in } \Omega \\ w = \psi & \text{on } \partial\Omega \end{cases} \tag{1.2.19}$$

Note that such a w exists in $H^1(\Omega)$ since $N(L_{a,q}) = \{0\}$. By the regularity result of Proposition 1.2.6 we have $w \in H^2(\Omega)$ and

$$\left\| a \frac{\partial w}{\partial n} \right\|_{H^{1/2}(\partial\Omega)} \leq c \|w\|_{H^2(\Omega)} \leq c \|\psi\|_{H^{3/2}(\partial\Omega)}.$$

Multiplying equation (1.2.19) by u and integrating by parts twice (or alternatively multiplying also equation (1.2.15) by w and then subtracting the resulted identities), one sees that

$$\begin{aligned} \int_{\partial\Omega} a \frac{\partial w(\sigma)}{\partial n(\sigma)} \varphi(\sigma)\, d\sigma &= \int_{\Omega} (a \nabla w \cdot \nabla u + q w u)\, dx \\ &= \int_{\partial\Omega} a \frac{\partial u(\sigma)}{\partial n(\sigma)} \psi(\sigma)\, d\sigma. \end{aligned}$$

Therefore

$$\begin{aligned} \left| \langle a \frac{\partial u}{\partial n}, \psi \rangle \right| &\leq \|\varphi\|_{H^{-1/2}(\partial\Omega)} \left\| a \frac{\partial w}{\partial n} \right\|_{H^{1/2}(\partial\Omega)} \\ &\leq c \|\varphi\|_{H^{-1/2}(\partial\Omega)} \|w\|_{H^2(\Omega)} \\ &\leq c \|\varphi\|_{H^{-1/2}(\partial\Omega)} \|\psi\|_{H^{3/2}(\partial\Omega)}. \end{aligned}$$

Finally from this we conclude that for any $\varphi \in H^{1/2}(\partial\Omega)$ the solution u of (1.2.15) satisfies

$$\left\| a \frac{\partial u}{\partial n} \right\|_{H^{-3/2}(\partial\Omega)} \leq c \|\varphi\|_{H^{-1/2}(\partial\Omega)}. \tag{1.2.20}$$

Now if we assume only $\varphi \in H^{-1/2}(\partial\Omega)$, since $H^{1/2}(\partial\Omega)$ is dense in $H^{-1/2}(\partial\Omega)$, a standard regularization procedure allows us to see that the estimates (1.2.18) and (1.2.20) go through and the Lemma is proved. □

1.2.8 Remark. One can also show that the reverse inequality

$$\|\varphi\|_{H^{-1/2}(\partial\Omega)} \leq c(\|\operatorname{div}(a \nabla u)\|_2 + \|u\|_2),$$

holds. It is important to observe that the estimate (1.2.14) cannot be replaced with an estimate of $a\, \partial u/\partial n$ in the norm of $H^{-1/2}(\partial\Omega)$, as it would be the case if we knew that $\nabla u \in (L^2(\Omega))^N$, upon using Lemma 1.2.2 : indeed here we have only $u \in L^2(\Omega)$ and $\operatorname{div}(a \nabla u) \in L^2(\Omega)$ but nothing is known about ∇u being possibly in a Lebesgue space. □

An important property of elliptic operators such as $L_{a,q}$ with Dirichlet boundary conditions is that their first eigenfunctions are positive functions.

1.2.9 Proposition. *Let Ω be a bounded domain of class C^2, the coefficients $a, \rho \in \mathcal{A}_{\mathbf{ad}}$ and $q \in L^\infty(\Omega)$ be given. Then there exist $\lambda_1 \in \mathbb{R}$ and $\varphi_1 \in H_0^1(\Omega)$ such that*

$$\text{(1.2.21)} \qquad \begin{cases} -\operatorname{div}(a\,\nabla\varphi_1) + q\varphi_1 = \lambda_1\rho\varphi_1 & \text{in } \Omega \\ \varphi_1 = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\Omega |\varphi_1(x)|^2\,dx = 1. \end{cases}$$

In particular $\varphi_1 \geq 0$, and if $a \in W^{1,\infty}(\Omega)$ we have $\varphi_1 > 0$ in Ω.

Proof. Indeed consider the functional $J : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$J(v) := \int_\Omega a(x)\,|\nabla v(x)|^2\,dx + \int_\Omega q(x)\,|v(x)|^2\,dx$$

for $v \in H_0^1(\Omega)$. Define also the set

$$S := \left\{ v \in H_0^1(\Omega)\ ;\ \int_\Omega |v(x)|^2\,\rho(x)\,dx = 1 \right\}.$$

Since the imbedding $H_0^1(\Omega) \subset L^2(\Omega)$ is compact, one checks easily that S is weakly sequentially compact, that is if $(v_n)_n$ is a sequence of elements in S that converges weakly to v in $H_0^1(\Omega)$ then $v \in S$. Note also that if $v \in S$ we have also $|v| \in S$.

Now consider the minimization problem

$$\lambda_1 = \inf\left\{ J(v)\ ;\ v \in S \right\}.$$

If $\varepsilon_0 > 0$ is such that $\min(a, \rho) \geq \varepsilon_0$ a.e. on Ω, then for $v \in S$

$$\int_\Omega |v(x)|^2\,dx \leq \frac{1}{\varepsilon_0}\int_\Omega |v(x)|^2\,\rho(x)\,dx = \frac{1}{\varepsilon_0},$$

and thus, writing $q = q^+ - q^-$ where $q^\pm = \max(\pm q, 0)$,

$$\int_\Omega q(x)\,|v(x)|^2\,dx \leq \int_\Omega q^+(x)\,|v(x)|^2\,dx \leq \frac{\|q^+\|_\infty}{\varepsilon_0}.$$

We conclude that

$$J(v) \geq \varepsilon_0\|\nabla v\|_{L^2}^2 - \frac{1}{\varepsilon_0}\|q^+\|_\infty \geq -\frac{1}{\varepsilon_0}\|q^+\|_\infty,$$

which shows that $\lambda_1 \in \mathbb{R}$ is finite and that any minimizing sequence $(v_n)_{n\geq 1}$ in S is bounded in $H_0^1(\Omega)$. Since $J(|v|) = J(v)$ we may assume that $v_n \geq 0$ and that $v_n \rightharpoonup \varphi_1$ weakly in $H_0^1(\Omega)$. Since $v_n \geq 0$ and $v_n \to \varphi_1$ strongly in $L^2(\Omega)$, we have

that $\varphi_1 \geq 0$ in Ω and $\varphi_1 \in S$. Also since $\int_\Omega q(x)\, v_n(x)^2\, dx \to \int_\Omega q(x)\, \varphi_1(x)^2\, dx$, we have

$$\begin{aligned}\int_\Omega a(x)\, |\nabla \varphi_1(x)|^2\, dx &\leq \liminf_{n\to\infty} \int_\Omega a(x)\, |\nabla v_n(x)|^2\, dx \\ &= \liminf_{n\to\infty} J(v_n) + \int_\Omega q(x)\, \varphi_1(x)^2\, dx \\ &= \lambda_1 + \int_\Omega q(x)\, \varphi_1(x)^2\, dx.\end{aligned}$$

This shows that $J(\varphi_1) \leq \lambda_1$ and since $\varphi_1 \in S$ we conclude that $J(\varphi_1) = \lambda_1$. Therefore by quadratic homogeneity of the functional J, and that of the mapping $v \mapsto \int_\Omega |v(x)|^2 \rho(x)\, dx$, we conclude that for all $v \in H_0^1(\Omega)$ we have

$$\lambda_1 \int_\Omega |v(x)|^2 \rho(x)\, dx \leq \int_\Omega a(x)\, |\nabla v(x)|^2\, dx + \int_\Omega q(x)\, |v(x)|^2\, dx.$$

Making the change $v = \varphi_1 + tu$ for $u \in H_0^1(\Omega)$ and $t \in \mathbb{R}$, after expanding the squared terms and using the facts that $\lambda_1 = J(\varphi_1)$ and $\int_\Omega |\varphi_1(x)|^2 \rho(x)\, dx = 1$, we end up with

$$\begin{aligned}2t\lambda_1 \int_\Omega \varphi_1(x)\, u(x)\, \rho(x)\, dx + t^2 \lambda_1 \int_\Omega |u(x)|^2 \rho(x)\, dx \leq \\ 2t \int_\Omega (a\, \nabla\varphi_1 \cdot \nabla u + q\varphi_1 u)\, dx \\ + t^2 \int_\Omega \left(a|\nabla u| + q|u|^2\right) dx.\end{aligned}$$

Dividing the above inequality by $2t > 0$ and then making $t \to 0^+$ we get

$$\lambda_1 \int_\Omega \varphi_1(x)\, u(x)\, \rho(x)\, dx \leq \int_\Omega (a\, \nabla\varphi_1 \cdot \nabla u + q\varphi_1 u)\, dx,$$

while dividing by $2t < 0$ and then making $t \to 0^-$ we obtain

$$\lambda_1 \int_\Omega \varphi_1(x)\, u(x)\, \rho(x)\, dx \geq \int_\Omega (a\, \nabla\varphi_1 \cdot \nabla u + q\varphi_1 u)\, dx.$$

This means that we have

$$\begin{aligned}&\varphi_1 \in H_0^1(\Omega), \quad \forall u \in H_0^1(\Omega), \\ &\lambda_1 \int_\Omega \varphi_1(x)\, u(x)\, \rho(x)\, dx = \int_\Omega (a\, \nabla\varphi_1 \cdot \nabla u + q\varphi_1 u)\, dx,\end{aligned}$$

which is the variational formulation of (1.2.21).

When $a \in W^{1,\infty}(\Omega)$, using the strong maximum principle (see for instance D. Gilbarg & N.S. Trudinger [13], Chapter 3) we have that $\varphi_1 > 0$ in Ω. □

When Ω and a are smooth enough, one can compare $\varphi_1(x)$ with $\mathrm{dist}(x,\partial\Omega)$, the distance of $x \in \Omega$ to the boundary $\partial\Omega$.

1.2.10 Proposition. *Assume that Ω is of class C^2 and that $a \in W^{2,\infty}(\Omega)$. Then there exist two constants $c_0, c_1 > 0$ such that*

$$c_0 \mathrm{dist}(x,\partial\Omega) \le \varphi_1(x) \le c_1 \mathrm{dist}(x,\partial\Omega)\,. \tag{1.2.22}$$

Proof. Considering the operator

$$u \mapsto \widetilde{L}u := -a\Delta u - \nabla a \cdot \nabla u + (q+\lambda)\,u$$

for $\lambda \in \mathbb{R}$ large enough to ensure that q + λ≥1 for instance, we know that $\widetilde{L}\varphi_1 = (\lambda+\lambda_1)\,\rho\varphi_1 > 0$ in Ω, and also $\varphi_1 \in C^1(\overline{\Omega})$. Then by using Lemma 3.4 in Chapter 3 of D. Gilbarg & N.S. Trudinger [13], we infer that for all $\sigma \in \partial\Omega$ we have

$$\lim_{t\to 0^+} \frac{\varphi_1(\sigma - t\boldsymbol{n}(\sigma)) - \varphi_1(\sigma)}{t} = -\frac{\partial\varphi_1(\sigma)}{\partial \boldsymbol{n}} > 0.$$

For $\varepsilon > 0$ small enough consider

$$\Omega_\varepsilon := \Big\{ x \in \Omega \;;\; \mathrm{dist}(x,\partial\Omega) < \varepsilon \Big\},$$

which is a neighbourhood of $\partial\Omega$. One can see that, since Ω sufficiently smooth, if ε is small enough then for $x \in \Omega_\varepsilon$ we have, for some $t_*(x) > 0$

$$\mathrm{dist}(x,\partial\Omega) = |x - t_*\sigma_*| = t_*(x)$$

for some $\sigma_* \in \partial\Omega$. Then setting

$$f(t) := \varphi_1(\sigma_* - t\boldsymbol{n}(\sigma_*))$$

we have $f'(0) = -\nabla\varphi_1(\sigma_*)\cdot\boldsymbol{n}(\sigma_*) = -\partial\varphi_1(\sigma_*)/\partial\boldsymbol{n}(\sigma_*) > 0$. Since $\varphi_1 \in C^1(\overline{\Omega})$, we may fix $\varepsilon > 0$, even smaller if necessary, such that for some $\alpha, \beta > 0$ and for all $x \in \Omega_\varepsilon$ and $0 \le t \le t_*(x)$ we have

$$0 < \alpha \le f'(t) \le \beta.$$

Now since

$$\varphi_1(x) = f(t_*) - f(0) = \int_0^{t_*} f'(t)\,dt,$$

we conclude that

$$\alpha\,\mathrm{dist}(x,\partial\Omega) = \alpha\, t_*(x) \le \varphi_1(x) \le \beta\, t_*(x) = \beta\,\mathrm{dist}(x,\partial\Omega)$$

for all $x \in \Omega_{\boldsymbol{\varepsilon}}$. Since $\varphi_{\mathbf{1}}$ and $x \mapsto \operatorname{dist}(x, \partial\Omega)$ do not vanish on $\Omega \setminus \Omega_{\boldsymbol{\varepsilon}}$ we may find two constants $\alpha', \beta' > 0$ so that

$$\forall x \in \Omega \setminus \Omega_{\boldsymbol{\varepsilon}}, \qquad \alpha' \leq \frac{\varphi_{\mathbf{1}}(x)}{\operatorname{dist}(x, \partial\Omega)} \leq \beta'.$$

Then choosing $c_{\mathbf{0}} \coloneqq \min(\alpha, \alpha')$ and $c_{\mathbf{1}} \coloneqq \max(\beta, \beta')$ we see that (1.2.22) is proved. □

26

26

27

2

IDENTIFICATION OF CONDUCTIVITIES

THIS chapter is dedicated to the determination of the electric conductivity of a body, through boundary measurements. This problem has a great many applications and is very interesting from a mathematical point of view.

2.1 A.P. CALDERÓN'S APPROACH

In this chapter we present the results of A.P. Calderón [10] which, despite being partial as far as the identification problem is concerned, contain some very interesting ideas, in particular the use of harmonic test functions of the type $u(x) := \exp(i\xi \cdot x + \eta \cdot x)$ where $\xi, \eta \in \mathbb{R}^N$ are such that $\xi \cdot \eta = 0$ and $|\xi| = |\eta|$ (see below section § 2.4).

We recall that by classical results on the resolution of elliptic boundary value problems (see section § 1.2) if $a \in \mathcal{A}_{ad}$, defined in (1.2.3), is given, then for each $\varphi \in H^{1/2}(\partial\Omega)$ there exists a unique $u \in H^1(\Omega)$ satisfying

$$\text{(2.1.1)} \qquad \begin{cases} -\operatorname{div}(a(x)\,\nabla u(x)) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

More precisely u may be characterized by the fact that it is the unique point in the convex set

$$\text{(2.1.2)} \qquad K_{\varphi} := \left\{ v \in H^1(\Omega)\ ;\ v = \varphi \ \text{ on } \ \partial\Omega \right\}$$

at which the strictly convex functional

$$\text{(2.1.3)} \qquad J(v) := J_a(v) := \int_{\Omega} a(x)\,|\nabla v(x)|^2 dx$$

achieves its minimum on K_{φ}. We will write

$$u = \operatorname*{Arg\,Min}_{v \in K_{\varphi}} J_a(v)$$

27

to mean that $u \in K_{\varphi}$ and for all $v \in K_{\varphi}$ we have $J(u) \le J(v)$. As the mapping $\varphi \mapsto u$ is linear continuous from $H^{1/2}(\partial\Omega)$ into $H^1(\Omega)$, it is a matter of simple algebra to check that $\varphi \mapsto J_a(u)$ defines a continuous quadratic form on $H^{1/2}(\partial\Omega)$ with values in $\mathbb{R}_+$. We will denote this quadratic form by

(2.1.4)
$$Q_a(\varphi) := \int_\Omega a(x)\,|\nabla u(x)|^2 dx, \quad \text{where } u \text{ satisfies (1.1.2).}$$

The *direct problem* would be : knowing the coefficient $a \in \mathcal{A}_{ad}$ study the mapping $a \mapsto Q_a(\varphi)$. It is clear that the *inverse problem*, namely trying to determine a assuming that the mapping $a \mapsto Q_a(\varphi)$ is known, would be better understood if one knows a good deal about the direct problem. In this regard our first observation is that, as one might expect, the mapping $(a, \varphi) \mapsto u$ is continuous, in fact analytic, on $\mathcal{A}_{ad} \times H^{1/2}(\partial\Omega)$.

2.1.1 Lemma. *For $(a, \varphi) \in \mathcal{A}_{ad} \times H^{1/2}(\partial\Omega)$ define $T(a, \varphi) := u$ where u satisfies (1.1.2). Then T is analytic on $\mathcal{A}_{ad} \times H^{1/2}(\partial\Omega)$.*

Proof. Let $(a_0, \varphi_0) \in \mathcal{A}_{ad} \times H^{1/2}(\partial\Omega)$ and assume that for some $\varepsilon_0 > 0$ we have $a_0 \ge \varepsilon_0$ a.e. We shall denote by $B := B_{a_0}$ the continuous operator defined for $f \in H^{-1}(\Omega)$ by $Bf := w$ where $w \in H_0^1(\Omega)$ satisfies $-\mathrm{div}(a_0 \nabla w) = f$. Let $a = a_0 + b \in \mathcal{A}_{ad}$ where $b \in L^\infty(\Omega)$ is such that $\|b\|_\infty < \varepsilon_0$. If for $w \in H^1(\Omega)$ we set

$$A_b w := \mathrm{div}(b \nabla w),$$

then we have $\|A_b w\|_{H^{-1}} \le \|b\|_\infty \|\nabla w\|_2$. Indeed

$$\begin{aligned}
\|A_b w\|_{H^{-1}} &= \sup\left\{ \langle A_b w, \psi\rangle \,;\ \psi \in H_0^1(\Omega),\ \|\psi\|_{H_0^1(\Omega)} = 1 \right\} \\
&= \sup\left\{ \int_\Omega b(x)\,\nabla w(x)\cdot\nabla\psi(x)\,dx \,;\ \psi \in H_0^1(\Omega),\ \|\psi\|_{H_0^1} = 1 \right\} \\
&\le \|b\|_\infty \|\nabla w\|_2.
\end{aligned}$$

In particular, when $w \in H_0^1(\Omega)$, using the norm $w \mapsto \|\nabla w\|_2$ on $H_0^1(\Omega)$, we infer that BA_b is a bounded operator from $H_0^1(\Omega)$ into itself and that $\|BA_b\| \le \|B\|\,\|b\|_\infty$. Now denote $u_0 := T(a_0, \varphi_0)$ and $u := T(a, \varphi_0)$. It is clear that $v := u - u_0$ belongs to $H_0^1(\Omega)$ and satisfies (recall that $\mathrm{div}(a_0 \nabla u_0) = 0$)

$$-\mathrm{div}(a_0 \nabla v) - \mathrm{div}(b \nabla v) = \mathrm{div}(b \nabla u_0) = A_b u_0,$$

which, after applying B to both sides, can be written $(I - BA_b)\,v = BA_b u_0$. If we assume that $\|a - a_0\|_\infty = \|b\|_\infty < \min(\|B\|^{-1}, \varepsilon_0) = \varepsilon_0$ (as we shall see below), then $\|BA_b\| < 1$ and $(I - BA_b)$ is invertible so we can write

$$v = (I - BA_b)^{-1} BA_b u_0 = \sum_{n=1}^{\infty} (BA_b)^n u_0,$$

and therefore

$$T(a, \varphi_0) = T(a_0, \varphi_0) + \sum_{n=1}^{\infty} (BA_{a-a_0})^n T(a_0, \varphi_0)\,.$$

Remarking that $\varphi \mapsto T(a, \varphi)$ is linear, from this we conclude that T is analytic at $(a_0, \varphi_0) \in \mathcal{A}_{\mathrm{ad}} \times H^{1/2}(\partial\Omega)$. □

2.1.2 Remark. We point out the following estimates concerning the dependance of u on the coefficient a. First observe that if we denote by ε_0 the essential infimum of $a_0 \in \mathcal{A}_{\mathrm{ad}}$, that is $\varepsilon_0 := \sup\{\varepsilon > 0\ ;\ \mathrm{meas}[a_0 < \varepsilon] = 0\}$, then $\|B_{a_0}\| \le \varepsilon_0^{-1}$. Indeed for $f \in H^{-1}(\Omega)$ and $B_{a_0} f := w \in H_0^1(\Omega)$ satisfying $-\mathrm{div}(a_0 \nabla w) = f$, we have

$$\varepsilon_0 \|Bf\|^2_{H_0^1(\Omega)} \le \int_\Omega a_0(x)\,|\nabla w(x)|^2 dx = \langle f, w\rangle \le \|f\|_{H^{-1}} \|Bf\|_{H_0^1(\Omega)}.$$

In particular $\|B\| \le \varepsilon_0^{-1}$ and so $\min(\|B\|^{-1}, \varepsilon_0) = \varepsilon_0$. Also if $\|a - a_0\|_\infty < \varepsilon_0$, then $a \in \mathcal{A}_{\mathrm{ad}}$ and, since $\|BA_{a-a_0}\|_{\mathcal{L}(H_0^1(\Omega))} \le \|a - a_0\|_\infty$, the above analysis shows that

(2.1.5)
$$\|T(a, \varphi_0) - T(a_0, \varphi_0)\|_{H_0^1(\Omega)} \le \frac{\|a - a_0\|_\infty}{\varepsilon_0 - \|a - a_0\|_\infty} \|T(a_0, \varphi_0)\|_{H_0^1(\Omega)}.$$

This inequality may be interpreted as the expression of the fact that if we set

$$\delta := \frac{\|T(a, \varphi_0) - T(a_0, \varphi_0)\|_{H_0^1(\Omega)}}{\|T(a_0, \varphi_0)\|_{H_0^1(\Omega)}} \qquad \text{then} \qquad \|a - a_0\|_\infty \ge \frac{\varepsilon_0 \delta}{1 + \delta}.$$

