Hierarchical control: controls with different objectives

Luz de Teresa,
Universidad Nacional Autónoma de México
DocCourse 2026
Sevilla

---

ldeteresa@im.unam.mx





Abstract Optimal Control Problem

Abstract Setting

Let H, U be Hilbert spaces. Consider:

- A **state equation**: $Ay = Bu$, where $A : H \to H'$, $B : U \to H'$.
- A **cost functional** $\mathcal{J} : U \to \mathbb{R}$, $u \mapsto \mathcal{J}(u)$.

Optimal Control Problem

Find $\hat{u} \in U_{\text{ad}}$ such that

$$\mathcal{J}(\hat{u}) = \min_{u \in U_{\text{ad}}} \mathcal{J}(u),$$

where $U_{\text{ad}} \subseteq U$ is the set of *admissible controls*.

We will work with $U_{\text{ad}} = U$ (unconstrained) and **quadratic** cost functionals, for which the theory is complete.

Existence of a Minimum

Theorem

Let $\mathcal{J} : U \to \mathbb{R}$ be

(i) ***continuous*** *(or lower semicontinuous),*

(ii) ***strictly convex****, and*

(iii) ***coercive****: $\mathcal{J}(u) \to +\infty$ as $\|u\|_U \to +\infty$.*

Then, there exists a unique minimizer $\hat{u} \in U$.

Reference: I. Ekeland, R. Temam, *Convex Analysis and Variational Problems,* SIAM, 1999.

Strategy for PDE problems

1. Verify the three conditions for the given $\mathcal{J}$.
2. Characterize $\hat{u}$ via the **optimality condition**:

$$\mathcal{J}'(\hat{u}) \cdot v = 0 \quad \forall v \in U.$$

3. Express $\hat{u}$ in terms of an **adjoint state**.

Example 1: Heat Equation

Example 1 – Heat Equation

Let Ω be an open, bounded, smooth set, $Q = (0, T) \times \Omega$, $y^0 \in L^2(\Omega)$, $y_d \in L^2(Q)$.

State equation

For a control $h \in L^2(\omega \times (0, T))$, the state $y = y(t, x)$ satisfies

$$\begin{cases} y_t - \Delta y = h 1_\omega & (t, x) \in Q, \\ y(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, x) = y^0(x) & x \in \Omega. \end{cases}$$

Cost functional

$$\mathcal{J}(h) = \frac{1}{2} \int_0^T \int_{\mathcal{O}_d} |y - y_d|^2 + \frac{\alpha}{2} \int_{\omega \times (0,T)} |h|^2, \quad \alpha > 0.$$

Example 1 – Verifying the Hypotheses

Well-posedness. The parabolic theory gives a unique solution $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, and $h \mapsto y(h)$ is *affine* (linear + constant):

$$y(h) = y_0 + Lh,$$

where y_0 solves the uncontrolled equation and L is a bounded linear operator.

Three conditions for $\mathcal{J}$:

- **Continuity.** $\mathcal{J}$ is a composition of continuous maps.
- **Strict convexity.** Since $h \mapsto Lh$ is linear, $h \mapsto \|Lh - y_d\|^2$ is convex; the term $\frac{\alpha}{2}\|h\|^2$ makes it *strictly* convex.
- **Coercivity.** $\mathcal{J}(h) \geq \frac{\alpha}{2}\|h\|^2 \to +\infty$.

$\Rightarrow$ There exists a unique minimizer $\hat{h}$.

Example 1 – Deriving the Necessary Condition

For $h, \ell \in L^2(\omega \times (0,T))$ and $\varepsilon > 0$, let $z = z^\ell$ be the **directional state**: the unique solution to

$$\begin{cases} z_t - \Delta z = \ell\, 1_\omega & \text{in } Q, \\ z = 0 & \text{on } \partial\Omega \times (0,T), \\ z(0,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

Since $y(h + \varepsilon\ell) = y(h) + \varepsilon z$, the Gâteaux derivative is:

$$\mathcal{J}'(h) \cdot \ell = \lim_{\varepsilon \to 0} \frac{\mathcal{J}(h + \varepsilon\ell) - \mathcal{J}(h)}{\varepsilon} = \int_{Q_d} (y - y_d)\, z + \alpha \int_{\omega \times (0,T)} h\ell.$$

Optimality condition: $\hat{h}$ is optimal iff

$$\int_{\mathcal{O}_d \times (0,T)} (y(\hat{h}) - y_d)\, z + \alpha \int_{\omega \times (0,T)} \hat{h}\ell = 0 \quad \forall\, \ell.$$

Example 1 – Introducing the Adjoint State

We want to express $\displaystyle\int_{\mathcal{O}_d\times(0,T)} (y - y_d)\, z$ in terms of ℓ directly.

Introduce the **adjoint state** $\phi = \phi(t, x)$ solving:

Adjoint (backward) equation

$$\begin{cases} -\phi_t - \Delta\phi = (y - y_d)\, 1_{\mathcal{O}_d} & \text{in } Q, \\ \phi(t, x) = 0 & (t, x) \in (0, T) \times \partial\Omega, \\ \phi(T, x) = 0 & x \in \Omega. \end{cases}$$

Key identity (multiply z-equation by ϕ, ϕ-equation by z, integrate by parts in Q):

$$\int_{\mathcal{O}_d\times(0,T)} (y - y_d)\, z = \int_Q (-\phi_t - \Delta\phi)\, z = \int_Q \phi\,(z_t - \Delta z) = \int_{\omega\times(0,T)} \phi\,\ell.$$