In order to obtain a *stability* result concerning the inverse problem, one should have a reverse inequality of the type $\|a - a_0\|_\infty \le CF(\delta)$ for some continuous function F such that $F(0) = 0$. We shall see that although such results may be proved in some cases, in general their proof is much more involved : actually this difficulty is typical of inverse and parameter identification problems (see for instance G. Alessandrini [3], [4]). □

2.1.3 Lemma. *Let $(a_0, \varphi) \in \mathcal{A}_{\mathbf{ad}} \times H^{1/2}(\partial\Omega)$ and $u_0 := T(a_0, \varphi)$. Then for any $b_0 \in L^\infty(\Omega)$ we have*

$$\lim_{t \to 0^+} \frac{Q_{a_0 + t b_0}(\varphi) - Q_{a_0}(\varphi)}{t} = \int_\Omega b_0(x) \, |\nabla u_0(x)|^2 dx.$$

Proof. Setting $b := t b_0$ and $a := a_0 + t b_0$, for t small enough the assumptions in the proof of Lemma 2.1.1 are satisfied and using the notations used there, we have

$$\begin{aligned} Q_a(\varphi) = Q_{a_0}(\varphi) &+ t \int_\Omega b_0(x) \, |\nabla u_0(x)|^2 dx \\ &+ 2 \int_\Omega a_0(x) \nabla u_0(x) \cdot \nabla v(x) \, dx \\ &+ 2 \int_\Omega b(x) \nabla u_0(x) \cdot \nabla v(x) \, dx \\ &+ \int_\Omega (a_0(x) + b(x)) \, |\nabla v(x)|^2 dx. \end{aligned}$$

Now, due to the fact that $\operatorname{div}(a_0 \nabla u_0) = 0$, multiplying this equation by $v \in H_0^1(\Omega)$ and integrating by parts, we remark that $\int_\Omega a_0(x) \nabla u_0(x) \cdot \nabla v(x) dx = 0$. On the other hand thanks to the estimate (2.1.5) we point out that $\|v\|_{H_0^1} \leq c\|b\|_\infty = O(t)$ and therefore

$$2 \int_\Omega b(x) \nabla u_0(x) \cdot \nabla v(x) \, dx + \int_\Omega (a_0(x) + b(x)) \, |\nabla v(x)|^2 dx = O(t^2).$$

Therefore

$$Q_{a_0 + t b_0}(\varphi) = Q_{a_0}(\varphi) + t \int_\Omega b_0(x) \, |\nabla u_0(x)|^2 dx + O(t^2),$$

and the proof of the Lemma is complete. □

2.1.4 Proposition. *For $a \in \mathcal{A}_{\mathbf{ad}}$ denote by $\Phi(a) := Q_a$ the quadratic form on $H^{1/2}(\partial\Omega)$ associated to a via (2.1.4). Then Φ is analytic on $\mathcal{A}_{\mathbf{ad}}$. In particular for $a_0 \in \mathcal{A}_{\mathbf{ad}}$ and $b_0 \in L^\infty(\Omega)$ the derivative of Φ at a_0 is given by $\Phi'(a_0) b_0 = q$ where the quadratic form q is given by*

$$q(\varphi) := \int_\Omega b_0(x) \, |\nabla u_0(x)|^2 dx,$$

where $u_0 \in H^1(\Omega)$ satisfies $\operatorname{div}(a_0 \nabla u_0) = 0$ in Ω and $u_0 = \varphi$ on $\partial\Omega$. If $a_0 \in \mathcal{A}_{\mathbf{ad}}$ is constant, then Φ' is injective at a_0, that is if $b_0 \in L^\infty(\Omega)$ is such that $\Phi'(a_0) b_0 = 0$ then $b_0 \equiv 0$.

(Here the space of quadratic forms on $H^{1/2}(\partial\Omega)$ is endowed with its natural underlying norm topology : if q is a continuous quadratic form on $H^{1/2}(\partial\Omega)$ then for some unique continuous self-adjoint operator Λ defined on $H^{1/2}(\partial\Omega)$ one has $q(\varphi) = (\Lambda\varphi|\varphi)_{H^{1/2}}$, and the norm of q is given by $\|\Lambda\|_{\mathcal{L}(H^{1/2}(\partial\Omega))}$).

Proof. The fact that Φ is differentiable is a consequence of the fact that by Lemma 2.1.3 the mapping $a_0 \mapsto Q_{a_0}$ is Gâteaux differentiable and its Gâteaux differential, namely $b \mapsto q$, is continuous. The fact that Φ is analytic follows from Lemma 2.1.1 and the expression of $\Phi'(a_0)$.

Now assume that a_0 is constant, so that $\operatorname{div}(a_0 \nabla u) = 0$ means $\Delta u = 0$. Recall that we have denoted (see section § 1.2 and (1.2.2)) $H_0(\Delta)$ the space of harmonic functions :

$$H_0(\Delta) := \left\{ v \in H^1(\Omega) \; ; \; \Delta v = 0 \right\},$$

which is actually the orthogonal space to $H_0^1(\Omega)$ in $H^1(\Omega)$. Note also that since a_0 is constant, by Lemma 2.1.3 we have :

$$(\Phi'(a_0)\, b_0)\,(\varphi) = \int_\Omega b_0(x)\, |\, \nabla u_0(x)\, |^2 dx,$$

for $u_0 \in H_0(\Delta)$ such that $u_0 = \varphi$ on $\partial\Omega$. Therefore if $b_0 \in L^\infty(\Omega)$ is such that for all $u_0 \in H_0(\Delta)$ one has $\int_\Omega b_0(x)\, |\, \nabla u_0(x)\, |^2 dx = 0$, then from this we conclude that

(2.1.6)

$$\text{for all } \; u_1, u_2 \in H_0(\Delta), \qquad \int_\Omega b_0(x)\, \nabla u_1(x) \cdot \nabla u_2(x)\, dx = 0.$$

(This is the *polarization* of the quadratic form q). Now consider the special functions u_1, u_2 defined to be

(2.1.7)

$$u_1(x) := \exp\left((\mathrm{i} \cdot \xi + \eta) \cdot x \right), \qquad u_2(x) := \exp\left((\mathrm{i}\xi - \eta) \cdot x \right),$$

where η, ξ are such that

(2.1.8)
$$\xi, \eta \in \mathbb{R}^N, \qquad \xi \cdot \eta = 0, \qquad |\xi| = |\eta| \neq 0.$$

(Recall that since $N \geq 2$, for any $\xi \neq 0$ there exists some $\eta \in \mathbb{R}^N$ satisfying the above conditions). One checks easily that thanks to (2.1.8) the functions u_j are harmonic in Ω and thus belong to $H_0(\Delta)$. However, since

$$\nabla u_1(x) \cdot \nabla u_2(x) = -2|\xi|^2 \exp(2\mathrm{i}\xi \cdot x),$$

identity (2.1.6) implies that

$$\forall \xi \in \mathbb{R}^N, \qquad \int_\Omega b_0(x)\, e^{2i\xi\cdot x} dx = 0 = \int_{\mathbb{R}^N} (1_\Omega(x)\, b_0(x))\, e^{2i\xi\cdot x} dx.$$

This means that the Fourier transform of $1_\Omega b_0$, namely $\mathcal{F}(1_\Omega b_0)$, is zero, which in turn, using the injectivity of the Fourier transform, yields $b_0 \equiv 0$. □

2.1.5 Remark. If we could prove that $\Phi'(a_0)$ is a homeomorphism, then the implicit functions theorem would yield that in a neighbourhood $B(a_0, \varepsilon_1)$ of a_0 the inverse problem can be solved : for $a \in B(a_0, \varepsilon_1)$ knowledge of Q_a would allow us to recover a. Unfortuantely at this point we are not able to prove such a result concerning $\Phi'(a_0)$... □

2.1.6 Remark. The above analysis contains the ingredients of an approximation process to recover a conductivity a, which is a small perturbation of a constant, when the quadratic form Q_a is known. Indeed let $a_0 \equiv 1$ (for instance) and assume that $a = 1 + b \in \mathcal{A}_{ad}$ is *close* to a_0. For $j = 1, 2$ and given $\varphi_j \in H^{1/2}(\partial\Omega)$ let $u_j, u_{0j} \in H^1(\Omega)$ satisfy

$$\operatorname{div}(a\nabla u_j) = \operatorname{div}(a_0 \nabla u_{0j}) = \Delta u_{0j} = 0 \text{ in } \Omega,$$
$$u_j = u_{0j} = \varphi_j \text{ on } \partial\Omega.$$

Since Q_a is known on $H^{1/2}(\partial\Omega)$, so is the bilinear form

$$M(\varphi_1, \varphi_2) := \frac{1}{2}\left[Q_a(\varphi_1 + \varphi_2) - Q_a(\varphi_1) - Q_a(\varphi_2)\right].$$

If we denote $v_j := u_j - u_{0j}$, as we did previously, we have $\int_\Omega \nabla u_{0j} \cdot \nabla v_k dx = 0$ since $v_k \in H_0^1(\Omega)$, and therefore

$$M(\varphi_1, \varphi_2) = \int_\Omega a(x)\, \nabla u_{01}(x) \cdot \nabla u_{02}(x)\, dx$$
$$+ \int_\Omega \left[(a - a_0)\,(\nabla u_{01} \cdot \nabla v_2 + \nabla u_{02} \cdot \nabla v_1) + a\, \nabla v_1 \cdot \nabla v_2\right] dx. \tag{2.1.9}$$

Now choose u_{0j} as in (2.1.7) with ξ, η satisfying (2.1.8) (this means that φ_j is the trace of u_{0j} on $\partial\Omega$). Then, assuming that $\Omega \subset B(0, R)$ for some $R > 0$, we obtain from (2.1.9)

$$2|\xi|^2 \hat{a}(2\xi) = -(2\pi)^{-N/2} M(\varphi_1, \varphi_2) + h(2\xi), \tag{2.1.10}$$

with

$$h(2\xi) := (2\pi)^{-N/2} \int_{\Omega} \Big[(a - a_0) (\nabla u_{01} \cdot \nabla v_2 + \nabla u_{02} \cdot \nabla v_1) + a \nabla v_1 \cdot \nabla v_2 \Big] dx.$$

Here we take as the difinition of the Fourier transform

$$\hat{f}(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} f(x) \exp(-ix \cdot \xi)\, dx.$$

Note also that as a matter of fact we should have written $\mathcal{F}(1_{\Omega} a)(2\xi)$ instead of $\hat{a}(2\xi)$, but this is not misleading in our analysis. The *error term* $h(2\xi)$ may be estimated by

$$|h(2\xi)| \le c_1 |\xi|^2 \|b\|_\infty^2 \, e^{2R|\xi|},$$

since $\|\nabla v_j\|_2 \le c\|b\|_\infty \|\nabla u_{0,j}\|_2 = c\|b\|_\infty |\xi| e^{R|\xi|}$, provided $\|b\|_\infty$ is small enough (actually if $\|b\|_\infty < 1$) to ensure that $a = a_0 + b \in \mathcal{A}_{ad}$ (see the proof of Lemma 2.1.1).

If F is such that

$$\hat{F}(2\xi) := \frac{-(2\pi)^{-N/2} M(\varphi_1, \varphi_2)}{2|\xi|^2},$$

then we have $\hat{a}(2\xi) = \hat{F}(2\xi) + (2|\xi|^2)^{-1} h(2\xi)$ and finally

$$\hat{a}(\xi) = \hat{F}(\xi) + \frac{2h(\xi)}{|\xi|^2}. \tag{2.1.11}$$

Now consider $\rho(x) := (2\pi)^{-N/2} \exp(-|x|^2/2)$ and $\rho_\sigma(x) := \sigma^N \rho(\sigma x)$, so that convolution with ρ_σ yields an approximation of the identity when σ is large. Since $\widehat{\rho_\sigma}(x) = \hat{\rho}(\xi/\sigma)$, and

$$\hat{a}(\xi)\hat{\rho}(\xi/\sigma) = \hat{F}(\xi)\hat{\rho}(\xi/\sigma) + \frac{2h(\xi)}{|\xi|^2}\hat{\rho}(\xi/\sigma),$$

we have

$$a * \rho_\sigma = F * \rho_\sigma + r,$$

where the remainder r is defined via $\hat{r}(\xi) := 2h(\xi)\hat{\rho}(\xi/\sigma)/|\xi|^2$. Since $\|r\|_\infty \le (2\pi)^{-N/2} \|\hat{r}\|_1$, it follows that

$$\|r\|_\infty \le c_2 \|b\|_\infty^2 \int_{\mathbb{R}^N} |\hat{\rho}(\xi/\sigma)| \exp(R|\xi|)\, d\xi = c_3 \|b\|_\infty^2 \sigma^N \exp(\sigma^2 R^2/2).$$

Finally we see that for a given large $\sigma > 0$, when $\|b\|_\infty$ is small enough, then $F * \rho_\sigma$ is a *reasonable* approximation of $a * \rho_\sigma$, which in turn, since σ is large, is a *good* approximation of a. □

2.2 Identification on the boundary

This section is devoted to the recovery of the coefficient on the boundary. To be more specific let us introduce the following notations. For $a \in \mathcal{A}_{ad}$ (which is defined in (1.2.3)) and $\varphi \in H^{1/2}(\partial\Omega)$ we denote by

$$\Lambda_a(\varphi) := a\frac{\partial u}{\partial n}, \tag{2.2.1}$$

where $u \in H^1(\Omega)$ satisfies (1.1.2). The operator Λ_a, called the *Dirichlet to Neumann map* by some authors, and the *Poincaré-Steklov operator* by others, is a self-adjoint operator mapping $H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$. Indeed if $-\mathrm{div}(a\nabla u_j) = 0$ in Ω and $u_j = \varphi_j$ on the boundary for $j = 1, 2$, then multiplying by u_2 the equation satisfied by u_1, and vice versa, after integrating by parts we get

$$\langle \Lambda_a(\varphi_1), \varphi_2\rangle = \int_\Omega a(x)\nabla u_1(x)\cdot\nabla u_2(x)\,dx$$
$$\langle \Lambda_a(\varphi_2), \varphi_1\rangle = \int_\Omega a(x)\nabla u_2(x)\cdot\nabla u_1(x)\,dx,$$

that is $\langle \Lambda_a(\varphi_1), \varphi_2\rangle = \langle \Lambda_a(\varphi_2), \varphi_1\rangle$ for all $\varphi_1, \varphi_2 \in H^{1/2}(\partial\Omega)$. This implies that Λ_a is a continuous self-adjoint operator mapping $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$. Note also that Q_a being defined in (2.1.4), we have for all $\varphi \in H^{1/2}(\partial\Omega)$

$$Q_a(\varphi) = \langle \Lambda_a(\varphi), \varphi\rangle. \tag{2.2.2}$$

In this section we will show that when $a_j \in \mathcal{A}_{ad}$ is of class C^1 near the boundary $\partial\Omega$ for $j = 0, 1$, and if one has

$$\Lambda_{a_0} = \Lambda_{a_1}, \tag{2.2.3}$$

then $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on the boundary. This result is due to R.V. Kohn & M. Vogelius [19] who prove actually much more : assuming that a_0 and a_1 are of class C^k near the boundary for some $k \geq 1$, then $\partial^\alpha a_0 = \partial^\alpha a_1$ on the boundary for $|\alpha| \leq k$, provided Ω has enough regularity. In particular, assuming that

the boundary and the coefficients a_0, a_1 are real analytic, their result implies that $a_0 = a_1$ on $\overline{\Omega}$.

The idea of the proof is the following : assuming that (2.2.3) holds, but for instance $a_0(\sigma_0) > a_1(\sigma_0)$ for some $\sigma_0 \in \partial\Omega$, one considers appropriately chosen electric potentials ψ_m having an $H^{1/2}(\partial\Omega)$ norm equal to 1 and small support around σ_0, and such that $\|\psi_m\|_{H^{-1/2}} = O(m^{-1})$. Then if

$$\begin{cases} -\operatorname{div}(a_j \nabla u_{jm}) = 0 & \text{in } \Omega \\ u_{jm} = \psi_m & \text{on } \partial\Omega, \end{cases}$$

after a minute analysis of the behaviour of u_{jm} as $m \to +\infty$, one shows that for large enough m's one has $Q_{a_0}(\psi_m) > Q_{a_1}(\psi_m)$.

We begin by the construction of the appropriate test functions $(\psi_m)_{m\geq 1}$.

2.2.1 Lemma. *Let Ω be a bounded domain and $\sigma_0 \in \partial\Omega$ be given ; assume also that Ω is $C^{1,1}$ in a neighbourhood of σ_0. Then there exists a sequence of functions $(\psi_m)_{m\geq 1}$ such that $\psi_m \in H^{3/2}(\partial\Omega) \cap C_c^1(\partial\Omega)$ and for some positive constants c_1, c_2 :*

(2.2.4)

(i) $\operatorname{supp}(\psi_{m+1}) \subset \operatorname{supp}(\psi_m)$ *and* $\cap_{m\geq 1} \operatorname{supp}(\psi_m) = \{\sigma_0\}$

(ii) $\|\psi_m\|_{H^{1/2}} = 1$

(iii) $\dfrac{c_1}{m^{(1+2s)/2}} \leq \|\psi_m\|_{H^{-s}(\partial\Omega)} \leq \dfrac{c_2}{m^{(1+2s)/2}}$ *for* $-1 \leq s \leq 1$.

Proof. Via a local diffeomorphism we may assume that $\partial\Omega \subset \mathbb{R}^{N-1} \times \{0\}$ and that σ_0 is the origin. Consider a function $f_* \in C_c^\infty(\mathbb{R})$ such that f_* is not identically zero and

$$\operatorname{supp}(f_*) \subset [-1, +1], \qquad \int_{-1}^{+1} f_*(t)\, dt = 0. \tag{2.2.5}$$

Now for $x' := (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$ and integers $m \geq 1$ define

$$f_m(x') := \prod_{i=1}^{N-1} f_*(m x_i).$$

Here for a function f defined on $\mathbb{R}^{N-1}$ we denote by

$$\hat{f}(\xi') := (2\pi)^{-(N-1)/2} \int_{\mathbb{R}^{N-1}} f(x')\, e^{-ix'\cdot\xi'} dx'$$

its Fourier transform on $\mathbb{R}^{N-1}$. We see that $\mathrm{supp}(f_m)$ is contained in $[-1/m, 1/m]^{N-1}$, and since $f_m(x') = f_1(mx')$, we have $\hat{f}_m(\xi') = m^{-(N-1)}\hat{f}_1(\xi'/m)$ and therefore for any $s \geq 0$

(2.2.6)
$$\begin{aligned}\|f_m\|^2_{H^{-s}(\mathbb{R}^{N-1})} &= \int_{\mathbb{R}^{N-1}} (1+|\xi'|^2)^{-s} |\hat{f}_m(\xi')|^2 d\xi' \\ &= \frac{1}{m^{N-1}} \int_{\mathbb{R}^{N-1}} (1+m^2|\xi'|^2)^{-s} |\hat{f}_1(\xi')|^2 d\xi'.\end{aligned}$$

Note that since $(1+m^2\theta^2)^{-1} \geq m^{-2}(1+\theta^2)^{-1}$ for $\theta \geq 0$, we have

(2.2.7)
$$\int_{\mathbb{R}^{N-1}} (1+m^2|\xi'|^2)^{-s} |\hat{f}_1(\xi')|^2 d\theta \geq \frac{c_3}{m^{2s}}.$$

On the other hand the mean value of f_* being zero, we have $\hat{f}_1(0) = 0$. Whence using the fact that $\hat{f}_1$ is analytic we have $|\cdot|^{-1}\hat{f}_1 \in S(\mathbb{R}^{N-1})$ (remember that f_1 has a compact support), and recalling the inequality $(1+m^2\theta^2)^{-1} \leq \theta^{-2}m^{-2}$ for $\theta > 0$, we obtain the upper bound for $0 \leq s \leq 1$

(2.2.8)
$$\int_{\mathbb{R}^{N-1}} (1+m^2|\xi'|^2)^{-s} |\hat{f}_1(\xi')|^2 d\xi' \leq \frac{1}{m^{2s}} \int_{\mathbb{R}^{N-1}} \frac{|\hat{f}_1(\xi')|^2 d\xi'}{|\xi'|^{2s}} = \frac{c_4}{m^{2s}}.$$

This, together with (2.2.7), plugged into (2.2.6) yield that for $0 \leq s \leq 1$ we have

(2.2.9)
$$c_5(s)\, m^{\frac{-(N-1)}{2}-s} \leq \|f_m\|_{H^{-s}(\mathbb{R}^{N-1})} \leq c_6(s)\, m^{\frac{-(N-1)}{2}-s}.$$

The same line of arguments shows that for any $s \geq 0$ we have

(2.2.10)
$$c_1(s)\, m^{\frac{-(N-1)}{2}+s} \leq \|f_m\|_{H^s(\mathbb{R}^{N-1})} \leq c_2(s)\, m^{\frac{-(N-1)}{2}+s}.$$

Finally from (2.2.10) and (2.2.9) we obtain for $s \geq -1$:

$$c_7(s)\, m^{\frac{-(N-1)}{2}+s} \leq \|f_m\|_{H^s(\mathbb{R}^{N-1})} \leq c_8(s)\, m^{\frac{-(N-1)}{2}+s},$$

and upon setting

$$\psi_m(x') := \frac{f_m(x')}{\|f_m\|_{H^{1/2}(\mathbb{R}^{N-1})}},$$

we conclude that for $-1 \leq s \leq 1$ we have

$$c_9(s)\, m^{-(1-2s)/2} \leq \|\psi_m\|_{H^s(\mathbb{R}^{N-1})} \leq c_{10}(s)\, m^{-(1-2s)/2},$$

and that the sequence $(\psi_m)_{m\geq 1}$ satisfies the conditions of the Lemma, at least when the boundary $\partial\Omega$ is a portion of $\mathbb{R}^{N-1} \times \{0\}$. The general case is

treated via a local diffeomorphism Φ and the resulting sequence $(\psi_m)_{m\geq 1}$ will inherit the regularity of Φ that is, with our assumptions, each ψ_m will be in $C^{1,1}(\partial\Omega)\cap H^{3/2}(\partial\Omega)$. □

In what follows, $\sigma_0\in\partial\Omega$ and $(\psi_m)_{m\geq 1}$ being as in Lemma 2.2.1 and $a\in L^\infty(\Omega)$ so that $a\geq\varepsilon_0$ for some $\varepsilon_0>0$, we shall denote by $u_m\in H^1(\Omega)$ the solution of

$$\text{(2.2.11)}\qquad \begin{cases} -\operatorname{div}(a\,\nabla u_m)=0 & \text{in } \Omega\\ u_m=\psi_m & \text{on } \partial\Omega. \end{cases}$$

Our purpose in the following lemma is to show that, in some sense, u_m concentrates around σ_0, more precisely that, far from σ_0, the H^1-norm of u_m is small.

2.2.2 Lemma. *Assume that Ω is a bounded $C^{1,1}$ domain and that the coefficient a in (2.2.11) is in $W^{1,\infty}(B(\sigma_0,R)\cap\Omega)$. Then if $\omega:=\Omega\setminus\overline{B}(\sigma_0,R)$, there exist $m_0\geq 1$ and a constant $c>0$ such that for all $m\geq m_0$ one has*

$$\text{(2.2.12)}\qquad \|u_m\|_{H^1(\omega)}\leq\frac{c}{m}\,.$$

Proof. First we notice that according to Lemma 1.2.7, the solution u_m of (2.2.11) satisfies

$$\text{(2.2.13)}\qquad \|u_m\|\leq c\|\psi_m\|_{H^{-1/2}(\partial\Omega)}\leq\frac{c}{m}\,.$$

Next, we may take $m_0\geq 1$ large enough so that the support of ψ_m is contained in $B(\sigma_0,R/8)\cap\partial\Omega$ for all $m\geq m_0$. Then we set $\omega_0:=\Omega\setminus\overline{B}(\sigma_0,R/4)$ and choose a cut-off function $\zeta\in C_c^\infty(\mathbb{R}^N)$ such that $0\leq\zeta\leq 1$, and

$$\zeta\equiv 1 \text{ in } \omega,\qquad \zeta\equiv 0 \text{ in } B(\sigma_0,R/2)\,.$$

Now consider $v_m:=\zeta u_m$ and observe that $v_m=0$ on $\partial\omega_0$ and, since $\operatorname{div}(a\,\nabla u_m)=0$,

$$-\operatorname{div}(a\,\nabla v_m)=-a\,\nabla\zeta\cdot\nabla u_m-\operatorname{div}(a u_m\,\nabla\zeta)\,.$$

Multiplying this equation by $v_m=\zeta u_m$, and integrating by parts over ω_0, we get

$$\int_{\omega_0} a(x)\,|\nabla v_m(x)|^2 dx = -\int_{\omega_0} a(\nabla\zeta\cdot\nabla u_m)\,u_m\zeta dx + \int_{\omega_0} a u_m\,\nabla\zeta\cdot\nabla(\zeta u_m)\,dx = \int_{\omega_0} a(x)\,u_m(x)^2\,|\nabla\zeta(x)|^2 dx,$$

and finally, observing that $u_m = v_m$ on $\omega \subset \omega_0$,

$$\int_{\omega} |\nabla u_m(x)|^2 dx \le c\int_{\Omega} |u_m(x)|^2\,|\nabla\zeta(x)|^2 dx \le c\|u_m\|^2,$$

which, together with the L^2 estimate (2.2.13), yields the claim of the Lemma. □

Our next step will be to show that in the ball $B(\sigma_0, R)$ the solution u_m is not *too small*:

2.2.3 Lemma. *With the notations and assumptions of Lemma 2.2.2 there exist $m_1 \ge m_0$ and a constant $c_0 > 0$ depending on R such that for all $m \ge m_1$*

$$\int_{B(\sigma_0,R)\cap\Omega} |\nabla u_m(x)|^2 dx \ge c_0.$$

Proof. Recall that if we denote by $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ the trace operator $u \mapsto u_{|\partial\Omega}$, there exists a positive constant c such that for all $u \in H^1(\Omega)$ we have (see section 1.2 and Lemma 1.2.1)

$$\|\gamma_0(u)\|^2_{H^{1/2}} \le c\left(\|\nabla u\|^2_2 + \|\gamma_0(u)\|^2_{L^2(\partial\Omega)}\right).$$

Applying this to u_m for $m \ge m_0$, one sees that since $\gamma_0(u_m) = \psi_m$ and $\|\psi_m\|_{L^2(\partial\Omega)} \le c m^{-1/2}$ (cf. Lemma 2.2.1):

$$1 = \|\gamma_0(u_m)\|^2_{H^{1/2}} \le c\left(\|\nabla u_m\|^2_2 + \frac{c}{m}\right) \le c\int_{B(\sigma_0,R)\cap\Omega} |\nabla u_m(x)|^2 dx + \frac{c}{m^2} + \frac{c}{m}.$$

Clearly, upon taking $m_1 \ge m_0$ large enough and $m \ge m_1$, this yields the claimed result of the Lemma. □

We can now state and prove the first identification result we had in mind: namely that if $\Lambda_{a_0} = \Lambda_{a_1}$ then $a_0 = a_1$ on the boundary.

2.2.4 Theorem. *Let Ω be a bounded domain and $\sigma_* \in \partial\Omega$ such that Ω is $C^{1,1}$ in a neighbourhood $B(\sigma_*, R_*)$ of σ_*. Assume that $a_j \in L^\infty(\Omega) \cap$*

$W^{1,\infty}(\Omega \cap B(\sigma_*, R_*))$ with $a_j \geq \varepsilon_0 > 0$ in Ω. Let Λ_{a_j} be as in (2.2.1) and assume that $\Lambda_{a_0} = \Lambda_{a_1}$. Then one has $a_0 = a_1$ on $B(\sigma_*, R_*) \cap \partial\Omega$.
In particular if Ω is $C^{1,1}$ and $a_j \in W^{1,\infty}(\Omega)$, the equality $\Lambda_{a_0} = \Lambda_{a_1}$ implies that $a_0 = a_1$ on $\partial\Omega$.

Proof. If there exists $\sigma_0 \in B(\sigma_*, R_*) \cap \partial\Omega$ such that (for instance) $a_0(\sigma_0) > a_1(\sigma_0)$, then the a_j's being continuous in a neighbourhood of σ_0, there exist $\beta > 0$ and $R > 0$ such that $B(\sigma_0, R) \subset B(\sigma_*, R_*)$ and

$$\forall x \in B(\sigma_0, R) \cap \Omega, \qquad a_0(x) \geq a_1(x) + \beta.$$

Now denote by $u_{jm} \in H^1(\Omega)$ the solution of

$$\begin{cases} -\operatorname{div}(a_j \nabla u_{jm}) = 0 & \text{in } \Omega \\ u_{jm} = \psi_m & \text{on } \partial\Omega. \end{cases}$$

Denoting $B := B(\sigma_0, R) \cap \Omega$ we have

$$\begin{aligned} Q_{a_0}(\psi_m) = \int_\Omega a_0(x) \,|\, \nabla u_{0m}|^2 dx &\geq \int_B a_0(x) \,|\, \nabla u_{0m}|^2 dx \\ &\geq \int_B a_1(x) \,|\, \nabla u_{0m}|^2 dx + \beta \int_B |\, \nabla u_{0m}|^2 dx \\ &\geq \int_B a_1(x) \,|\, \nabla u_{0m}|^2 dx + \beta c_0 \\ &\geq \int_\Omega a_1(x) \,|\, \nabla u_{0m}|^2 dx - \frac{c}{m^2} + \beta c_0, \end{aligned}$$

provided $m \geq m_1$, where m_1 is given by Lemma 2.2.3. Observe that in the last estimate we have used Lemma 2.2.2, that is

$$\int_{\Omega \setminus B} a_1(x) \,|\, \nabla u_{0m}(x) \,|^2 dx \leq c m^{-2}.$$

If we denote by K_{ψ_m} the convex set

$$K_{\psi_m} := \left\{ v \in H^1(\Omega) \,;\, v = \psi_m \text{ on } \partial\Omega \right\},$$

by definition of u_{1m} we have

$$\begin{aligned} \int_\Omega a_1(x) \,|\, \nabla u_{0m}|^2 dx &\geq \min_{v \in K_{\psi_m}} \int_\Omega a_1(x) \,|\, \nabla v|^2 dx \\ &= \int_\Omega a_1(x) \,|\, \nabla u_{1m}|^2 dx = Q_{a_1}(\psi_m). \end{aligned}$$

Therefore when $m_2 \geq m_1$ is large enough so that for $m \geq m_2$ we have $c m^{-2} \leq \beta c_0/2$, we infer that

(2.2.14)
$$Q_{a_0}(\psi_m) \geq \int_\Omega a_1(x)\,|\,\nabla u_{0m}|^2 dx + \frac{1}{2}\beta c_0 \geq Q_{a_1}(\psi_m) + \frac{1}{2}\beta c_0.$$

However, since $\Lambda_{a_0} = \Lambda_{a_1}$, we have $Q_{a_0}(\psi_m) = Q_{a_1}(\psi_m)$, which is in contradiction with the above inequality. Therefore we have $a_0 = a_1$ on $B(\sigma_*, R_*) \cap \partial\Omega$ and the Proposition is proved. □

In order to prove that the assumption $\Lambda_{a_0} = \Lambda_{a_1}$ implies that $a_0 = a_1$ at order one on the boundary, that is $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on $\partial\Omega$, we need to show a little bit more than the estimate of Lemma 2.2.3. Indeed, since we know that $a_0 = a_1$ on the boundary, the tangential derivatives of a_0 and a_1 coincide and so it remains to show that $\partial a_0/\partial n = \partial a_1/\partial n$. If this were not so, we would find $\beta > 0$ such that in a neighbourhood $B := B(\sigma_0, R) \cap \Omega$ of some $\sigma_0 \in \partial\Omega$:

$$\forall x \in B, \qquad a_0(x) \geq a_1(x) + \beta \mathrm{dist}(x, \partial\Omega)\,.$$

(Here $\mathrm{dist}(x, \partial\Omega)$ denotes the distance of x to the boundary). After a detailed inspection of the above proof of Proposition 2.2.4, one is convinced that if we knew that $\int_B \mathrm{dist}(x, \partial\Omega)\,|\,\nabla u_{0m}|^2 dx$ has a *greater magnitude* than

$$\int_{\Omega\setminus B} a_1(x)\,|\,\nabla u_{0m}|^2 dx$$

(which is $O(m^{-2})$), then the whole line of the above argument might be carried out, and we would obtain a contradiction to the fact that $\Lambda_{a_0} = \Lambda_{a_1}$, or equivalently $Q_{a_0} = Q_{a_1}$.

In order to obtain the estimate we are lacking, we shall use a special function equivalent to $\mathrm{dist}(x, \partial\Omega)$. Indeed let us denote by $\varphi_1 \in H_0^1(\Omega)$ the first eigenfunction of the operator $u \mapsto -\mathrm{div}(a\,\nabla u)$ with Dirichlet boundary condition, that is

(2.2.15)
$$\begin{cases} -\mathrm{div}(a\,\nabla\varphi_1) = \lambda_1\varphi_1 > 0 \ \text{ in } \Omega, \\ \varphi_1 \in H_0^1(\Omega), \qquad \displaystyle\int_\Omega \varphi_1(x)^2\,dx = 1. \end{cases}$$

It is a classical consequence of the Hopf maximum principle (see for instance D. Gilbarg & N.S. Trudinger [13]) that on a piece $B(\sigma_0, R) \cap \partial\Omega$ of the boundary which is $C^{1,1}$ one has $\partial\varphi_1/\partial n \leq -c_*(R, \Omega) < 0$ and

$$\forall x \in B(\sigma_0, R) \cap \overline{\Omega}, \qquad c_0\,\mathrm{dist}(x, \partial\Omega) \leq \varphi_1(x) \leq c_1\,\mathrm{dist}(x, \partial\Omega)\,.$$

2.2.5 Lemma. *With the notations and assumptions of Lemma 2.2.2 there exist an integer $m_1 \geq m_0$ and a constant $c_0 > 0$ depending on R such that for all $m \geq m_1$*

$$\int_{B(\sigma_0,R)\cap\Omega} \operatorname{dist}(x,\partial\Omega)\,|\,\nabla u_m(x)\,|^2 dx \geq \frac{c_0}{m}\,.$$

Proof. Let φ_1 be as in (2.2.15). Then we know that for some positive constants c_*, c_{1*}, c_{2*} we have (here again $B := B(\sigma_0, R) \cap \Omega$)

(2.2.16)
$$\begin{cases} \forall \sigma \in B(\sigma_0,R)\cap\partial\Omega, \qquad -\dfrac{\partial\varphi_1}{\partial n} \geq c_* > 0 \\ \forall x \in B, \qquad c_{1*}\operatorname{dist}(x,\partial\Omega) \leq \varphi_1(x) \leq c_{2*}\operatorname{dist}(x,\partial\Omega)\,. \end{cases}$$

Assuming that $m \geq m_0$ is large enough so that $\operatorname{supp}(\psi_m) \subset B(\sigma_0,R)\cap\partial\Omega$, and multiplying (2.2.11) by $u_m\varphi_1$, which belongs to $H_0^1(\Omega)$ since $\varphi_1 \in W^{1,\infty}(\Omega)$ and vanishes on the boundary $\partial\Omega$, we obtain

(2.2.17)
$$\int_\Omega a(x)\,|\,\nabla u_m|^2\varphi_1(x)\,dx + \int_\Omega a(x)\,u_m(x)\,\nabla u_m(x)\cdot\nabla\varphi_1(x)\,dx = 0.$$

But noting that $u_m\nabla u_m = \nabla(u_m^2)/2$, we may integrate by parts the second term of the above equality to see that (we use also the equation satisfied by φ_1) :

$$\int_\Omega a(x)\,u_m(x)\,\nabla u_m(x)\cdot\nabla\varphi_1(x)\,dx = \frac{1}{2}\int_{\partial\Omega}\psi_m(\sigma)^2 a(\sigma)\frac{\partial\varphi_1(\sigma)}{\partial n(\sigma)}\,d\sigma + \frac{\lambda_1}{2}\int_\Omega u_m(x)^2\varphi_1(x)\,dx.$$

Inserting this into (2.2.17) yields :

(2.2.18)
$$\int_\Omega a(x)\,|\,\nabla u_m|^2\varphi_1(x)\,dx = -\frac{\lambda_1}{2}\int_\Omega u_m(x)^2\varphi_1(x)\,dx - \frac{1}{2}\int_{\partial\Omega}\psi_m(\sigma)^2 a(\sigma)\frac{\partial\varphi_1}{\partial n(\sigma)}\,d\sigma.$$

Now we may use the Hopf maximum principle expressed in the first inequality of (2.2.16) to see that for some positive constant c, due to property (iii) of Lemma 2.2.1 with $s = 0$, we have

(2.2.19)
$$-\int_{\partial\Omega}\psi_m(\sigma)^2 a(\sigma)\frac{\partial\varphi_1}{\partial n(\sigma)}\,d\sigma \geq c_*\int_{\partial\Omega} a(\sigma)\,\psi_m(\sigma)^2\,d\sigma \geq \frac{c}{m}\,.$$

As far as the first term in the right hand side of (2.2.18) is concerned, observe that according to Lemma 1.2.7 we have

$$\int_\Omega u_m(x)^2 \varphi_1(x)\,dx \le c\|u_m\|^2 \le c\|\psi_m\|^2_{H^{-1/2}(\partial\Omega)} \le \frac{c}{m^2}.$$

It is clear that plugging this estimate together with (2.2.19) into (2.2.18), yields the result of the Lemma. □

At this point we can prove the following identification result.

2.2.6 Theorem. *Let Ω be a bounded domain and $\sigma_* \in \partial\Omega$ such that Ω is $C^{1,1}$ in a neighbourhood $B(\sigma_*, R_*)$ of σ_*. Assume that $a_j \in L^\infty(\Omega) \cap C^1(\overline{\Omega} \cap B(\sigma_*, R_*))$ with $a_j \ge \varepsilon_0 > 0$ in Ω. Let Λ_{a_j} be as in (2.2.1) and assume that $\Lambda_{a_0} = \Lambda_{a_1}$. Then one has $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on $B(\sigma_*, R_*) \cap \partial\Omega$. In particular if Ω is $C^{1,1}$ and $a_j \in C^1(\overline{\Omega})$, the equality $\Lambda_{a_0} = \Lambda_{a_1}$ implies that $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on $\partial\Omega$.*

Proof. By Proposition 2.2.6 we know that $a_0 = a_1$ on $B(\sigma_*, R_*) \cap \partial\Omega$, therefore in order to prove that $\nabla a_0 = \nabla a_1$ on this portion of the boundary, it is enough to show that $\partial a_0/\partial n = \partial a_1/\partial n$. If there exists $\sigma_0 \in B(\sigma_*, R_*) \cap \partial\Omega$ such that $\partial a_0(\sigma_0)/\partial \mathbf{n}(\sigma_0) \ne \partial a_1(\sigma_0)/\partial n(\sigma_0)$, then the a_j's being C^1 and $a_0 = a_1$ on $B(\sigma_*, R_*) \cap \partial\Omega$, there exist $\beta > 0$ and $R > 0$ such that $B(\sigma_0, R) \subset B(\sigma_*, R_*)$ and (for instance)

(2.2.20)
$$\forall x \in B(\sigma_0, R) \cap \Omega, \qquad a_0(x) \ge a_1(x) + \beta\,\mathrm{dist}(x, \partial\Omega).$$

We denote again by u_{jm} the solution of $\mathrm{div}(a_j \nabla u_{jm}) = 0$ and $u_{jm} = \psi_m$ on $\partial\Omega$, and by $\varphi_1 \in H^1_0(\Omega)$ the eigenfunction considered in (2.2.15) with $a := a_0$. If $B := B(\sigma_0, R) \cap \Omega$ we have (here we use Lemma 2.2.5)

$$\begin{aligned}
\int_\Omega a_0(x)\,|\nabla u_{0m}|^2 dx &\ge \int_B a_0(x)\,|\nabla u_{0m}|^2 dx \\
&\ge \int_B a_1(x)\,|\nabla u_{0m}|^2 dx + \beta \int_B \mathrm{dist}(x, \partial\Omega)\,|\nabla u_{0m}|^2 dx \\
&\ge \int_B a_1(x)\,|\nabla u_{0m}|^2 dx + \frac{\beta c_0}{m} \\
&\ge \int_\Omega a_1(x)\,|\nabla u_{0m}|^2 dx - \frac{c}{m^2} + \frac{\beta c_0}{m},
\end{aligned}$$

provided $m \ge m_1$, given by Lemma 2.2.5 (in the last estimate we have used Lemma 2.2.2). It follows that since by definition of u_{1m}

$$\int_{\Omega} a_1(x)\,|\nabla u_{0m}|^2 dx \geq Q_{a_1}(\psi_m),$$

for $m_2 \geq m_1$ large enough so that for $m \geq m_2$ we have $cm^{-1} \leq \beta c_0/2$,

$$Q_{a_0}(\psi_m) \geq \int_{\Omega} a_1(x)\,|\nabla u_{0m}|^2 dx + \frac{\beta c_0}{2m} \geq Q_{a_1}(\psi_m) + \frac{\beta c_0}{2m}.$$

This is a contradiction with the fact that $Q_{a_0}(\psi_m) = Q_{a_1}(\psi_m)$. Therefore we have $\partial a_0/\partial n = \partial a_1/\partial n$ on $B(\sigma_*, R_*) \cap \partial\Omega$ and the Theorem is proved. □

2.2.7 Remark. It is interesting to note that the arguments used to prove Theorem 2.2.4 may be used to give a stability estimate of the type

(2.2.21)
$$\|a_0 - a_1\|_{L^\infty(\partial\Omega)} \leq c(M)\,\|\Lambda_{a_0} - \Lambda_{a_1}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))},$$

for all $a_j \in W^{1,\infty}(\Omega)$ such that $\|a_j\|_{W^{1,\infty}(\Omega)} \leq M$, where $c(M)$ is a constant depending only on Ω and M. Indeed assume that $a_0, a_1 \in W^{1,\infty}(\Omega)$ are such that $a_0 - a_1 \not\equiv 0$ on the boundary and that if $2\beta := \|a_0 - a_1\|_{L^\infty(\partial\Omega)}$, for some $\sigma_0 \in \partial\Omega$ and $R > 0$ we have

$$\forall\, x \in B(\sigma_0, R) \cap \Omega, \qquad a_0(x) \geq a_1(x) + \beta.$$

Then according to (2.2.14) for some constant $c_0 > 0$ we have that

$$\frac{1}{2}\beta c_0 \leq Q_{a_0}(\psi_m) - Q_{a_1}(\psi_m) = \langle (\Lambda_{a_0} - \Lambda_{a_1})\,\psi_m, \psi_m \rangle$$

$$\leq \|\Lambda_{a_0} - \Lambda_{a_1}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}$$

since $\|\psi_m\|_{H^{1/2}} = 1$. It follows that (2.2.21) holds with $c(M) := 4/c_0$. □

2.3 Identification of potentials in Schrödinger operators

In this section will treat a few more general identification problems for elliptic operators, through appropriate boundary measurements. For $a \in \mathcal{A}_{ad}$ and $q \in L^\infty(\Omega)$ denote by $L := L_{a,q}$ the (formal) elliptic operator

(2.3.1)
$$Lu := L_{a,q}u := -\mathrm{div}(a\,\nabla u) + qu.$$

It is known that (see section § 2) the subspace $N(L)$ or $N(L_{a,q})$ defined to be

(2.3.2)
$$N(L_{a,q}) := \left\{ \varphi \in H_0^1(\Omega)\ ;\ L_{a,q}\varphi = 0 \ \text{ in } H^{-1}(\Omega) \right\},$$

is finite dimensional. Then we introduce the following way of saying :

2.3.1 Definition. *We shall say that zero is not an eigenvalue of $L_{a,q}$ if*

$$N(L_{a,q}) = \{0\}.$$

If zero is not an eigenvalue of $L_{a,q}$ then for any given $\varphi \in H^{1/2}(\partial\Omega)$ there exists a unique $u \in H^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(a\nabla u) + qu = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \partial\Omega, \end{cases} \tag{2.3.3}$$

and we may define the corresponding Poincaré-Steklov operator $\Lambda_{a,q}$

$$\Lambda_{a,q}(\varphi) = a\frac{\partial u}{\partial n}, \qquad \Lambda_{a,q} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega). \tag{2.3.4}$$

Now, the question we may ask is the following : if for two sets of coefficients a_j, q_j (where $j = 0, 1$) we have $\Lambda_{a_0,q_0} = \Lambda_{a_1,q_1}$ can we conclude that $a_0 = a_1$ and $q_0 = q_1$?

Before trying to answer this question, let us show that $\Lambda_{a,q}$ is a continuous self-adjoint operator on $H^{1/2}(\partial\Omega)$.

2.3.2 Lemma. *For given $a \in \mathcal{A}_{\mathrm{ad}}$ and $q \in L^\infty(\Omega)$ assume that zero is not an eigenvalue of $\Lambda_{a,q}$ defined by (2.3.4). Then $\Lambda_{a,q}$ is a self-adjoint operator mapping $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$.*

Proof. . The fact that $\Lambda_{a,q}$ is continuous from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$ is a consequence of Lemma 1.2.2. In order to see that $\Lambda_{a,q}$ is self-adjoint, let u solve (2.3.3) and for $\psi \in H^{1/2}(\partial\Omega)$ let $v \in H^1(\Omega)$ solve

$$-\operatorname{div}(a\nabla v) + qv = 0 \text{ in } \Omega, \qquad v = \psi \text{ on } \partial\Omega.$$

Upon multiplying this equation by u, equation (2.3.3) by v and integrating by parts, one sees that :

$$\int_\Omega a(x)\nabla v(x)\cdot\nabla u(x)\,dx + \int_\Omega q(x)v(x)u(x)\,dx = \int_{\partial\Omega} a(\sigma)\frac{\partial v(\sigma)}{\partial n(\sigma)}\varphi(\sigma)\,d\sigma$$
$$\int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega q(x)u(x)v(x)\,dx = \int_{\partial\Omega} a(\sigma)\frac{\partial u(\sigma)}{\partial n(\sigma)}\psi(\sigma)\,d\sigma,$$

which yields that $\langle\Lambda_{a,q}\psi, \varphi\rangle = \langle\Lambda_{a,q}\varphi, \psi\rangle$, and hence $\Lambda^*_{a,q} = \Lambda_{a,q}$. □

This gives the following important identity :

2.3.3 Corollary. *If $a_j \in \mathcal{A}_{\mathbf{ad}}$ and $q_j \in L^\infty(\Omega)$ (for $j = 0, 1$) are such that zero is not an eigenvalue of Λ_{a_j,q_j}, then for all $\varphi_j \in H^{1/2}(\partial\Omega)$ if $u_j \in H^1(\Omega)$ solves (2.3.3) with $a := a_j$ and $q := q_j$ and $\varphi := \varphi_j$, we have :*

$$\langle (\Lambda_{a_0,q_0} - \Lambda_{a_1,q_1})\varphi_0, \varphi_1 \rangle = \int_\Omega (a_0(x) - a_1(x))\,\nabla u_0(x)\cdot\nabla u_1(x)\,dx + \int_\Omega (q_0(x) - q_1(x))\,u_0(x)\,u_1(x)\,dx. \tag{2.3.5}$$

Proof. Multiplying by u_1 the equation satisfied by u_0, and vice versa, integrating by parts gives :

$$\langle \Lambda_{a_0,q_0}\varphi_0, \varphi_1 \rangle = \int_\Omega a_0(x)\,\nabla u_0(x)\cdot\nabla u_1(x)\,dx + \int_\Omega q_0(x)\,u_0(x)\,u_1(x)\,dx$$
$$\langle \Lambda_{a_1,q_1}\varphi_1, \varphi_0 \rangle = \int_\Omega a_1(x)\,\nabla u_1(x)\cdot\nabla u_0(x)\,dx + \int_\Omega q_1(x)\,u_1(x)\,u_0(x)\,dx.$$

Subtracting the second equality from the first, and using the fact that according to Lemma 2.3.2 one has

$$\langle \Lambda_{a_1,q_1}\varphi_1, \varphi_0 \rangle = \langle \Lambda_{a_1,q_1}\varphi_0, \varphi_1 \rangle,$$

we see that identity (2.3.5) is proved. □

This corollary, although quite simple to prove, is of the utmost importance in the study of inverse or identification problems. Indeed it shows that identification of the coefficients a_j, q_j boils down to the proof of an abstract density result. Indeed denote by $E_0 := E_0(a_0, q_0, a_1, q_1)$ the space

$$E_0 := \operatorname{span}\Big\{ (\nabla u_0\cdot\nabla u_1, u_0 u_1)\ ;\ u_j \in H^1(\Omega),\ L_{a_j,q_j}u_j = 0 \Big\}, \tag{2.3.6}$$

in $L^1(\Omega)\times L^1(\Omega)$. If one could prove that E_0 is dense in $L^1(\Omega)\times L^1(\Omega)$, one could conclude that when $\Lambda_{a_0,q_0} = \Lambda_{a_1,q_1}$ then necessarily $a_0 = a_1$ and $q_0 = q_1$: indeed by identity (2.3.5) we would infer that $(a_0 - a_1, q_0 - q_1)$ is orthogonal to E_0, and the density of this space in $L^1(\Omega)\times L^1(\Omega)$ would imlpy that $(a_0 - a_1, q_0 - q_1) = (0,0)$ in $L^\infty(\Omega)\times L^\infty(\Omega)$. More precisely, rather than the density of E_0 in $L^1(\Omega)\times L^1(\Omega)$, since E_0 depends on a_0, q_0, a_1, q_1, we need to know that if

$$(a_0 - a_1, q_0 - q_1) \in E_0(a_0, a_1, q_0, q_1)^\perp,$$

then $(a_0, q_0) = (a_1, q_1)$. At this point we are not able to show such a general result for any two pairs of coefficients (a_j, q_j). However we may state the following :

2.3.4 Theorem. *For $a_j \in \mathcal{A}_{\mathrm{ad}}$ and $q_j \in L^\infty(\Omega)$ so that zero is not an eigenvalue of L_{a_j,q_j}, define the spaces*

$$V_0(q_0, q_1) := \operatorname{span}\Big\{ u_0 u_1 \,;\; u_j \in H^1(\Omega),\; L_{a_j,q_j} u_j = 0 \Big\}$$
$$V_1(a_0, a_1) := \operatorname{span}\Big\{ \nabla u_0 \cdot \nabla u_1 \,;\; u_j \in H^1(\Omega),\; L_{a_j,q_j} u_j = 0 \Big\}.$$

Then if $a_0 = a_1$ and if $V_0(q_0, q_1)$ is dense in $L^1(\Omega)$, the equality $\Lambda_{a_0,q_0} = \Lambda_{a_0,q_1}$ implies that $q_0 = q_1$ in Ω.
Analogously if $q_0 = q_1$ and if $V_1(a_0, a_1)$ is dense in $L^1(\Omega)$, the equality $\Lambda_{a_0,q_0} = \Lambda_{a_1,q_0}$ implies that $a_0 = a_1$ in Ω.

Proof. Indeed if for instance $a_0 = a_1$ and $\Lambda_{a_0,q_0} = \Lambda_{a_0,q_1}$ then thanks to identity (2.3.5) for any $\varphi_j \in H^{1/2}(\partial\Omega)$ we have

$$\int_\Omega (q_0(x) - q_1(x))\, u_0(x)\, u_1(x)\, dx = 0.$$

This means that $q_0 - q_1$ is orthogonal to $V_0(q_0, q_1)$, and since the latter is dense in $L^1(\Omega)$ we conclude that $q_0 = q_1$. The other claim of the Theorem is seen to be true in an analogous manner. □

We observe now that a practical way to prove that some coefficients may be identified in inverse problems, would be to prove that the spaces V_0 or V_1 are dense in $L^1(\Omega)$. In these lecture notes, due to lack of space we shall not consider this general problem (see [<KavianODet>] for a detailed treatment), but rather consider the simpler case $a_0 = a_1 \equiv 1$. The following results are essentially due to J. Sylvester & G. Uhlmann [30], although in part we shall use a simpler proof of one of their arguments, a simplification due to P. Hähner [14].

We begin with a result which says that there is a direct connection between the study of the general case and the special case $a_0 = a_1 \equiv 1$ and $q_0, q_1 \in L^\infty(\Omega)$.

2.3.5 Proposition. *Let $a_j \in \mathcal{A}_{\mathrm{ad}} \cap W^{2,\infty}(\Omega)$ and $q_j \in L^\infty(\Omega)$ (for $j = 0, 1$) be such that zero is not an eigenvalue of L_{a_j,q_j}. Define*

$$\widetilde{q}_j := \frac{q_j}{a_j} + \frac{\Delta a_j^{1/2}}{a_j^{1/2}}$$

and assume that $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on the boundary $\partial\Omega$. Then

$$\Lambda_{a_0,q_0} = \Lambda_{a_1,q_1} \iff \Lambda_{1,\tilde{q}_0} = \Lambda_{1,\tilde{q}_1}\,.$$

Proof. For a given $\varphi \in H^{1/2}(\partial\Omega)$, if $u_j \in H^1(\Omega)$ satisfies

$$-\operatorname{div}(a_j \nabla u_j) + q_j u_j = 0 \text{ in } \Omega, \qquad u_j = \varphi \text{ on } \partial\Omega\,,$$

then using the Liouville's transform, that is setting $v_j := a_j^{1/2} u_j$, or equivalently $u_j = a_j^{-1/2} v$, one checks that

$$\begin{aligned}\operatorname{div}(a_j \nabla u_j) = \operatorname{div}\left(a_j^{1/2}\nabla v_j - \frac{\nabla a_j}{2a_j^{1/2}}\, v_j\right) &= \operatorname{div}\left(a_j^{1/2}\nabla v_j - v_j \nabla a_j^{1/2}\right)\\ &= a_j^{1/2}\left(\Delta v_j - \frac{\Delta a_j^{1/2}}{a_j^{1/2}}\, v_j\right).\end{aligned}$$

Hence v_j satisfies

$$-\Delta v_j + \left(\frac{q_j}{a_j} + \frac{\Delta a_j^{1/2}}{a_j^{1/2}}\right) v_j = 0 \text{ in } \Omega, \qquad v_j = a_j^{1/2}\varphi \text{ on } \partial\Omega\,.$$

Observe also that since v_j satisfies the above equation, it follows that

$$\begin{aligned}\Lambda_{1,\tilde{q}_j}(a_j^{1/2}\varphi) = \frac{\partial v_j}{\partial \boldsymbol{n}} &= a_j^{1/2}\frac{\partial u_j}{\partial \boldsymbol{n}} + \frac{1}{2}a_j^{-1/2}\frac{\partial a_j}{\partial \boldsymbol{n}}\,\varphi\\ &= a_j^{-1/2}\Lambda_{a_j,q_j}(\varphi) + \frac{1}{2}a_j^{-1/2}\frac{\partial a_j}{\partial \boldsymbol{n}}\,\varphi \qquad (2.3.7)\end{aligned}$$

Since $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on $\partial\Omega$, and due to the fact that $\varphi \mapsto a_j^{1/2}\varphi$ is a homeomorphism on $H^{1/2}(\partial\Omega)$, we conclude that saying $\Lambda_{a_0,q_0} = \Lambda_{a_1,q_1}$ is equivalent to say that $\Lambda_{1,\tilde{q}_0} = \Lambda_{1,\tilde{q}_1}$. □

2.3.6 Corollary. *Let Ω be a bounded $C^{1,1}$ domain. Assume that for $q_j \in L^\infty(\Omega)$ (with $j = 0, 1$) such that zero is not an eigenvalue of L_{1,q_j} we have the following property :*

$$\Lambda_{1,q_0} = \Lambda_{1,q_1} \Longrightarrow q_0 = q_1 \text{ in } \Omega. \qquad (2.3.8)$$

Then if $a_j \in \mathcal{A}_{\mathbf{ad}} \cap W^{2,\infty}(\Omega)$ are such that $\Lambda_{a_0,0} = \Lambda_{a_1,0}$, we have $a_0 = a_1$ in $\overline{\Omega}$.

Proof. First according to Theorem 2.2.4 and Theorem 2.2.6, we know that whenever $\Lambda_{a_0,0} = \Lambda_{a_1,0}$, then $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on the boundary $\partial\Omega$. Next setting

$$\widetilde{q}_j := \frac{\Delta a_j^{1/2}}{a_j^{1/2}},$$

then clearly zero is not an eigenvalue of $L_{1,\widetilde{q}_j}$, because saying that $v \in H_0^1(\Omega)$ satisfies $-\Delta v + \widetilde{q}_j v = 0$ is equivalent to say that $\operatorname{div}(a_j \nabla u) = 0$ with $u := a^{-1/2} v \in H_0^1(\Omega)$, and the latter equality implies $u = 0 = v$. Now Proposition 2.3.5 shows that if $\Lambda_{a_0,0} = \Lambda_{a_1,0}$, since a_0 and a_1, as well as ∇a_0 and ∇a_1, are equal on the boundary, we have $\Lambda_{1,\widetilde{q}_0} = \Lambda_{1,\widetilde{q}_1}$. Property (2.3.8) implies that $\widetilde{q}_0 = \widetilde{q}_1$ in Ω, but it is clear that if we set $q := \widetilde{q}_j$, the functions $a_j^{1/2}$ (which are also in $H^1(\Omega)$) satisfy the elliptic equation

$$\begin{cases} -\Delta a_j^{1/2} + q a_j^{1/2} = 0 & \text{in } \Omega \\ a_j = a_0 & \text{on } \partial\Omega. \end{cases}$$

Since zero is not an eigenvalue of $L_{1,q}$ it follows that the above equation has a unique solution, that is $a_0 = a_1$ in Ω, and the Corollary is proved. □

Now our goal will be to prove property (2.3.8), which is true when the dimension $N \geq 3$, and its proof is due to J. Sylvester & G. Uhlmann [30]. We state the following density theorem which will be proved in the next section.

2.3.7 Theorem. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain with $N \geq 3$. If $q_j \in L^\infty(\Omega)$ are such that zero is not an eigenvalue of L_{1,q_j}, then the space*

(2.3.9) $V_0(q_0, q_1) :=$

$$\operatorname{span}\Big\{ u_0 u_1 \,;\; u_j \in H^1(\Omega) \text{ and } -\Delta u_j + q_j u_j = 0 \text{ in } \Omega \Big\}$$

is dense in $L^1(\Omega)$. As a consequence if the Poincaré-Steklov operators Λ_{1,q_0} and Λ_{1,q_1} are equal, then $q_0 = q_1$ in Ω.

2.3.8 Remark. The case of dimension $N = 2$ is particular in two respects. First, by a result of A.I. Nachman [28], which uses a completely different approach, one knows that if a_0 and a_1 are smooth enough then the equality of the Poincaré-Steklov operators $\Lambda_{a_0,0} = \Lambda_{a_1,0}$ implies that $a_0 = a_1$ in $\overline{\Omega}$. Using this result one may show that if q_0, q_1 are such that the first eigenvalue of the operator $(L_{1,q_j}, D(L_{1,q_j}))$ is positive, then the fact that $\Lambda_{1,q_0} = \Lambda_{1,q_1}$ implies $q_0 = q_1$.

▶ **To complete…** □

2.4 Density of products of solutions

In this last section we prove the density result invoked at the end of the previous section. In order to establish the density Theorem 2.3.7 the main idea, as pointed out in the original paper of J. Sylvester & G. Uhlmann [30], is the following. Since the exponential functions $\exp((i\xi \pm \eta)\cdot x)$ used by A.P. Calderón (see section § 3, in particular (2.1.7) and (2.1.8)) are such that the space spanned by their product is dense, each of them being a harmonic function, one may consider the operators $u \mapsto -\Delta u + q_j u$ as perturbations of the Laplacian and prove that the space $E(q_0, q_1)$ defined in (2.3.9) is dense. The point is that this clever idea works, at least when $N \geq 3$. However here we will follow the point of view presented in a later work by P. Hähner [14].

In what follows we consider a potential $q \in L^\infty(\Omega)$ such that zero is not an eigenvalue of the operator $L_{1,q}$. We wish to construct solutions of

$$u \in H^1(\Omega), \qquad -\Delta u + qu = 0 \tag{2.4.1}$$

such that

$$u(x) = e^{\zeta\cdot x}(1 + \psi(x)), \quad \text{with } \zeta = i\xi + \eta,$$

and

$$\xi, \eta \in \mathbb{R}^N, \quad |\xi| = |\eta|, \quad \xi \cdot \eta = 0. \tag{2.4.2}$$

Since $L_{1,q}u = e^{\zeta\cdot x}[-\Delta\psi - 2\zeta\cdot\nabla\psi + q(1+\psi)]$, in order for u to be a solution of (2.4.1), the new unknown function ψ should be a solution of

$$\psi \in H^1(\Omega), \quad -\Delta\psi - 2\zeta\cdot\nabla\psi + q\psi = -q. \tag{2.4.3}$$

Moreover it is desirable to have ψ *small* in some appropriate norm, at least when $|\zeta| \to +\infty$, so that u is close to $e^{(i\xi+\eta)\cdot x}$. We are going to solve (2.4.3) when $\Omega \subset (-R, R)^N$ for some $R > 0$ and in the special case $\zeta = s e_1 + i\xi$ for some $s \in \mathbb{R}$, $s \neq 0$, and $\xi \in \mathbb{R}^N$ such that $e_1 \cdot \xi = 0$. Note that this is not any restriction since the operators are essentially invariant under translations and rotations of $\mathbb{R}^N$; also we should say that we denote $e_1 := (1, 0, \dots, 0)$, and analogously e_k will be the k-th vector of the standard basis of $\mathbb{R}^N$. We denote by Q the cube $(-R, R)^N$ and we introduce

$$Z_0 := \left\{ \alpha \in \mathbb{R}^N \,;\, \frac{\alpha_1 R}{\pi} - \frac{1}{2} \in \mathbb{Z}, \quad \frac{\alpha_j R}{\pi} \in \mathbb{Z} \text{ for } j \geq 2 \right\}. \tag{2.4.4}$$

Observe that when $\alpha \in Z_0$ then $\alpha_1 \neq 0$, more precisely $|\alpha_1| \geq \pi/2R$ and, as we shall see below, this is the reason for the choice of a shifted grid. Also note that we may write

$$Z_0 = \frac{\pi}{2R} e_1 + \frac{\pi}{R} \mathbb{Z}^N. \tag{2.4.5}$$

We recall here the classical definition of Q-periodic functions :

2.4.1 Definition. *A function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is said to be Q-periodic if for all $1 \leq k \leq N$ it satisfies $u(\cdot + 2R\mathbf{e_k}) = u(\cdot)$. Such a function is completely determined by its values on Q and the subspace of Q-periodic functions of $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is denoted by $H^1_{\mathrm{per}}(Q)$.*

Equivalently one can give a local definition of $H^1_{\mathrm{per}}(Q)$. A function $u \in H^1(Q)$ is said to belong to $H^1_{\mathrm{per}}(Q)$, or to be Q-periodic, if for all $1 \leq k \leq N$ we have $u(\sigma + 2Re_k) = u(\sigma)$ for $\sigma \in \partial Q$ (since the trace $u_{|\partial Q}$ is defined, the condition makes sense). In this case, a Q-periodic function in the latter sense can be extended by periodicity to all $\mathbb{R}^N$, and one may check that the two definitions of $H^1_{\mathrm{per}}(Q)$ coincide.

The space $H^1_{\mathrm{per}}(Q)$ is a closed subspace of $H^1(Q)$ and it is endowed with the natural topology of $H^1(Q)$. Analogously the space $H^2_{\mathrm{per}}(Q)$ is defined to be :

$$H^2_{\mathrm{per}}(Q) := \left\{ u \in H^1_{\mathrm{per}}(Q) \,;\, \partial_k u \in H^1_{\mathrm{per}}(Q) \text{ for } 1 \leq k \leq N \right\}.$$

Next we define the notion of Z_0-quasiperiodic functions.

2.4.2 Definition. *A function $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is said to be Z_0-quasiperiodic if the function $x \mapsto \exp(-i\pi x_1/2R)\, u(x)$ is Q-periodic. The space of Z_0-quasiperiodic functions which are in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ will be denoted by $H^1_{Z_0}(Q)$.*

One may check that this is a closed subspace of $H^1(Q)$. Analogously the space $H^2_{Z_0}(Q)$ is the space of Z_0-quasiperiodic functions which are in $H^2_{\mathrm{loc}}(\mathbb{R}^N)$. Equivalently, the space $H^1_{Z_0}(Q)$ (resp. $H^2_{Z_0}(Q)$) can be defined as the space generated by $\exp(i\pi x_1/2R)\varphi$ for $\varphi \in H^1_{\mathrm{per}}(Q)$ (resp. $\varphi \in H^2_{\mathrm{per}}(Q)$).

For $\alpha \in Z_0$ we denote by φ_α the function

$$\varphi_\alpha(x) := (2R)^{-N/2} e^{i\alpha \cdot x}. \tag{2.4.6}$$

One checks easily that each φ_α is Z_0-quasiperiodic, that $(\varphi_\alpha)_{\alpha \in Z_0}$ is a Hilbert basis of $L^2(Q)$ and that for any $\alpha \in \mathbb{Z}^N$

$$\nabla\varphi_{\boldsymbol{\alpha}} = \mathrm{i}\alpha\varphi_{\boldsymbol{\alpha}}, \qquad \Delta\varphi_{\boldsymbol{\alpha}} = -|\alpha|^2\varphi_{\boldsymbol{\alpha}}.$$

A useful observation about Q-periodic functions is that when $u, v \in C^1(\overline{Q}) \cap H^1_{\mathbf{per}}(Q)$ then we have $\int_{\partial Q} u(\sigma)\,\overline{v(\sigma)}\,\boldsymbol{n_k}(\sigma)\,d\sigma = 0$ (because $\sigma \mapsto \boldsymbol{n_k}(\sigma)$ is *anti-periodic* on ∂Q) and therefore $\int_Q \nabla u(x)\;\overline{v(x)}dx = -\int_Q u(x)\;\nabla\overline{v(x)}dx$. Also for $u, v \in C^2(\overline{Q}) \cap H^2_{\mathbf{per}}(Q)$, since $\sigma \mapsto \nabla u(\sigma)\cdot\boldsymbol{n}(\sigma)$ is anti-periodic, we have

$$\int_{\partial Q} \frac{\partial u(\sigma)}{\partial \boldsymbol{n}(\boldsymbol{\sigma})}\overline{v(\sigma)}d\sigma = 0,$$

and

$$\int_{\partial Q}\left(\frac{\partial u(\sigma)}{\partial \boldsymbol{n}(\sigma)}\overline{v(\sigma)} - u(\sigma)\frac{\partial\overline{v(\sigma)}}{\partial \boldsymbol{n}(\sigma)}\right)d\sigma = 0,$$

and hence

$$\int_Q \Delta u(x)\;\overline{v(x)}dx = \int_Q u(x)\,\Delta\overline{v(x)}dx.$$

For our study of equation (2.4.3) it is useful to point out that if $u, v \in C^2(\overline{Q})$ are $Z_{\mathbf{0}}$-quasiperiodic, then setting

$$\varphi(x) \coloneqq \exp(-\mathrm{i}\pi x_{\mathbf{1}}/2R)\,u(x), \qquad \psi(x) \coloneqq \exp(-\mathrm{i}\pi x_{\mathbf{1}}/2R)\,v(x),$$

we have two Q-periodic functions and

$$\nabla u = \exp(\mathrm{i}\pi x_{\mathbf{1}}/2R)\,\nabla\varphi(x) + \frac{\mathrm{i}\pi}{2R}\exp(\mathrm{i}\pi x_{\mathbf{1}}/2R)\,\varphi(x)\,\boldsymbol{e_1}$$

$$\nabla v = \exp(\mathrm{i}\pi x_{\mathbf{1}}/2R)\,\nabla\psi(x) + \frac{\mathrm{i}\pi}{2R}\exp(\mathrm{i}\pi x_{\mathbf{1}}/2R)\,\psi(x)\,\boldsymbol{e_1}.$$

Hence one can easily verify that, since $\int_{\partial Q}\varphi(\sigma)\,\overline{\psi(\sigma)}\,\boldsymbol{e_1}\cdot\boldsymbol{n}(\sigma)\,d\sigma = 0$,

$$\begin{aligned}\int_{\partial Q}\left(\frac{\partial u(\sigma)}{\partial \boldsymbol{n}(\sigma)}\overline{v(\sigma)} - u(\sigma)\frac{\partial\overline{v(\sigma)}}{\partial \boldsymbol{n}(\sigma)}\right)d\sigma &= \int_{\partial Q}\left(\frac{\partial\varphi(\sigma)}{\partial \boldsymbol{n}(\sigma)}\overline{\psi(\sigma)} - \varphi(\sigma)\frac{\partial\overline{\psi(\sigma)}}{\partial \boldsymbol{n}(\sigma)}\right)d\sigma\\ &\quad + \frac{\mathrm{i}\pi}{R}\int_{\partial Q}\varphi(\sigma)\,\overline{\psi(\sigma)}\,\boldsymbol{e_1}\cdot\boldsymbol{n}(\sigma)\,d\sigma\\ &= 0.\end{aligned}$$

Therefore the following rules for integration by parts hold for all $u, v \in C^1(\overline{Q}) \cap H^1_{Z_{\mathbf{0}}}(Q)$:

$$\int_Q \nabla u(x)\;\overline{v(x)}dx = -\int_Q u(x)\;\nabla\overline{v(x)}dx, \tag{2.4.7}$$

and for all $u, v \in C^2(\overline{Q}) \cap H^2_{Z_0}(Q)$ we have

$$\int_Q \Delta u(x)\, \overline{v(x)} dx = \int_Q u(x)\, \Delta\overline{v(x)} dx. \tag{2.4.8}$$

Invoking a density argument, one is convinced that the above relations are still valid when the dervivatives exist in the weak sense : the first equality (2.4.7) holds for $u, v \in H^1_{Z_0}(Q)$, while (2.4.8) is valid for $u, v \in H^2_{Z_0}(Q)$.

We are now in a position to solve an elliptic equation which corresponds to the differential part of equation (2.4.3).

2.4.3 Proposition. . *Let $s \in \mathbb{R}$, $s \neq 0$ and $\xi \in \mathbb{R}^N$ such that $\xi \cdot \boldsymbol{e_1} = 0$ and set $\zeta := i\xi + s\boldsymbol{e_1}$. Then for any $f \in L^2(Q)$ there exists a unique $\psi \in H^1_{Z_0}(Q) \cap H^2(Q)$ such that*

$$-\Delta\psi - 2\zeta \cdot \nabla\psi = f. \tag{2.4.9}$$

Moreover we have

$$\|\psi\|_{L^2(Q)} \leq \frac{R}{\pi|s|} \|f\|_{L^2(Q)}.$$

Proof. Using the Hilbert basis $(\varphi_\alpha)_{\alpha \in Z_0}$ defined in (2.4.6), one sees that equation (2.4.9) is equivalent to :

$$\forall \alpha \in Z_0, \qquad (-\Delta\psi - 2\zeta \cdot \nabla\psi | \varphi_\alpha) = (f|\varphi_\alpha),$$

where $(\cdot|\cdot)$ denotes the scalar product of $L^2(Q)$. Using the above remarks summarized in (2.4.7)–(2.4.8) and the fact that $\nabla\varphi_\alpha = i\alpha\varphi_\alpha$, we have

$$(-\Delta\psi - 2\zeta \cdot \nabla\psi | \varphi_\alpha) = (\alpha \cdot \alpha - 2i\zeta.\alpha)\ (\psi|\varphi_\alpha).$$

It follows that, provided $\alpha \cdot \alpha - 2i\zeta \cdot \alpha \neq 0$, we can express $(\psi|\varphi_\alpha)$ in terms of $(f|\varphi_\alpha)$. However this is the case, since $i\zeta \cdot \alpha = is\alpha_1 - \xi \cdot \alpha$ and therefore, as $|R\alpha_1/\pi| \geq 1/2$ for $\alpha \in Z_0$,

$$|\alpha \cdot \alpha - 2i\zeta \cdot \alpha| \geq |\Im\mathrm{m}(\alpha \cdot \alpha - 2i\zeta \cdot \alpha)| = |2s\alpha_1| \geq \frac{|s|\pi}{R}.$$

Therefore we have $(\psi|\varphi_\alpha) = (f|\varphi_\alpha)/(\alpha \cdot \alpha - 2i\zeta \cdot \alpha)$ and

$$\psi = \sum_{\alpha \in Z_0} \frac{(f|\varphi_\alpha)}{\alpha \cdot \alpha - 2i\zeta \cdot \alpha} \varphi_\alpha.$$

This, since $(\varphi_\alpha)_\alpha$ is a Hilbert basis of $L^2(Q)$, yields the estimate

$$\|\psi\|^2_{L^2(Q)} = \sum_{\alpha\in Z_0} \frac{|(f|\varphi_\alpha)|^2}{|\alpha\cdot\alpha - 2i\zeta\cdot\alpha|^2} \le \frac{R^2}{|s|^2\pi^2}\,\|f\|^2_{L^2(Q)},$$

and, using the regularity results for elliptic equations, we have also $\psi \in H^2(Q)$. □

2.4.4 Corollary. *Let $\zeta := i\xi + \eta$ where $\xi, \eta \in \mathbb{R}^N$ are such that $\xi\cdot\eta = 0$ and $\eta \neq 0$. Then if $\Omega \subset \mathbb{R}^N$ with $N \ge 2$ is a bounded domain, there exists a linear continuous operator $B_\zeta : L^2(\Omega) \longrightarrow H^2(\Omega)$ such that for any $f \in L^2(\Omega)$, the function $\psi := B_\zeta(f)$ satisfies*

(2.4.10) $$\psi \in H^2(\Omega), \qquad -\Delta\psi - 2\zeta\cdot\nabla\psi = f.$$

Moreover for some constant $c_0 > 0$, independent of ξ, η and f, we have

$$\|B_\zeta(f)\|_2 \le \frac{c_0}{|\eta|}\,\|f\|_2.$$

Proof. Without loss of generality we may assume that $0 \in \Omega$, and we may fix $R > 0$ such that for any rotation S of $\mathbb{R}^N$ around the origin we have $S(\overline{\Omega}) \subset (-R, R)^N$. For a given $\eta \in \mathbb{R}^N \setminus \{0\}$, we may set $s := |\eta|$ and find a rotation S in $\mathbb{R}^N$ such that $Se_1 = \eta/s$. If $f : \Omega \longrightarrow \mathbb{C}$ is a function in $L^2(\Omega)$, first we may extend it to Q by setting $\tilde f(x) := 0$ when $x \in Q \setminus \Omega$ and $\tilde f(x) = f(x)$ for $x \in \Omega$. Note also that if necessary, $\tilde f$ can be extended to all $\mathbb{R}^N$ by periodicity. Next one checks that if $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a function in $H^1_{loc}(\mathbb{R}^N)$, upon setting $v(x) := u(Sx)$ one has $\nabla v(x) = S^*(\nabla u)(Sx)$ and

$$\zeta_0\cdot\nabla v(x) = (S\zeta_0)\cdot(\nabla u)(Sx), \qquad \Delta v(x) = \Delta u(Sx).$$

It follows that if we set $\zeta_0 := iS^*\xi + S^*\eta$ (recall that $S^* = S^{-1}$ and $S^*\eta = se_1$), then in order to solve (2.4.10) in Ω it is enough to set $g(x) := \tilde f(Sx)$ and solve

$$v \in H^2(Q) \cap H^1_{Z_0}(Q), \qquad -\Delta v - 2\zeta_0\cdot\nabla v = g.$$

Since this has been done in Proposition 2.4.3 with $\|v\|_{L^2(Q)} \le c\|g\|_{L^2(Q)}/|\eta|$, one sees that if we set $B_\zeta(f) := v_{|\Omega}$, all the requirements of the Corollary are satisfied. □

Now we can solve equation (2.4.3) provided $|\zeta|$ is large enough.

2.4.5 Proposition. *Let $\zeta := i\xi + \eta$ where $\xi, \eta \in \mathbb{R}^N$ are such that $\xi\cdot\eta = 0$ and $\eta \neq 0$. Then if $\Omega \subset \mathbb{R}^N$ (with $N \ge 2$) is a bounded domain and $q \in L^\infty(\Omega)$, the constant $c_0 > 0$ being as in Corollary 2.4.4, for $|\eta| > c_0\|q\|_\infty$ there*

exists a linear continuous operator $M_\zeta : L^2(\Omega) \longrightarrow H^2(\Omega)$ *such that for any* $f \in L^2(\Omega)$, *the function* $\psi := M_\zeta(f)$ *satisfies*

(2.4.11) $\qquad \psi \in H^2(\Omega), \qquad -\Delta\psi - 2\zeta\cdot\nabla\psi + q\psi = f.$

Moreover for $|\eta| \geq 2c_0\|q\|_\infty$ *we have*

$$\|\psi\|_2 = \|M_\zeta(f)\|_2 \leq \frac{2c_0}{|\eta|}\,\|f\|_2\,.$$

Proof. The operator B_ζ being as in Corollary 2.4.4, we note that solving (2.4.11) is equivalent to solve $-\Delta\psi - 2\zeta\cdot\nabla\psi = f - q\psi$ and hence

(2.4.12) $$\psi = B_\zeta(f) - B_\zeta(q\psi)\,.$$

If we show that for $|\eta|$ large enough $\psi \mapsto B_\zeta(q\psi)$ is a contraction from $L^2(\Omega)$ into itself, then the mapping $\psi \mapsto B_\zeta(f) - B_\zeta(q\psi)$ has a unique fixed point and equation (2.4.12), and thus (2.4.11), has a unique solution.

But setting $A_\zeta(\psi) := B_\zeta(q\psi)$, we have clearly

$$\|A_\zeta(\psi)\|_2 \leq c_0|\eta|^{-1}\|q\psi\|_2 \leq c_0|\eta|^{-1}\|q\|_\infty\|\psi\|_2,$$

and therefore when $c_0\|q\|_\infty < |\eta|$ the operator $\psi \mapsto B_\zeta(f) - A_\zeta(\psi)$ is a (strict) contraction and equation (2.4.12) has a unique solution. Also if $|\eta| \geq 2c_0\|q\|_\infty$, we have $\|A_\zeta\| \leq 1/2$ and the solution ψ of (2.4.12) is given by

$$\psi = (I + A_\zeta)^{-1}B_\zeta(f)\,.$$

Whence if we set $M_\zeta := (I + A_\zeta)^{-1}B_\zeta$, we have

$$\|\psi\|_2 = \|M_\zeta(f)\|_2 \leq \|(I + A_\zeta)^{-1}\|\,\|B_\zeta(f)\|_2 \leq \frac{2c_0}{|\eta|}\,\|f\|_2\,,$$

since $\|(I + A_\zeta)^{-1}\| \leq 2$. Finally using classical regularity results in equation (2.4.11) one sees that $\psi \in H^2(\Omega)$, and the Proposition is proved. □

We are now in a position to prove the result announced in Theorem 2.3.7.

2.4.6 Theorem. . *Assume that* $\Omega \subset \mathbb{R}^N$ *is a bounded domain and that* $N \geq 3$. *For* $q_j \in L^\infty(\Omega)$ *(with* $j = 0, 1$*) let* $V_0(q_0, q_1)$ *be the space*

$$V_0(q_0, q_1) := \Big\{u_0u_1\,;\ u_j \in H^1(\Omega),\ \ \textit{and}\ \ -\Delta u_j + q_ju_j = 0\Big\}.$$

Then $V_0(q_0, q_1)$ *is dense in* $L^1(\Omega)$.

Proof. Let $\xi \in \mathbb{R}^N$ with $\xi \neq 0$ be given. It is interesting to emphasize that the assumption $N \geq 3$ comes in at this point : when the dimension $N \geq 3$, there

exist at least two other directions η, σ orthogonal to ξ and such that η and σ are mutually orthogonal. That is we may fix η and $\sigma \in \mathbb{R}^N$ such that for a given $t > 0$ we have :

$$\eta \cdot \xi = \xi \cdot \sigma = 0, \quad |\sigma| = 1, \quad |\eta|^2 = |\xi + t\sigma|^2 = |\xi|^2 + t^2.$$

Then consider

$$\zeta_0 := \eta + i(\xi + t\sigma), \qquad \zeta_1 := -\eta + i(\xi - t\sigma),$$

so that $x \mapsto \exp(\zeta_j \cdot x)$ is harmonic in $\mathbb{R}^N$, and let ψ_j be the solution given by Proposition 2.4.5 solving (2.4.3). More precisely let ψ_j satisfy :

$$\psi_j \in H^2(\Omega), \qquad -\Delta\psi_j - 2\zeta_j \cdot \nabla\psi_j + q_j\psi_j = -q_j,$$

in such a way that $u_j := (1 + \psi_j)\exp(\zeta_j \cdot x)$ solves

$$u_j \in H^2(\Omega), \qquad -\Delta u_j + q_j u_j = 0,$$

and $u_0 u_1 \in V_0(q_0, q_1)$. In order to show that $V_0(q_0, q_1)$ is dense in $L^1(\Omega)$, we have to prove that if for some $h \in L^\infty(\Omega)$ and for all $w \in V_0(q_0, q_1)$ we have $\int_\Omega h(x)\, w(x)\, dx = 0$, then necessarily $h = 0$. Now choosing $w := u_0 u_1$, we have

$$\int_\Omega h(x)\, u_0(x)\, u_1(x)\, dx =$$

$$\int_\Omega h(x)\, (1 + \psi_0(x))\, (1 + \psi_1(x))\exp((\zeta_0 + \zeta_1) \cdot x)\, dx = 0,$$

and since $\zeta_0 + \zeta_1 = 2i\xi$, we conclude that

$$\int_\Omega h(x)\, e^{2i\xi \cdot x} dx = -\int_\Omega h(x)\, e^{2i\xi \cdot x}(\psi_0(x) + \psi_1(x) + \psi_0(x)\,\psi_1(x))\, dx.$$

Therefore

$$\left|\int_\Omega h(x)\, e^{2i\xi \cdot x} dx\right| \le \|h\|_\infty(\|\psi_0\|_2 + \|\psi_1\|_2 + \|\psi_0\|_2\,\|\psi_1\|_2)\,,$$

and using the estimates $\|\psi_j\|_2 \le c|\eta|^{-1}$ and the fact that $|\eta| \ge t$ we conclude, as $t \to +\infty$, that $\int_\Omega h(x)\, e^{2i\xi \cdot x} dx = 0$, and hence $h = 0$. □

56

56

3

Inverse spectral problems

We address here inverse spectral problems for a certain family of elliptic operators. Let H be an infinite dimensional Hilbert space, with the scalar product denoted $(\cdot|\cdot)$ and norm $\|\cdot\|$. Consider a selfadjoint operator $(L, D(L))$ acting on H, that is a linear operator with a dense domain $D(L) \subset H$ and $L : D(L) \longrightarrow H$ such that $L^* = L$. Assume also that L has a compact resolvent and that for some $\lambda_0 \in \mathbb{R}$ we have $(Lu|u) \geq \lambda_0 \|u\|^2$. Then there is a sequence of eigenvalues and eigenfunctions (λ_k, φ_k) such that for $k, j \in \mathbb{N}^*$

$$L\varphi_k = \lambda_k \varphi_k, \qquad (\varphi_k|\varphi_j) = \delta_{kj}.$$

Given the operator L, determining the sequence $(\lambda_k, \varphi_k)_{k\geq 1}$ is the *direct spectral problem* for L.

The *inverse spectral problem* would be : given the sequence of eigenvalues $(\lambda_k)_{k\geq 1}$ what kind of information can we obtain about the operator L ? Clearly, in general, the only knowledge of the eigenvalues is far from being sufficient to determine entirely the operator. For instance, for any unitary operator $U : H \longrightarrow H$ (that is $U^*U = UU^* = I$) such that $U : D(L) \longrightarrow D(L)$ is bounded for the domain norm $\|u\|_{D(L)} := \|u\| + \|Lu\|$, then U^*LU and L have exactly the same eigenvalues. Therefore, in order to obtain more informations about L, it is natural to add some other insights, such as partial knowledge of eigenfunctions, or some informations about the nature of the operator L.

For instance when $H = L^2(\Omega)$ and L is a differential operator such as

$$Lu := -\mathrm{div}(a\,\nabla u) + qu$$

with a domain $D(L)$ which ensures $(D, D(L))$ to be self-adjoint with a compact resolvent, the inverse spectral problem consists in determining the coefficients a, q through the knowledge (or partial knowledge) of the eigenvalues λ_k and the eigenfunctions on the boundary $\partial\Omega$.

In case the knowledge of the eigenvalues is not sufficient to determine the coefficients of the differential operator, we shall be interested to find what kind of (minimal) information on the eigenfunctions would allow us to determine them. For practical resaons arising from applications to physics or engineering, as far as differential operators are concerned, one adds some informations about the behaviour of the eigenfunctions on the boundary.

In order to illustrate the motivations behind the study of inverse spectral problems, consider the following wave equation,

$$\begin{cases} \partial_{tt} u - \Delta u + q(x)\, u = 0 & \text{in } (0,T)\times\Omega \\ u(0,x) = f(x) & \text{in } \Omega \\ \partial_t u(0,x) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases} \tag{3.13}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and $N \geq 1$ is an integer. One might imagine that u describes the oscillations or an object when it is slightly disturbed from its rest position and left oscillating for a certain time T. We wish to identify the potential $q \in L^\infty(\Omega)$, which describes some physical property of the material in Ω, by observing the oscillations on the boundary $\partial\Omega$.

Denote by $(\lambda_k, \varphi_k)_{k\geq 1}$ the eigenvalues and eigenfunctions of the operator $-\Delta + q$ with Neumann boundary conditions, that is

$$-\Delta\varphi_k + q\varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega, \qquad \frac{\partial \varphi_k}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

We may assume that the eigenfunctions $(\varphi_k)_k$ form a Hilbert basis of $L^2(\Omega)$. Then setting

$$\omega_k := \sqrt{\lambda_k}, \qquad \text{and} \qquad (f|\varphi_k) := \int_\Omega f(x)\,\varphi_k(x)\,dx,$$

the solution of the wave equation (3.13) can be written as

$$u(t,x) = \sum_{k\geq 1} \cos(\omega_k t)\ (f|\varphi_k)\ \varphi_k(x)$$

which, « observed » on the boundary, can be formally written as being

$$\gamma_0(u)\,(t,\sigma) = \sum_{k\geq 1} \cos(\omega_k t)\ (f|\varphi_k)\ \gamma_0(\varphi_k)\,(\sigma).$$

It is reasonable to expect that through the observation of $\gamma_0(u)$ for various values of the initial data f and on a sufficiently long interval of time $(0,T)$, one can determine ω_k and $\gamma_0(\varphi_k)$, and then ask whether these boundary spectral data can lead to the determination of the potential q.

In the remainder of this chapter we give an overview of such problems, beginning with a one dimensional case which however will not be addressed here.

3.1 Sturm-Liouville problems

Let us consider the case of a Sturm-Liouville differential operator : if $H := L^2(0,\pi)$, consider a potential $q \in L^1(0,\pi)$ and set

$$Lu := -u'' + qu,$$

for

$$u \in D(L) := \left\{ v \in H_0^1(0,\pi) \; ; \; -v'' + qv \in L^2(0,\pi) \right\},$$

corresponding to Dirichlet boundary conditions, or with the domain

$$u \in D(L) := \left\{ v \in H^1(0,\pi) \; ; \; -v'' + qv \in L^2(0,\pi), \; v'(0) = v'(\pi) = 0 \right\},$$

corresponding to Neumann boundary conditions.

Then, with either of the domains $D(L)$, it is known that $(L, D(L))$ is a selfadjoint operator and has a sequence of eigenvalues and eigenfunctions $(\lambda_k, \varphi_k)_k$.

The *inverse spectral problem* for this operator L would be : knowing the eigenvalues $(\lambda_k)_k$ what can be said about the function q ?

Upon considering the function $\widetilde{q}(x) := q(\pi - x)$ and the operator

$$\widetilde{L}u := -u'' + \widetilde{q}u, \qquad \text{with } D(\widetilde{L}) := D(L),$$

one checks easily that L and $\widetilde{L}$ have the same eigenvalues, and thus if $\widetilde{q} \not\equiv q$ one sees that the only knowledge of the eigenvalues is not sufficient in order to determine q.

As a matter of fact, if one considers the eigenvalue problem

$$\begin{cases} -\varphi_k'' + q(x)\,\varphi_k = \lambda_k \varphi_k & \text{in } (0,\pi) \\ \varphi_k(0) = \varphi_k(\pi) = 0 & , \end{cases} \tag{3.1.1}$$

then one has the following expansion of the eigenvalues and eigenfunctions :

$$\lambda_k = k^2 + \frac{1}{\pi}\int_0^\pi q(x)\,dx - \frac{1}{\pi}\int_0^\pi q(x)\cos(2kx)\,dx + O\left(\frac{1}{k}\right)$$
$$\varphi_k(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\sin(kx) + O\left(\frac{1}{k}\right)$$
$$\varphi_k'(x) = \frac{k\sqrt{2}}{\sqrt{\pi}}\cos(kx) + O(1),$$

provided $q \in L^2(0,\pi)$. (See for instance A. Kirsch [18] chapter 4, page 135). However the above expansion of λ_k is not sufficient to solve the spectral inverse problem. Nevertheless, one shows that if for $j = 0, 1$ and $q_j \in L^2(0,\pi)$ one has

$$-\varphi_{j,k}'' + q_j\varphi_{j,k} = \lambda_k\varphi_{j,k}, \qquad \varphi_{j,k}(0) = \varphi_{j,k}(\pi) = 0,$$

and moreover

$$\frac{\varphi_{0,k}'(0)}{\varphi_{0,k}'(\pi)} = \frac{\varphi_{1,k}'(0)}{\varphi_{1,k}'(\pi)}\,,$$

then one can prove that $q_0 \equiv q_1$.

In particular if $q_j(\pi - x) = q_j(x)$, then the eigenfunctions are either symmetric or anti-symmetric, that is they satisfy either

$$\varphi_{j,k}(\pi - x) = \varphi_{j,k}(x) \qquad \text{or} \qquad \varphi_{j,k}(\pi - x) = -\varphi_{j,k}(x)$$

and thus

$$\frac{\varphi_{0,k}'(0)}{\varphi_{0,k}'(\pi)} = \frac{\varphi_{1,k}'(0)}{\varphi_{1,k}'(\pi)} = \pm 1,$$

for all $k \geq 1$, and so one may conclude that $q_0 \equiv q_1$.

In a paper dating back to 1929, V. Ambarzumian [5] shows that if we consider the eigenvalues corresponding to the Neumann boundary conditions

$$\begin{cases} -\varphi_k'' + q(x)\,\varphi_k = \lambda_k\varphi_k \quad \text{in } (0,\pi) \\ \varphi_k'(0) = \varphi_k'(\pi) = 0, \end{cases} \tag{3.1.2}$$

and if

$$\forall\, k \geq 1, \quad \lambda_k = (k-1)^2,$$

then one must have $q \equiv 0$.

However, in general, the only knowledge of the spectrum is not enough to determine the potential. G. Borg [8], N. Levinson [22], I.M. Gel'fand & B.M. Levitan [12] consider

$$\text{(3.1.3)} \qquad \begin{cases} -(a(x)\varphi_k')' + q(x)\varphi_k = \lambda_k \varphi_k & \text{in } (0,\pi) \\ \varphi_k(0) = \varphi_k(\pi) = 0 \end{cases}$$

They show that knowledge of the set

$$\left\{ (\lambda_k, \varphi_k'(0)) \; ; \; k \geq 1 \right\}$$

implies that of the coefficients a, q, provided certain regularity conditions are satisfied.

Another approach to inverse spectral problems for Sturm-Liouville operators consists in assuming that two sets of eigenvalues for the operator $L : u \mapsto Lu := -(au')' + qu$, for instance $\lambda_{k,D}$ for the Dirichlet boundary conditions and $\lambda_{k,N}$ for the Neumann boundary conditions, are known and then one wants to determine the coefficients a and q. The interested reader may refer to the vast literature available on the subject, for instance the books by B.M. Levitan [23], V.A. Marchenko [26], V.V. Kravchenko [20], C. Bennewitz, M. Brown & R. Weikard [6].

3.2 Higher Dimensions and Boundary Spectral Data

Let $\Omega \subset \mathbb{R}^N$ be a sufficiently smooth domain, and $q \in L^\infty(\Omega)$, and consider the eigenvalue problem

$$\text{(3.2.1)} \qquad \begin{cases} -\Delta\varphi_k + q(x)\varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\ \dfrac{\partial \varphi_k}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

As we did in section § 1.2, let us denote, for a function $\varphi \in H^1(\Omega)$, the trace operator $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$, that is $\gamma_0(\varphi) := \varphi_{|\partial\Omega}$. The boundary spectral data, associated to q, is defined by

$$\mathrm{Dbsd}(q) := \{ (\lambda_k, \gamma_0(\varphi_k)) \; ; \; k \geq 1 \},$$

Roman G. Novikov (1987–1988) and A.I. Nachman, J. Sylvester & G. Uhlmann (1988) show that

$$\mathrm{Dbsd}(q_0) = \mathrm{Dbsd}(q_1) \quad \Longrightarrow \quad q_0 = q_1.$$

Actually instead of $(\lambda_k, \gamma_0(\varphi_k))$ one should consider the set

$$\mathrm{Dbsd}(q) := \left\{ \left(\lambda_k, \mathrm{span}\left\{ \gamma_0(\varphi_{k,j}) \; ; \; 1 \le j \le m_k \right\} \right) \; ; \; k \ge 1 \right\}$$

where m_k is the multiplicity of λ_k, as we shall see later.

The same question can be studied for the Dirichlet eigenvalues, that is

$$\begin{cases} -\Delta\varphi_k + q(x)\,\varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\ \qquad\qquad \varphi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.2}$$

in which case one should consider the Neumann boundary spectral data

$$\mathrm{Nbsd}(q) := \left\{ \left(\lambda_k, \mathrm{span}\left\{ \gamma_1(\varphi_{k,j}) \; ; \; 1 \le j \le m_k \right\} \right) \; ; \; k \ge 1 \right\}$$

where

$$\gamma_1(\varphi) := \frac{\partial \varphi}{\partial \mathbf{n}}.$$

In order to study more general problems, let us introduce some notations : for $a, q, \rho \in L^\infty(\Omega)$ with $\min(a, \rho) \ge \varepsilon_0 > 0$ a.e. on Ω, consider the eigenvalue problem :

$$\begin{cases} -\mathrm{div}(a\,\nabla\varphi_k) + q\varphi_k = \lambda_k \rho \varphi_k & \text{in } \Omega \\ \qquad\qquad\qquad \varphi_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2.3}$$

If we endow $L^2(\Omega)$ with the scalar product

$$(u|v)_\rho := \int_\Omega u(x)\,v(x)\,\rho(x)\,dx, \tag{3.2.4}$$

then the operator

$$Lu := -\frac{1}{\rho}\mathrm{div}(a\,\nabla u) + \frac{q}{\rho}u,$$

with the domain

$$D(L) := \left\{ u \in H_0^1(\Omega) \; ; \; Lu \in L^2(\Omega) \right\},$$

is a self-adjoint operator on $(L^2(\Omega), (\cdot|\cdot)_\rho)$, has a compact resolvent, and thus admits a sequence of eigenvalues λ_k, each having finite multiplicity m_k, and eigenfunctions $\varphi_{k,i}$ for $1 \le i \le m_k$, which can be chosen to form a Hilbert basis of $L^2(\Omega)$ endowed with the scalar product $(\cdot|\cdot)_\rho$.

As usual, when $\varphi \in H^1(\Omega)$ and $\operatorname{div}(a\nabla\varphi) \in L^2(\Omega)$, we set

$$\gamma_1(\varphi) := (a\nabla\varphi)\cdot \boldsymbol{n} = a\frac{\partial\varphi}{\partial \boldsymbol{n}},$$

and we know that $\gamma_1(\varphi) \in H^{-1/2}(\partial\Omega)$ (see section § 1.2). The *Neumann boundary spectral data* associated to a, q, ρ is defined by

(3.2.5)
$$\operatorname{Nbsd}(a,q,\rho) := \left\{ \left(\lambda_k, \operatorname{span}\left\{\gamma_1(\varphi_{k,i}\right\}\,;1\le i\le m_k\right))\,;\ k\ge 1\right\}.$$

We can also consider the spectral problem associated to the Neumann boundary value problem, that is

(3.2.6)
$$\begin{cases} -\operatorname{div}(a\nabla\psi_{k,i}) + q\psi_{k,i} = \lambda_k\rho\psi_{k,i} & \text{in } \Omega \\ a\dfrac{\partial\psi_{k,i}}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

to which we associate the *Dirichlet boundary spectral data* defined by

(3.2.7)
$$\operatorname{Dbsd}(a,q,\rho) := \left\{ \left(\lambda_k, \operatorname{span}\left\{\gamma_0(\psi_{k,i}\right\}\,;1\le i\le m_k\right))\,;\ k\ge 1\right\},$$

where m_k is the multiplicity of λ_k and $\psi_{k,i}$ for $1\le i\le m_k$ are its corresponding eigenfunctions.

In the coming sections our aim is to show that under appropriate conditions on the coefficients a, q, ρ, the knowledge of the boundary spectral data implies that the associated Poincaré-Steklov operator (see (2.3.4) and Lemma 2.3.2). More precisely we will show that we can fix λ_* so that for instance $q+\lambda_*\rho \ge 1$, and then for all $\lambda \ge \lambda_*$ we have

$$\operatorname{Dbsd}(a_0,q_0,\rho_0) = \operatorname{Dbsd}(a_1,q_1,\rho_1) \Longrightarrow \Lambda_{a_0,q_0+\lambda\rho_0} = \Lambda_{a_1,q_1+\lambda\rho_1}.$$

We should add that the same result holds with Nbsd, that is the Neumann boundary spectral data.

In the following Sections § 3.3 and § 3.4 we follow the approach presented in the paper B. Canuto & O. Kavian [11].

3.3 The case of Dirichlet Boundary Spectral Data

In this section we assume that the eigenvalues λ_k of the Neumann boundary value problem (3.2.6) are known, each eigenvalue λ_k having multiplicity $m_k \ge 1$,

and that the family $\psi_{k,i}$ for $k \geq 1$ and $1 \leq i \leq m_k$ form a Hilbert basis of $L^2(\Omega)$, endowed with the scalar product $(\cdot|\cdot)_\rho$ defined in (3.2.4). We assume also that the Dirichlet boundary spectral data (3.2.7) is known.

We fix λ_* so that

$$q + \lambda_* \rho \geq 1 \quad \text{a.e. in } \Omega. \tag{3.3.1}$$

Then for any $\lambda \geq \lambda_*$ and $\psi \in H^{-1/2}(\partial\Omega)$, there exists a unique $u_\lambda \in H^1(\Omega)$ solution to the Neumann problem

$$\begin{cases} -\operatorname{div}(a \nabla u_\lambda) + (q + \lambda\rho)\, u_\lambda = 0 & \text{in } \Omega \\ a \dfrac{\partial u_\lambda}{\partial n} = \psi & \text{on } \partial\Omega. \end{cases} \tag{3.3.2}$$

Then we define the operator $\widetilde{\Lambda}_{a,q+\lambda\rho} : H^{-1/2}(\partial\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ by setting

$$\widetilde{\Lambda}_{a,q+\lambda\rho}\psi := \gamma_0(u_\lambda). \tag{3.3.3}$$

This operator is in fact the inverse of the Poincaré-Steklov $\Lambda_{a,q+\lambda\rho}$ introduced in (2.3.4) (though associated to the coefficients a and $q + \lambda\rho$). Recall that $\widetilde{\Lambda}_{a,q+\lambda\rho}$ is bounded and self-adjoint.

3.3.1 Lemma. *Let $\lambda \geq \lambda_*$ and $\psi \in H^{-1/2}(\partial\Omega)$. Then $u_\lambda \in H^1(\Omega)$ being solution to (3.3.2), we have*

$$\lim_{\lambda \to \infty} \|u_\lambda\|_{L^2(\Omega)} = 0.$$

Proof. Multiplying equation (3.3.2) by an eigenfunction $\psi_{k,i}$ and integrating by parts twice one finds that

$$\begin{aligned} \langle \psi, \gamma_0(\psi_{k,i}) \rangle &= \int_\Omega (a \nabla u \cdot \nabla \psi_{k,i} + (q + \lambda\rho)\, u \psi_{k,i})\, dx \\ &= \int_\Omega (-\operatorname{div}(a \nabla \psi_{k,i}) + (q + \lambda\rho)\, \psi_{k,i})\, u_\lambda dx \\ &= (\lambda_k + \lambda)\, (u_\lambda | \psi_{k,i})_\rho \end{aligned}$$

where we use the scalar product $(\cdot|\cdot)_\rho$ defined in (3.2.4) on $L^2(\Omega)$. Thus we have

$$(u_\lambda | \psi_{k,i})_\rho = \frac{\langle \psi, \gamma_0(\psi_{k,i}) \rangle}{\lambda_k + \lambda}, \tag{3.3.4}$$

and

$$u_\lambda = \sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{\langle \psi, \gamma_0(\psi_{k,i})\rangle}{\lambda_k + \lambda}\psi_{k,i}.$$

Now for $\lambda \geq \lambda_*$ we have

$$\left|\frac{\langle \psi, \gamma_0(\psi_{k,i})\rangle}{\lambda_k + \lambda}\right|^2 \leq \left|\frac{\langle \psi, \gamma_0(\psi_{k,i})\rangle}{\lambda_k + \lambda_*}\right|^2,$$

and since each term of the left hand side of the above inequality converges to zero, and since

$$\sum_{k\geq 1}\sum_{i=1}^{m_k}\left|\frac{\langle \psi, \gamma_0(\psi_{k,i})\rangle}{\lambda_k + \lambda_*}\right|^2 < \infty,$$

using Lebesgue dominated convergence theorem we conclude that

$$\lim_{\lambda\to\infty}\|u_\lambda\|^2 = \lim_{\lambda\to\infty}\sum_{k\geq 1}\sum_{i=1}^{m_k}\left|\frac{\langle \psi, \gamma_0(\psi_{k,i})\rangle}{\lambda_k + \lambda}\right|^2 = 0.$$

3.3.2 Lemma. *Let $\lambda \geq \lambda_*$ and $\psi \in H^{-1/2}(\partial\Omega)$. Then $u_\lambda \in H^1(\Omega)$ being solution to (3.3.2), for a fixed constant $c > 0$ and for all $\lambda \geq \lambda_*$ and $\psi \in H^{-1/2}(\partial\Omega)$ we have*

$$\|u_\lambda\|_{H^1(\Omega)} \leq c\|\psi\|_{H^{-1/2}(\partial\Omega)},$$

and $u_\lambda \to 0$ as $\lambda \to +\infty$.

Proof. Multiplying (3.3.2) by u_λ and integrating by parts we have

$$\begin{aligned}(3.3.5)\quad \int_\Omega a|\nabla u_\lambda|^2 dx + \int_\Omega (q+\lambda\rho)\,|u|^2 dx &= \langle \psi, \gamma_0(u_\lambda)\rangle \\ &\leq \|\psi\|_{H^{-1/2}}\|\gamma_0(u_\lambda)\|_{H^{1/2}(\partial\Omega)} \\ &\leq c_0\|\psi\|_{H^{-1/2}}\|u_\lambda\|_{H^1(\Omega)},\end{aligned}$$

for some constant $c_0 > 0$ depending on Ω. Assuming that $a \geq \varepsilon_0$ a.e. in Ω, we have (since $q + \lambda\rho \geq q \neq \lambda_*\rho \geq 1$)

$$\int_\Omega a|\nabla u_\lambda|^2 dx + \int_\Omega (q+\lambda\rho)\,|u|^2 dx \geq \min(\varepsilon_0, 1)\,\|u_\lambda\|^2_{H^1(\Omega)},$$

and from (3.3.5) we conclude that

$$\|u_\lambda\|_{H^1(\Omega)} \leq \frac{c_0}{\min(\varepsilon_0, 1)}\|\psi\|_{H^{-1/2}(\partial\Omega)}.$$

Now, since the family $(u_\lambda)_{\lambda\geq\lambda_*}$ is bounded and $u_\lambda \to 0$ strongly in $L^2(\Omega)$ as $\lambda \to +\infty$ along any subsequence, the unique weak limit point of the family is zero, that is $u_\lambda \rightharpoonup 0$ as $\lambda \to +\infty$. □

Next we compare the functions u_λ and u_{λ_*}.

3.3.3 Lemma. *Set $z_\lambda := u_\lambda - u_{\lambda_*}$ for $\lambda \geq \lambda_*$. Then we have*

$$\gamma_0(z_\lambda) = -\sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{(\lambda-\lambda_*)\,\langle \psi, \gamma_0(\psi_{k,i})\rangle}{(\lambda_k+\lambda)\,(\lambda_k+\lambda_*)}\,\gamma_0(\psi_{k,i}) \tag{3.3.6}$$

Proof. One easily verifies that z_λ satisfies the equation

$$\begin{cases} -\operatorname{div}(a\,\nabla z_\lambda) + (q+\lambda\rho)\,z_\lambda = -(\lambda-\lambda_*)\,u_{\lambda_*} & \text{in } \Omega \\ a\,\dfrac{\partial z_\lambda}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3.7}$$

Multiplying this equation by $\psi_{k,i}$ we see that

$$(\lambda_k+\lambda)\,(z_\lambda|\psi_{k,i})_\rho = -(\lambda-\lambda_*)\,(u_{\lambda_*}|\psi_{k,i})_\rho,$$

and hence, using (3.3.4) for $\lambda = \lambda_*$,

$$z_\lambda = -\sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{(\lambda-\lambda_*)\,\langle \psi, \gamma_0(\psi_{k,i})\rangle}{(\lambda+\lambda_k)\,(\lambda_*+\lambda_k)}\,\psi_{k,i}. \tag{3.3.8}$$

Now consider the operator $(L, D(L))$ defined on $L^2(\Omega)$, endowed with the scalar product $(\cdot|\cdot)_\rho$, defined by

$$Lu := -\frac{1}{\rho}\operatorname{div}(a\,\nabla u) + \frac{1}{\rho}(q+\lambda_*\rho)\,u,$$

with the domain

$$D(L) := \left\{ u \in H^1(\Omega)\ ;\ Lu \in L^2(\Omega) \quad \text{and } a\,\frac{\partial u}{\partial n} = 0 \right\}.$$

As mentioned previously, $(L, D(L))$ is self-adjoint on $L^2(\Omega)$ endowed with the scalar product $(\cdot|\cdot)_\rho$ and its associated norm $\|\cdot\|_\rho$, and in particular is a closed operator, making the domain $D(L)$, endowed with the graph norm

$$u \mapsto \|u\|_{D(L)} := \left(\|u\|_\rho^2 + \|Lu\|_\rho^2\right)^{1/2}$$

a Banach space (in fact, a Hilbert space) and moreover, since $(Lu|u)_\rho \geq \|u\|^2_{H^1(\Omega)}$, for some constant $c > 0$, for all $u \in D(L)$ we have

$$\|u\|_{H^1(\Omega)} \leq c\,\|u\|_{D(L)}. \tag{3.3.9}$$

As a matter of fact, one can verify that (since L is a diagonal operator in the Hilbert basis $\{\psi_{k,i}\,;\ k\geq 1,\ 1\leq i\leq m_k\}$)

$$\|u\|^2_{D(L)}=\sum_{k\geq 1}\sum_{i=1}^{m_k}\Big(\big|(u|\psi_{k,i})_\rho\big|^2+\lambda_k^2\big|(u|\psi_{k,i})_\rho\big|^2\Big).$$

From these observations, one can easily see that the series obtained in (3.3.8) converges in $D(L)$ endowed with its graph norm $\|\cdot\|_{D(L)}$. We infer from (3.3.9) that the series converges also in $H^1(\Omega)$, and since the trace operator $\gamma_0: H^1(\Omega)\longrightarrow H^{1/2}(\partial\Omega)$ is a continuous linear operator we have (3.3.6). □

Recall that the set of admissible coefficients is defined in (1.2.3) as being

$$\mathcal{A}_{\mathrm{ad}}:=\Big\{a\in L^\infty(\Omega)\,;\ \exists\varepsilon_0>0 \text{ such that } a\geq\varepsilon_0 \text{ a.e. on } \Omega\Big\}.$$

3.3.4 Lemma. *For $j=0,1$ assume that $a_j,\rho_j\in\mathcal{A}_{\mathrm{ad}}$ and $q_j\in L^\infty(\Omega)$. Fix $\lambda_*\in\mathbb{R}$ such that $q_j+\lambda_*\rho_j\geq 1$. Assuming moreover that $a_j\in W^{1,\infty}(\Omega)$, if*

$$\mathrm{Dbsd}(a_0,q_0,\rho_0)=\mathrm{Dbsd}(a_1,q_1,\rho_1),$$

then for all $\lambda\geq\lambda_*$ we have

(3.3.10)

$$\widetilde{\Lambda}_{a_0,q_0+\lambda_*\rho_0}-\widetilde{\Lambda}_{a_1,q_1+\lambda_*\rho_1}=\widetilde{\Lambda}_{a_0,q_0+\lambda\rho_0}-\widetilde{\Lambda}_{a_1,q_1+\lambda\rho_1}.$$

Proof. First, we know that since $a_j\in W^{1,\infty}(\Omega)$, the unique continuation principle applies and thus the spaces

$$\mathrm{span}\Big\{\gamma_0(\psi_{j,k,i})\,;\ 1\leq i\leq m_k\Big\}$$

have precisely dimension m_k (see Lemma 1.2.3). Also since for each k we have

$$\mathrm{span}\Big\{\gamma_0(\psi_{0,k,i})\,;\ 1\leq i\leq m_k\Big\}=\mathrm{span}\Big\{\gamma_0(\psi_{1,k,i})\,;\ 1\leq i\leq m_k\Big\}$$

we may assume that a common basis for these spaces have been chosen so that for all $k\geq 1$

$$\gamma_0(\psi_{0,k,i})=\gamma_0(\psi_{1,k,i})\qquad\text{for } 1\leq i\leq m_k. \tag{3.3.11}$$

Next, fix a $\psi\in H^{-1/2}(\partial\Omega)$ and let $u_{j,\lambda}$ solve

$$\begin{cases}-\mathrm{div}(a_j\nabla u_{j,\lambda})+(q_j+\lambda\rho_j)\,u_{j,\lambda}=0 & \text{in } \Omega\\ \qquad\qquad a_j\dfrac{\partial u_{j,\lambda}}{\partial n}=\psi & \text{on } \partial\Omega,\end{cases} \tag{3.3.12}$$

for $j=0$ and $j=1$. Upon setting

$$z_{j,\lambda} := u_{j,\lambda} - u_{j,\lambda_*},$$

thanks to Lemma 3.3.3 we know that

(3.3.13)
$$\gamma_0(z_{j,\lambda}) = -\sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{\lambda-\lambda_*}{(\lambda_k+\lambda)(\lambda_k+\lambda_*)} \left\langle \psi, \gamma_0(\psi_{j,k,i})\right\rangle \gamma_0(\psi_{j,k,i}).$$

Now using (3.3.11) we see that, as a matter of fact, the right hand side of the above expression is the same for $j=0$ and $j=1$. Thus for all $\lambda \geq \lambda_*$ we have

$$\gamma_0(z_{0,\lambda}) = \gamma_0(z_{1,\lambda})$$

which means that

$$\left(\widetilde{\Lambda}_{a_0,q_0+\lambda\rho_0} - \widetilde{\Lambda}_{a_0,q_0+\lambda_*\rho_0}\right)(\psi) = \left(\widetilde{\Lambda}_{a_1,q_1+\lambda\rho_1} - \widetilde{\Lambda}_{a_1,q_1+\lambda_*\rho_1}\right)(\psi)$$

for all $\psi \in H^{-1/2}(\partial\Omega)$, from which (3.3.10) follows. □

We can now state the main result of this section.

3.3.5 Theorem. *For $j=0,1$ assume that $a_j, \rho_j \in \mathcal{A}_{\mathrm{ad}}$ and $q_j \in L^\infty(\Omega)$. Fix $\lambda_* \in \mathbb{R}$ such that $q_j+\lambda_*\rho_j \geq 1$. Assuming also that $a_j \in W^{1,\infty}(\Omega)$, and*

$$\mathrm{Dbsd}(a_0,q_0,\rho_0) = \mathrm{Dbsd}(a_1,q_1,\rho_1).$$

Then for all $\lambda \geq \lambda_*$ we have

(3.3.14)
$$\widetilde{\Lambda}_{a_0,q_0+\lambda\rho_0} = \widetilde{\Lambda}_{a_1,q_1+\lambda\rho_1}.$$

Proof. With the above notations we know that for a fixed $\psi \in H^{-1/2}(\partial\Omega)$ and all $\lambda \geq \lambda_*$, thanks to (3.3.10) we have

$$\left(\widetilde{\Lambda}_{a_0,q_0+\lambda_*\rho_0} - \widetilde{\Lambda}_{a_1,q_1+\lambda_*\rho_1}\right)(\psi) = \gamma_0(u_{0,\lambda}) - \gamma_0(u_{1,\lambda}).$$

Thanks to Lemma 3.3.2 we know that $u_{j,\lambda} \rightharpoonup 0$ in $H^1(\Omega)$ as $\lambda \to \infty$, and since $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$ is a continuous linear operator, we infer that $\gamma_0(u_{j,\lambda}) \rightharpoonup 0$ in $H^{1/2}(\partial\Omega)$. Therefore

$$\left(\widetilde{\Lambda}_{a_0,q_0+\lambda_*\rho_0} - \widetilde{\Lambda}_{a_1,q_1+\lambda_*\rho_1}\right)(\psi) = 0,$$

for all ψ, that is $\widetilde{\Lambda}_{a_0,q_0+\lambda_*\rho_0} = \widetilde{\Lambda}_{a_1,q_1+\lambda_*\rho_1}$, from which thanks to (3.3.10), we deduce (3.3.14). □

We can now state the result concerning the inverse spectral problem for the Neumann boundary value problem, by proving that the knowledge of the Dirichlet boundary spectral data implies that of two coefficients out of three :

3.3.6 Theorem. *For $j = 0, 1$ let $a_j, \rho_j \in \mathcal{A}_{\mathbf{ad}}$ and $q_j \in L^\infty(\Omega)$ be given and moreover assume that $a_j \in W^{2,\infty}(\Omega)$ and that $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on the boundary $\partial\Omega$. Then if*

$$\mathrm{Dbsd}(a_0, q_0, \rho_0) = \mathrm{Dbsd}(a_1, q_1, \rho_1),$$

we have :

1) *if $a_0 \equiv a_1$, then $q_0 \equiv q_1$ and $\rho_0 \equiv \rho_1$.*
2) *If $q_0 \equiv q_1$, then $a_0 \equiv a_1$ and $\rho_0 \equiv \rho_1$.*
3) *If $\rho_0 \equiv \rho_1$, then $a_0 \equiv a_1$ and $q_0 \equiv q_1$.*

Proof. By Theorem 3.3.5 we know that

$$\widetilde{\Lambda}_{a_0, q_0+\lambda\rho_0} = \widetilde{\Lambda}_{a_1, q_1+\lambda\rho_1}.$$

Let us introduce the potentials

$$\widetilde{q}_j := \frac{q_j + \lambda\rho_j}{a_j} + \frac{\Delta a_j^{1/2}}{a_j^{1/2}},$$

for $j = 0, 1$. Then using the Liouville transform (see proof of Proposition 2.3.5), and the fact that $a_0 = a_1$ and $\nabla a_0 = \nabla a_1$ on the boundary $\delta\Omega$, we have that

$$\widetilde{\Lambda}_{a_0, q_0+\lambda\rho_0} = \widetilde{\Lambda}_{a_1, q_1+\lambda\rho_1} \iff \widetilde{\Lambda}_{1, \widetilde{q}_0} = \widetilde{\Lambda}_{1, \widetilde{q}_1}.$$

When $N \geq 3$, by Theorem 2.3.7 we know that the space

$$E_0 := \mathrm{span}\Big\{ u_0 u_1 \,;\; u_j \in H^1(\Omega),\; -\Delta u_j + \widetilde{q}_j u_j = 0,\; j = 0, 1 \Big\}$$

is dense in $L^1(\Omega)$ and therefore

$$\widetilde{\Lambda}_{1, \widetilde{q}_0} = \widetilde{\Lambda}_{1, \widetilde{q}_1} \Longrightarrow \widetilde{q}_0 \equiv \widetilde{q}_1. \tag{3.3.15}$$

When $N = 2$, there are several results (see for instance A.L. Bukhgeim [9], O.Y. Imanuvilov & M. Yamamoto [16], E. Blåsten, O.Y. Imanuvilov & M. Yamamoto [7]) which prove that E_0 is dense in $L^1(\Omega)$ and thus (3.3.15) holds.

Therefore, what we know is that for all $\lambda \geq \lambda_*$

$$\frac{q_0 + \lambda\rho_0}{a_0} + \frac{\Delta a_0^{1/2}}{a_0^{1/2}} = \frac{q_1 + \lambda\rho_1}{a_1} + \frac{\Delta a_1^{1/2}}{a_1^{1/2}}. \tag{3.3.16}$$

Proof of claim 1). If $a_0 = a_1$, then $q_0 + \lambda\rho_0 = q_1 + \lambda\rho_1$ for all $\lambda \geq \lambda_*$, from which we deduce claim **1)** of the Theorem.

Proof of claim 2). If $q_0 = q_1$, then for all $\lambda \geq \lambda_*$ we have

$$\frac{q_0+\lambda\rho_0}{a_0}+\frac{\Delta a_0^{1/2}}{a_0^{1/2}}=\frac{q_0+\lambda\rho_1}{a_1}+\frac{\Delta a_1^{1/2}}{a_1^{1/2}}.$$

Therefore

$$\frac{\rho_0}{a_0}=\frac{\rho_1}{a_1}\qquad\text{and}\qquad\frac{q_0}{a_0}+\frac{\Delta a_0^{1/2}}{a_0^{1/2}}=\frac{q_0}{a_1}+\frac{\Delta a_1^{1/2}}{a_1^{1/2}}. \tag{3.3.17}$$

Now consider the potential

$$q(x)\coloneqq\frac{q_0}{a_0}+\frac{\Delta a_0^{1/2}}{a_0^{1/2}}=\frac{q_0}{a_1}+\frac{\Delta a_1^{1/2}}{a_1^{1/2}}.$$

Upon setting $v_j\coloneqq a_j^{1/2}$, we have that

$$-\Delta v_0+qv_0-\frac{q_0}{v_0}=0\quad\text{in }\Omega$$

$$-\Delta v_1+qv_1-\frac{q_0}{v_1}=0\quad\text{in }\Omega.$$

If we set $v\coloneqq v_1-v_0$ one sees that

$$-\Delta v+\left(q+\frac{q_0}{v_0v_1}\right)v=0\quad\text{in }\Omega,\qquad v=\frac{\partial v}{\partial n}=0\quad\text{on }\partial\Omega.$$

At this point, since the potential $q+(q_0/v_0v_1)\in L^\infty(\Omega)$, using the unique continuation Theorem 1.2.3, we conclude that $v\equiv 0$ in Ω, that is $a_0=a_1$. Using this in (3.3.17) implies $\rho_0=\rho_1$, and the proof of claim **2)** is complete.

Proof of claim 3). If $\rho_0=\rho_1$, then the identity (3.3.16) for all $\lambda\geq\lambda_*$ implies that $a_0=a_1$, which in turn implies that $q_0=q_1$. □

3.3.7 Remark. As a matter of fact, proceeding exactly as in Section § 2.2 and the proofs of Theorem 2.2.4 and Theorem 2.2.6, one can show that when one assumes that Ω is of class $C^{1,1}$ and $a_j\in C^1(\overline{\Omega})$, and if $q_j+\lambda\rho_j>0$, then

$$\widetilde{\Lambda}_{a_0,q_0+\lambda\rho_0}=\widetilde{\Lambda}_{a_1,q_1+\lambda\rho_1}$$

implies that $a_0=a_1$ and $\nabla a_0=\nabla a_1$ on the boundary $\partial\Omega$. Therefore in the statement of Theorem 3.3.6 these assumptions may be relaxed. □

3.4 The case of Neumann Boundary Spectral Data

In this section we assume that the eigenvalues λ_k of the Dirichlet boundary value problem (3.2.3) are known, each eigenvalue λ_k having multiplicity $m_k\geq 1$,

and that the family $\varphi_{k,i}$ for $k \geq 1$ and $1 \leq i \leq m_k$ form a Hilbert basis of $L^2(\Omega)$, endowed with the scalar product $(\cdot|\cdot)_\rho$ defined in (3.2.4). We assume also that the Nirichlet boundary spectral data (3.2.5) is known.

We fix λ_* so that

$$q + \lambda_* \rho \geq 1 \quad \text{a.e. in } \Omega. \tag{3.4.1}$$

Then for any $\lambda \geq \lambda_*$ and $\varphi \in H^{1/2}(\partial\Omega)$, there exists a unique $u_\lambda \in H^1(\Omega)$ solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a \nabla u_\lambda) + (q + \lambda\rho)\, u_\lambda = 0 & \text{in } \Omega \\ u_\lambda = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3.4.2}$$

Then we define the operator $\Lambda_{a,q+\lambda\rho} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ by setting

$$\Lambda_{a,q+\lambda\rho}\varphi := \gamma_1(u_\lambda) := a\frac{\partial u_\lambda}{\partial n}. \tag{3.4.3}$$

Recall that $\Lambda_{a,q+\lambda\rho}$ is bounded and self-adjoint.

As we did previously, we denote by $(\cdot|\cdot)_\rho$ the weighted scalar product defined in (3.2.4), and we recall that the elliptic operator

$$Lu := L_{a,q,\rho}u := -\frac{1}{\rho}\operatorname{div}(a \nabla u) + \frac{q}{\rho}u \tag{3.4.4}$$

with the domain

$$D(L) := D(L_{a,q,\rho}) := \left\{ u \in H_0^1(\Omega)\ ;\ Lu \in L^2(\Omega) \right\}. \tag{3.4.5}$$

is a self-adjoint operator with a compact resolvent on $L^2(\Omega)$ endowed with the scalar product $(\cdot|\cdot)_\rho$, and its associated norm $\|\cdot\|_\rho$. Thus denoting the graph norm of L by

$$\|u\|_{D(L)} := \left(\|u\|_\rho^2 + \|Lu\|_\rho^2 \right)^{1/2}, \tag{3.4.6}$$

the space $D(L)$ endowed with the norm $\|\cdot\|_{D(L)}$ is a Banach space (actually a Hilbert space), and thanks to the J.L. Lions Lemma 1.2.2 there exists a constant $c > 0$ depending on a, q, ρ and Ω such that

$$\forall u \in D(L), \qquad \|\gamma_1(u)\|_{H^{-1/2}} \leq c\|u\|_{D(L)}, \qquad \text{where } \gamma_1(u) := a\frac{\partial u}{\partial n}. \tag{3.4.7}$$

We begin by proving the analogous of Lemma 3.3.1, that is the fact that $\|u_\lambda\|_{L^2}$ is small as $\lambda \to \infty$.

3.4.1 Lemma. *Let $\lambda \geq \lambda_*$ and $\varphi \in H^{1/2}(\partial\Omega)$. Then $u_\lambda \in H^1(\Omega)$ being solution to (3.4.2), we have*

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{L^2(\Omega)} = 0.$$

Proof. Multiplying equation (3.4.2) by an eigenfunction $\varphi_{k,i}$ and integrating by parts twice one finds that

$$\begin{aligned} -\langle \gamma_1(\varphi_{k,i}), \varphi\rangle &= \int_\Omega (a\,\nabla u\cdot\nabla\varphi_{k,i} + (q+\lambda\rho)\,u\varphi_k)\,dx \\ &= \int_\Omega (-\operatorname{div}(a\,\nabla\varphi_{k,i}) + (q+\lambda\rho)\,\varphi_{k,i})\,u_\lambda dx \\ &= (\lambda_k+\lambda)\,(u_\lambda|\varphi_{k,i})_\rho \end{aligned}$$

where we use the scalar product $(\cdot|\cdot)_\rho$ defined in (3.2.4) on $L^2(\Omega)$. Thus we have

$$(u_\lambda|\varphi_{k,i})_\rho = -\frac{\langle \gamma_1(\varphi_{k,i}), \varphi\rangle}{\lambda_k+\lambda}, \tag{3.4.8}$$

and

$$u_\lambda = -\sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{\langle \gamma_1(\varphi_{k,i}), \varphi\rangle}{\lambda_k+\lambda}\varphi_{k,i}.$$

Now for $\lambda \geq \lambda_*$ we have

$$\left|\frac{\langle \gamma_1(\varphi_{k,i}), \varphi\rangle}{\lambda_k+\lambda}\right|^2 \leq \left|\frac{\langle \gamma_1(\varphi_{k,i}), \varphi\rangle}{\lambda_k+\lambda_*}\right|^2,$$

and since each term of the left hand side of the above inequality converges to zero, and since

$$\sum_{k\geq 1}\sum_{i=1}^{m_k}\left|\frac{\langle \gamma_1(\varphi_{k,i}), \varphi\rangle}{\lambda_k+\lambda_*}\right|^2 < \infty,$$

using Lebesgue dominated convergence theorem we conclude that

$$\lim_{\lambda\to\infty}\|u_\lambda\|^2 = \lim_{\lambda\to\infty}\sum_{k\geq 1}\sum_{i=1}^{m_k}\left|\frac{\langle \gamma_1(\varphi_{k,i}), \varphi\rangle}{\lambda_k+\lambda}\right|^2 = 0.$$

Compare the following to Lemma 3.3.2.

3.4.2 Lemma. *Let $\lambda \geq \lambda_*$ and $\varphi \in H^{1/2}(\partial\Omega)$, and $\varphi \not\equiv 0$. Then $u_\lambda \in H^1(\Omega)$ being solution to (3.4.2), we have*

$$\lim_{\lambda\to+\infty} \|u_\lambda\|_{H^1(\Omega)} = +\infty.$$

Proof. Indeed, otherwise if the family $(u_\lambda)_{\lambda\geq\lambda_*}$ were bounded, there would exist a subsequence $(u_{\lambda^{(n)}})_{n\geq 1}$ with $\lambda^{(n)} \to +\infty$ and $u^* \in H^1(\Omega)$ so that $u_{\lambda^{(n)}} \rightharpoonup u^*$ weakly in $H^1(\Omega)$. Since $u_{\lambda^{(n)}} \to 0$ strongly in $L^2(\Omega)$, according to Lemma 3.4.1, this would imply $u^* = 0$. However, the trace operator being linear and continuous from $H^1(\Omega)$ into $H^{1/2}(\partial\Omega)$, we would have $\varphi = \gamma_0(u_{\lambda^{(n)}}) \rightharpoonup 0$ in $H^{1/2}(\partial\Omega)$, while we have assumed that $\varphi \neq 0$. □

Note that the above result means that $\|\nabla u_\lambda\|_{L^2(\Omega)} \to +\infty$ as $\lambda \to +\infty$. As a matter of fact, even though we are not going to use the following result in our study of inverse spectral problem, since the result is a good example of *localization* of a solution to a boundary value problem inside Ω, we are going to present the proof of the fact that *inside* Ω and *far from* the boundary $\partial\Omega$, the gradient $|\nabla u_\lambda|$ converges to zero. Recall that we denote the diameter of Ω by

$$\operatorname{diam}(\Omega) := \sup_{x,y\in\Omega} |x-y|.$$

3.4.3 Lemma. *Assume that $\varphi \neq 0$ and let $0 < \varepsilon < \operatorname{diam}(\Omega)/2$ and define*

$$\Omega_\varepsilon := \left\{x \in \Omega \,;\, \operatorname{dist}(x, \Omega^c) < \varepsilon\right\}.$$

Then

$$\lim_{\lambda\to+\infty} \int_{\Omega\setminus\Omega_\varepsilon} |\nabla u_\lambda|^2 dx = 0.$$

Proof. Consider a function $\zeta \in C_c^\infty(\Omega)$ such that $0 \leq \zeta \leq 1$ and

$$\zeta(x) = 1 \text{ in } \Omega\setminus\Omega_\varepsilon, \qquad \zeta(x) = 0 \text{ in } \Omega_{\varepsilon/2}.$$

Multiplying equation (3.4.2) by $\zeta^2 u_\lambda$ and integrating by parts we obtain

$$\begin{aligned}\int_\Omega a|\nabla u_\lambda|^2 dx + \int_\Omega (q+\lambda\rho)|u|^2 dx &= -2\int_\Omega a\zeta\,\nabla\zeta\cdot u_\lambda\nabla u_\lambda dx \\ &\leq c\|u_\lambda\|_{L^2}\left(\int_\Omega a\zeta^2|\nabla u_\lambda|^2 dx\right)^{1/2}.\end{aligned}$$

Therefore, since $q+\lambda\rho \geq 1$, and $\zeta = 1$ on $\Omega\setminus\Omega_\varepsilon$, we have

$$\int_{\Omega\setminus\Omega_\varepsilon} |\nabla u_\lambda|^2 dx \leq \int_\Omega a\zeta^2|\nabla u_\lambda|^2 dx \leq c\|u_\lambda\|_{L^2}^2$$

from which the claimed result of the Lemma follows, since $\|u_\lambda\|_{L^2} \to 0$ as $\lambda \to +\infty$. □

With the same approach one can show the following result :

3.4.4 Lemma. *Assume that $\sigma_* \in \partial\Omega$ and $\varphi \neq 0$ is such that for a ball $B(\sigma_*, R)$ we have* $\mathrm{supp}(\varphi) \subset B(\sigma_*, R) \cap \partial\Omega$. *Then setting $B := B(\sigma_*, R)$, we have*

$$\lim_{\lambda \to +\infty} \int_{\Omega \setminus B} |\nabla u_\lambda|^2 \, dx = 0.$$

Since the proof is similar to that of our previous lemma, we may leave it to the reader. □

In order to overcome the difficulty arising from the unboundedness of u_λ in $H^1(\Omega)$, we are going to establish some identities which will be used later.

3.4.5 Lemma. *Assume that $\varphi \neq 0$ and denote by $\varphi_1 \in H_0^1(\Omega)$ the first eigenfunction of the operator $(L, D(L))$ defined in (3.4.4) and (3.4.5), with $\|\varphi_1\|_\rho = 1$ and $\varphi_1 > 0$ in Ω.Then we have*

(3.4.9)
$$\int_\Omega a\varphi_1 |\nabla u_\lambda|^2 \, dx + \frac{1}{2} \int_\Omega (q + (2\lambda + \lambda_1)\rho)\, \varphi_1 |u_\lambda|^2 \, dx = -\frac{1}{2} \int_{\partial\Omega} a(\sigma) \frac{\partial \varphi_1(\sigma)}{\partial n} \varphi(\sigma)^2 \, d\sigma,$$

and

(3.4.10)
$$\int_\Omega a(x)\, \varphi_1^2 |\nabla u_\lambda|^2 \, dx + (\lambda + \lambda_1) \int_\Omega \varphi_1^2 |u_\lambda|^2 \rho \, dx = \int_\Omega a(x)\, |u_\lambda(x)|^2 \, |\nabla \varphi_1(x)|^2 \, dx,$$

Proof. Indeed multiplying (3.4.2) by $\varphi_1 u_\lambda$ and integrating by parts we get

(3.4.11)
$$\int_\Omega a\varphi_1 |\nabla u_\lambda|^2 \, dx + \int_\Omega a u_\lambda \nabla u_\lambda \cdot \nabla \varphi_1 \, dx + \int_\Omega (q + \lambda\rho)\, \varphi_1 u_\lambda^2 \, dx = 0.$$

The second term in the above equality can be written

$$\int_\Omega a u_\lambda \nabla u_\lambda \cdot \nabla \varphi_1 \, dx = \frac{1}{2} \int_\Omega \nabla\left(u_\lambda^2\right) \cdot a \nabla \varphi_1 \, dx = \frac{1}{2} \int_{\partial\Omega} u_\lambda^2(\sigma)\, a(\sigma) \frac{\partial \varphi_1(\sigma)}{\partial n} \, d\sigma - \frac{1}{2} \int_\Omega u_\lambda^2 \mathrm{div}(\nabla \varphi_1) \, dx.$$

Now φ_1 being an eigenfunction we have

$$\operatorname{div}(a\nabla\varphi_1) = (q-\lambda_1\rho)\varphi_1, \tag{3.4.12}$$

and also $u_\lambda = \varphi$ on $\partial\Omega$. Therefore reporting these into (3.4.11) we deduce (3.4.9).

In order to obtain (3.4.10) we multiply (3.4.2) by $\varphi_1^2 u_\lambda$ and integrate by parts to obtain

$$\int_\Omega a\varphi_1^2|\nabla u_\lambda|^2 dx + 2\int_\Omega u_\lambda \nabla u_\lambda\cdot\varphi_1 a\nabla\varphi_1 dx + \int_\Omega (q+\lambda\rho)\varphi_1^2 u_\lambda^2 dx = 0. \tag{3.4.13}$$

Again, the second term in the above equality can be written as

$$\begin{aligned} 2\int_\Omega u_\lambda\nabla u_\lambda\cdot\varphi_1 a\nabla\varphi_1 dx &= \int_\Omega \nabla\left(u_\lambda^2\right)\cdot\varphi_1 a\nabla\varphi_1 dx \\ &= -\int_\Omega u_\lambda^2 a|\nabla\varphi_1|^2 dx - \int_\Omega u_\lambda^2\varphi_1\operatorname{div}(a\nabla\varphi_1)dx, \end{aligned}$$

where we have performed an integration by parts and used the fact that $\varphi_1 = 0$ on the boundary. Using again (3.4.12) and reporting the result into (3.4.13) one sees that (3.4.10) holds. □

Next we compare the functions u_λ and u_{λ_*}, as we did in Lemma 3.3.3.

3.4.6 Lemma. *Set $z_\lambda := u_\lambda - u_{\lambda_*}$ for $\lambda \geq \lambda_*$. Then we have*

(3.4.14)

$$\gamma_1(z_\lambda) = -\sum_{k\geq 1}\sum_{i=1}^{m_k}\frac{(\lambda-\lambda_*)\langle\gamma_1(\varphi_{k,i}),\varphi\rangle}{(\lambda_k+\lambda)(\lambda_k+\lambda_*)}\gamma_1(\varphi_{k,i})$$

Proof. One verifies that z_λ satisfies the equation

$$\begin{cases} -\operatorname{div}(a\nabla z_\lambda) + (q+\lambda\rho)z_\lambda = -(\lambda-\lambda_*)u_{\lambda_*} & \text{in } \Omega \\ z_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4.15}$$

Multiplying this equation by $\varphi_{k,i}$ and integrating by parts twice, we see that

$$(\lambda_k+\lambda)(z_\lambda|\varphi_{k,i})_\rho = -(\lambda-\lambda_*)(u_{\lambda_*}|\varphi_{k,i})_\rho,$$

and hence, using (3.4.8) for $\lambda = \lambda_*$,

$$z_\lambda = -\sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{(\lambda-\lambda_*)\,\langle \gamma_1(\varphi_{k,i}),\varphi\rangle}{(\lambda+\lambda_k)\,(\lambda_*+\lambda_k)}\,\varphi_{k,i}. \tag{3.4.16}$$

Using the graph norm of the operator $(L, D(L))$ defined in (3.4.6) one sees that as a matter of fact the series in (3.4.16) converges in $D(L)$, and thus, $\gamma_1 : D(L) \longrightarrow H^{-1/2}(\partial\Omega)$ being a continuous linear operator, thanks to (3.4.7), we obtain (3.4.15). □

3.4.7 Lemma. *For $j = 0, 1$ assume that $a_j, \rho_j \in \mathcal{A}_{\mathrm{ad}}$ and $q_j \in L^\infty(\Omega)$. Fix $\lambda_* \in \mathbb{R}$ such that $q_j + \lambda_*\rho_j \geq 1$. Assuming moreover that $a_j \in W^{1,\infty}(\Omega)$, if*

$$\mathrm{Nbsd}(a_0, q_0, \rho_0) = \mathrm{Nbsd}(a_1, q_1, \rho_1),$$

then for all $\lambda \geq \lambda_*$ we have

(3.4.17)

$$\Lambda_{a_0,q_0+\lambda_*\rho_0} - \Lambda_{a_1,q_1+\lambda_*\rho_1} = \Lambda_{a_0,q_0+\lambda\rho_0} - \Lambda_{a_1,q_1+\lambda\rho_1}.$$

Proof. As we observed in the proof of Lemma 3.3.4, when $a_j \in W^{1,\infty}(\Omega)$, we have

$$\dim \mathrm{span}\Big\{\gamma_1(\varphi_{j,k,i})\ ;\ 1 \leq i \leq m_k\Big\} = m_k,$$

and since for each k we have

$$\mathrm{span}\Big\{\gamma_1(\varphi_{0,k,i})\ ;\ 1 \leq i \leq m_k\Big\} = \mathrm{span}\Big\{\gamma_1(\varphi_{1,k,i})\ ;\ 1 \leq i \leq m_k\Big\}$$

we may assume that a common basis for these spaces have been chosen so that for all $k \geq 1$

$$\gamma_1(\varphi_{0,k,i}) = \gamma_1(\varphi_{1,k,i}) \qquad \text{for } 1 \leq i \leq m_k. \tag{3.4.18}$$

Next, fix a $\varphi \in H^{1/2}(\partial\Omega)$ and let $u_{j,\lambda}$ solve

$$\begin{cases} -\mathrm{div}(a_j\nabla u_{j,\lambda}) + (q_j+\lambda\rho_j)\,u_{j,\lambda} = 0 & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad u_{j,\lambda} = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.4.19}$$

for $j = 0$ and $j = 1$. Upon setting

$$z_{j,\lambda} := u_{j,\lambda} - u_{j,\lambda_*},$$

thanks to Lemma 3.4.6 we know that

(3.4.20)

$$\gamma_1(z_{j,\lambda}) = -\sum_{k\geq 1}\sum_{i=1}^{m_k} \frac{\lambda-\lambda_*\Big\langle \gamma_1(\varphi_{j,k,i}),\varphi\Big\rangle}{(\lambda_k+\lambda)\,(\lambda_k+\lambda_*)}\,\gamma_1(\varphi_{j,k,i}).$$

Now using (3.4.18) we see that, as a matter of fact, the right hand side of the above expression is the same for $j = 0$ and $j = 1$. Thus for all $\lambda \geq \lambda_*$ we have

$$\gamma_1(z_{0,\lambda}) = \gamma_1(z_{1,\lambda})$$

which means that

$$\left(\Lambda_{a_0,q_0+\lambda\rho_0} - \Lambda_{a_0,q_0+\lambda_*\rho_0}\right)(\varphi) = \left(\Lambda_{a_1,q_1+\lambda\rho_1} - \Lambda_{a_1,q_1+\lambda_*\rho_1}\right)(\varphi)$$

for all $\varphi \in H^{1/2}(\partial\Omega)$, from which (3.4.17) follows. □

Due to the fact that $\left\|\nabla u_{j,\lambda}\right\|_{L^2}$ is not bounded, in order to prove that from the equality $\mathrm{Nbsd}(a_0, q_0, \rho_0) = \mathrm{Nbsd}(a_1, q_1, \rho_1)$ we deduce the equality $\Lambda_{a_0,q_0+\lambda\rho_0} = \Lambda_{a_1,q_1+\lambda\rho_1}$ we need further assumptions on $a_0 - a_1$ and $\rho_0 - \rho_1$ on the boundary.

3.4.8 Lemma. *Assume that for $j = 0, 1$ the coefficients $a_j, \rho_j \in \mathcal{A}_{ad}$, and $a_j \in W^{1,\infty}(\Omega)$, $q_j \in L^\infty(\Omega)$. Moreover assume that for some constant $c > 0$*

(3.4.21) $$\forall x \in \Omega, \qquad |a_0(x) - a_1(x)| \leq c(\mathrm{dist}(x, \partial\Omega))^2,$$

and

(3.4.22) $$\forall x \in \Omega, \qquad |\rho_0(x) - \rho_1(x)| \leq c(\mathrm{dist}(x, \partial\Omega))^2.$$

Then

(3.4.23) $$\lim_{\lambda\to+\infty} \int_\Omega (a_0 - a_1)\, \nabla u_{0,\lambda} \cdot \nabla u_{1,\lambda} dx = 0,$$

and

(3.4.24) $$\lim_{\lambda\to+\infty} \lambda \int_\Omega (\rho_0 - \rho_1)\, u_{0,\lambda} u_{1,\lambda} dx = 0.$$

Proof. With the notations of Lemma 3.4.5, thanks to (3.4.10), for some constant $c > 0$ and all $\lambda \geq \lambda_*$, we have

$$\int_\Omega a(x)\, \varphi_1(x)^2\, |\nabla u_\lambda(x)|^2 dx \leq c\|u_\lambda\|^2_{L^2(\Omega)},$$

from which, upon using Lemma 3.4.1, we infer that

(3.4.25) $$\lim_{\lambda\to+\infty} \int_\Omega a(x)\, \varphi_1(x)^2\, |\nabla u_\lambda(x)|^2 dx = 0.$$

Here we are assuming moreover that $a \in W^{1,\infty}(\Omega)$, so by Lemma 1.2.10 we have

(3.4.26) $$c_1 \mathrm{dist}(x, \partial\Omega) \geq \varphi_1(x) \geq c_0 \mathrm{dist}(x, \partial\Omega).$$

Therefore, using (3.4.25) we have

$$\lim_{\lambda \to +\infty} \int_\Omega (\mathrm{dist}(x, \partial\Omega))^2 |\nabla u_\lambda(x)|^2 dx = 0.$$

In particular, using (3.4.21), for $j = 0$ or $j = 1$ we have

$$\lim_{\lambda \to +\infty} \int_\Omega |a_0(x) - a_1(x)| \left|\nabla u_{j,\lambda}(x)\right|^2 dx = 0.$$

It is clear that since

$$\left|\int_\Omega (a_0 - a_1)\, \nabla u_0 \cdot \nabla u_1 dx\right| \leq \left(\int_\Omega |a_0 - a_1|\, |\nabla u_{0,\lambda}|^2 dx\right)^{1/2} \times \left(\int_\Omega |a_0 - a_1|\, |\nabla u_{1,\lambda}|^2 dx\right)^{1/2},$$

the result claimed in (3.4.23) follows.

In order to prove (3.4.24), again using the notations of Lemma 3.4.5 and the equality (3.4.10), we have, for some constante $c > 0$,

$$(\lambda + \lambda_1) \int_\Omega \varphi_1(x)^2 |u_\lambda(x)|^2 \rho(x)\, dx \leq c \|u_\lambda\|^2_{L^2},$$

so that, thanks to Lemma 3.4.1, we have

(3.4.27) $$\lim_{\lambda \to +\infty} \lambda \int_\Omega \varphi_1(x)^2 |u_\lambda(x)|^2 \rho(x)\, dx = 0.$$

At this point, using the assumption (3.4.22) together with (3.4.26), for $j = 0$ or $j = 1$ thanks to the above observations we have

$$\lim_{\lambda \to +\infty} \lambda \int_\Omega |\rho_0(x) - \rho_1(x)| \left|u_{j,\lambda}(x)\right|^2 dx = 0.$$

It is clear that using this and the Cauchy-Schwarz inequality

$$\left|\int_\Omega (\rho_0 - \rho_1)\, u_0 u_1 dx\right| \leq \left(\int_\Omega |\rho_0 - \rho_1|\, |u_{0,\lambda}|^2 dx\right)^{1/2} \times \left(\int_\Omega |\rho_0 - \rho_1|\, |u_{1,\lambda}|^2 dx\right)^{1/2},$$

implies (3.4.24). □

We can now state the analogous of Theorem 3.3.5.

3.4.9 Theorem. *For $j = 0, 1$ assume that $a_j, \rho_j \in \mathcal{A}_{\mathbf{ad}}$, with $a_j \in W^{1,\infty}(\Omega)$, and $q_j \in L^\infty(\Omega)$. Fix $\lambda_* \in \mathbb{R}$ such that $q_j + \lambda_* \rho_j \geq 1$. Assume also that conditions (3.4.21) and (3.4.22) are satisfied, and*

$$\mathrm{Nbsd}(a_0, q_0, \rho_0) = \mathrm{Nbsd}(a_1, q_1, \rho_1).$$

Then for all $\lambda \geq \lambda_*$ we have

(3.4.28) $$\Lambda_{a_0, q_0+\lambda\rho_0} = \Lambda_{a_1, q_1+\lambda\rho_1}.$$

Proof. With the above notations we know that for a fixed $\varphi \in H^{1/2}(\partial\Omega)$ and all $\lambda \geq \lambda_*$, using (3.4.17) we have

(3.4.29)
$$\left(\Lambda_{a_0, q_0+\lambda_*\rho_0} - \Lambda_{a_1, q_1+\lambda_*\rho_1}\right)\varphi = \left(\Lambda_{a_0, q_0+\lambda\rho_0} - \Lambda_{a_1, q_1+\lambda\rho_1}\right)\varphi.$$

On the other hand we have :

$$\begin{aligned}\left\langle \left(\Lambda_{a_0, q_0+\lambda\rho_0} - \Lambda_{a_1, q_1+\lambda\rho_1}\right)\varphi, \varphi \right\rangle &= \int_\Omega (a_0 - a_1)\, \nabla u_{0,\lambda} \cdot \nabla u_{1,\lambda} dx \\ &+ \int_\Omega (q_0 - q_1)\, u_{0,\lambda} u_{1,\lambda} dx \\ &+ \lambda \int_\Omega (\rho_0 - \rho_1)\, u_{0,\lambda} u_{1,\lambda} dx.\end{aligned}$$

Using (3.4.23), Lemma 3.4.1 and (3.4.24) we know that each of the three terms on the right hand side of the above equality converges to zero as $\lambda \to +\infty$. Therefore using this together with (3.4.29) we infer that for all $\varphi \in H^{1/2}(\partial\Omega)$ we have

$$\left\langle \left(\Lambda_{a_0, q_0+\lambda_*\rho_0} - \Lambda_{a_1, q_1+\lambda_*\rho_1}\right)\varphi, \varphi \right\rangle = 0.$$

Since the above equality is valid for all $\varphi \in H^{1/2}(\partial\Omega)$ and the operator

$$\Lambda_{a_0, q_0+\lambda_*\rho_0} - \Lambda_{a_1, q_1+\lambda_*\rho_1}$$

is self-adjoint, we conclude that

$$\Lambda_{a_0, q_0+\lambda_*\rho_0} - \Lambda_{a_1, q_1+\lambda_*\rho_1} = 0$$

and the proof of the Theorem is complete. □

The result concerning the inverse spectral problem for the Dirichlet boundary value problem, is analogous to Theorem 3.3.6, that is the knowledge of the Neumann boundary spectral data implies that of two coefficients out of three :

3.4.10 Theorem. *For $j = 0, 1$ let $a_j, \rho_j \in \mathcal{A}_{\mathbf{ad}}$ and $q_j \in L^\infty(\Omega)$ be given and moreover assume that $a_j \in W^{2,\infty}(\Omega)$, and conditions (3.4.21) and (3.4.22) are satisfied. Then if*

$$\mathrm{Nbsd}(a_0, q_0, \rho_0) = \mathrm{Nbsd}(a_1, q_1, \rho_1),$$

we have :

1) *if $a_0 \equiv a_1$, then $q_0 \equiv q_1$ and $\rho_0 \equiv \rho_1$.*
2) *If $q_0 \equiv q_1$, then $a_0 \equiv a_1$ and $\rho_0 \equiv \rho_1$.*
3) *If $\rho_0 \equiv \rho_1$, then $a_0 \equiv a_1$ and $q_0 \equiv q_1$.*

Proof. The proof follows that of Theorem 3.3.6 and can be omitted. □

BIBLIOGRAPHY

[1] R.A. ADAMS : *Sobolev Spaces*, Academic Press, New York (NY), USA, (1975).

[2] V. ADOLFSSON & L. ESCAURIAZA : *$C^{1,\alpha}$ domains and unique continuation at the boundary*, Comm. Pure Appl. Math., vol. **50** (n° 10), pp. 935–969, (1997).

[3] G. ALESSANDRINI : *Stable determination of conductivity by boundary measurements*, Applicable Analysis, vol. **27** (n° 1-3), pp. 153–172, (1988).

[4] G. ALESSANDRINI : *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, Journal of Differential Equations, vol. **84** (n° 2), pp. 252–272, (1990).

[5] V. AMBARZUMIAN : *Über eine Frage der Eigenwerttheorie*, Zeitschrift für Physik, n° 53, pp. 690–695, (1929).

[6] C. BENNEWITZ, M. BROWN & R. WEIKARD : *Spectral and Scattering Theory for Ordinary Differential Equations : Vol. I : Sturm–Liouville Equations*, Springer International Publishing, (2020).

[7] E. BLÅSTEN, O.Y. IMANUVILOV & M. YAMAMOTO : *Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials*, Inverse Problems and Imaging, vol. **9** (n° 3), pp. 709–723, (2015).

[8] G. BORG : *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte*, Acta Mathematica, vol. **78**, pp. 1–96, (1946).

[9] A.L. BUKHGEIM : *Recovering a potential from Cauchy data in the two-dimensional case*, jiip, vol. **16** (n° 1), pp. 19–33, (2008).

Chapter 1. Bibliography

[10] A.-P. CALDERÓN : *On an inverse boundary value problem*, Sociedade Brasileira di Matematica, Rio de Janeiro, (1980).

[11] B. CANUTO & O. KAVIAN : *Determining two coefficients in elliptic operators via boundary spectral data : a uniqueness result*, Bollettino della Unione Matematica Italiana. Serie VIII. Sezione B. Articoli di Ricerca Matematica, vol. **7** (n° 1), pp. 207–230, (2004).

[12] I.M. GEL'FAND & B.M. LEVITAN : *On the determination of a differential equation from its spectral function*, American Mathematical Society Translations (2), vol. **1**, pp. 253–304, (1955).

[13] D. GILBARG & N.S. TRUDINGER : *Elliptic partial differential equations of second order* (Second ed., vol. 224), Springer-Verlag, Berlin, (1983).

[14] P. HÄHNER : *A periodic Faddeev-type solution operator*, Journal of Differential Equations, vol. **128** (n° 1), pp. 300–308, (1996).

[15] L.V. HÖRMANDER : *The Analysis of Linear Partial Differential Operators III, Springer.* (vol. III), Springer-Verlag, Berlin, (1983).

[16] O.Y. IMANUVILOV & M. YAMAMOTO : *Inverse Boundary Value Problem for Schrödinger Equation in Two Dimensions*, SIAM Journal on Mathematical Analysis, vol. **44** (n° 3), pp. 1333–1339, (2012).

[17] O. KAVIAN : *Introduction à la théorie des points critiques et applications aux problèmes elliptiques* (vol. 13), Springer-Verlag, Paris, (1993).

[18] A. KIRSCH : *An Introduction to the Mathematical Theory of Inverse Problems* (n° 120), Springer-Verlag, New York, Montréal, (1996).

[19] R.V. KOHN & M. VOGELIUS : *Determining conductivity by boundary measurements*, Communications on Pure and Applied Mathematics, vol. **37** (n° 3), pp. 289–298, (1984).

[20] V.V. KRAVCHENKO : *Direct and Inverse Sturm-Liouville Problems : A Method of Solution*, Springer International Publishing, (2020).

[21] O.A. LADYŽENSKAJA & N.N. URALcprimeCEVA : *Équations aux dérivées partielles de type elliptique*, Dunod, Paris, (1968).

[22] N. LEVINSON : *The inverse Sturm-Liouville problem*, Mat. Tidsskr. B.,, pp. 25–30, (1949).

[23] B.M. LEVITAN : *Inverse Sturm-Liouville problems*, VSP, Zeist, (1987).

[24] J.-L. LIONS : *Problèmes aux Limites dans les équations aux Dérivées Partielles*, Presses de l'Université de Montréal, Montréal, (1965).

[25] J.-L. LIONS & E. MAGENES : *Problèmes aux Limites Non Homogènes et Applications* (vol. 1), Editions Dunod, Paris, Paris, France, (1968).

[26] V. MARCHENKO : *Sturm-Liouville Operators and Applications*, American Mathematical Society, (2011).

[27] V.G. MAZJA : *Sobolev Spaces*, Springer-Verlag, Berlin, Berlin, (1985).

[28] A.I. NACHMAN : *Global uniqueness for a two-dimensional inverse boundary value problem*, Annals of Mathematics. Second Series, vol. **143** (n° 1), pp. 71–96, (1996).

[29] G. STAMPACCHIA : *Équations Elliptiques du Second Ordre à Coefficients Discontinus* (n° 16), Presses de l'Université de Montréal, Montreal, (1965).

[30] J. SYLVESTER & G. UHLMANN : *A global uniqueness theorem for an inverse boundary value problem*, Annals of Mathematics. Second Series, vol. **125** (n° 1), pp. 153–169, (1987).

[31] K. YOSIDA : *Functional Analysis* (n° 123), Springer-Verlag, New York, New York, (1974).

▶ **To complete**…

84

84