Substituting the key identity, the optimality condition becomes:

$$\int_{\omega\times(0,T)} \left(\phi + \alpha\,\hat{h}\right) \ell = 0 \quad \forall\, \ell \in L^2(\omega \times (0,T)).$$

Example 1 – Optimality System

Since ℓ is arbitrary:

Optimal control

$$\boxed{\hat{h} = -\frac{1}{\alpha}\,\phi\big|_{\omega\times(0,T)}}$$

The triple $(y, \phi, \hat{h})$ satisfies the **optimality system**:

$$\begin{cases} y_t - \Delta y = \hat{h}\,1_\omega & \text{(state)}, \\ -\phi_t - \Delta\phi = (y - y_d)\,1_{\mathcal{O}_d} & \text{(adjoint)}, \\ \hat{h} = -\frac{1}{\alpha}\phi|_{\omega\times(0,T)} & \text{(optimality)}. \end{cases}$$

This is a *coupled forward-backward* system for (y, ϕ).

Example 1 – The Coupled System

Substituting $\hat{h} = -\frac{1}{\alpha}\phi$ into the state equation, the optimality system reduces to finding (y, ϕ) solving:

$$\begin{cases} y_t - \Delta y + \dfrac{1}{\alpha}\phi\, 1_\omega = 0 & \text{in } Q, \\ -\phi_t - \Delta\phi - (y - y_d)\, 1_{\mathcal{O}_d} = 0 & \text{in } Q, \\ y(t,x) = \phi(t,x) = 0, (t,x) \in (0,T) \times \partial\Omega \\ y(0,x) = y^0(x), \quad \phi(T,x) = 0. \end{cases}$$

Solvability

This system has a unique solution (y, ϕ). It can be solved by a **fixed-point argument** or by reformulating it as a minimization problem on $L^2(\Omega)$.

Variant: Penalizing the Terminal State

Variant – Penalizing the Terminal State

Traditionally there is a term measuring how far $y(T)$ is from a **desired final state** $y_T \in L^2(\Omega)$, $\alpha, \beta > 0$:

Modified cost functional

$$\mathcal{J}_\beta(h) = \frac{1}{2}\int_0^T \int_{\mathcal{O}_d} |y - y_d|^2 + \frac{\alpha}{2}\int_{\omega\times(0,T)} |h|^2 + \frac{\beta}{2}\int_\Omega |y(T) - y_T|^2.$$

- The new term is still continuous, convex, and does not affect coercivity (the $\frac{\alpha}{2}\|h\|^2$ term dominates).
- Hence $\mathcal{J}_\beta$ admits a **unique minimizer** $\hat{h}$ by the same argument as before.
- The derivation of the necessary condition follows the same steps — only the **adjoint terminal condition** changes.

Variant – Gâteaux Derivative

With the same directional state $z = z^{\ell}$ (zero initial data), the Gâteaux derivative now picks up an extra boundary-in-time term:

$$\mathcal{J}'_{\beta}(h)\cdot\ell = \underbrace{\int_{\mathcal{O}_d\times(0,T)} (y-y_d)\,z}_{\text{tracking}} + \underbrace{\alpha\int_{\omega\times(0,T)} h\,\ell}_{\text{control cost}} + \underbrace{\beta\int_{\Omega}(y(T)-y_T)\,z(T)}_{\text{terminal cost}}.$$

Optimality condition at $\hat{h}$:

$$\int_{\mathcal{O}_d\times(0,T)} (y-y_d)\,z + \beta\int_{\Omega}(y(T)-y_T)\,z(T) + \alpha\int_{\omega\times(0,T)} \hat{h}\,\ell = 0 \quad \forall\,\ell.$$

We must absorb *both* integrals involving z into a single duality pairing with ℓ — this determines the new adjoint.

Variant – Modified Adjoint Equation

Introduce the adjoint state ϕ solving:

Modified adjoint equation

$$\begin{cases} -\phi_t - \Delta\phi = (y - y_d)\,1_{\mathcal{O}_d} & \text{in } Q, \\ \phi(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ \boxed{\phi(T,x) = \beta\,(y(T,x) - y_T(x))} & x \in \Omega. \end{cases}$$

Here we obtain,

$$\hat{h} = -\frac{1}{\alpha}\,\phi|_{\omega\times(0,T)}.$$

Variant – Optimality System

The only difference lies in the adjoint terminal condition:

	Original ($\beta = 0$)	**Variant** ($\beta > 0$)
Adjoint eq.	$-\phi_t - \Delta\phi = (y - y_d)1_{\mathcal{O}_d}$	$-\phi_t - \Delta\phi = (y - y_d)1_{\mathcal{O}_d}$
Terminal cond.	$\phi(T, x) = 0$	$\phi(T, x) = \beta\,(y(T) - y_T)$
Optimal control	$\hat{h} = -\frac{1}{\alpha}\phi\|_\omega$	$\hat{h} = -\frac{1}{\alpha}\phi\|_\omega$

Remark

As $\beta \to +\infty$, we force $y(T) \to y_T$: this is the link between optimal control and **approximate controllability**.

NULL CONTROLLABILITY

Null controllability

$$(Hc) \qquad \begin{cases} y_t - \Delta y = h\chi_\omega & (t,x) \in Q, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ y(0,x) = y^0, & x \in \Omega. \end{cases}$$

- Target: find $h \in L^2(0,T,L^2(\omega))$ such that

$$\boxed{y(T) = 0}.$$

Observability inequality

Lemma

The adjoint equation

$$(Adj)\begin{cases} v_t + \Delta v = 0 \quad (t,x) \in \Omega_T, \\ v(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega \\ v(T,x) = v^T, \quad x \in (0,\pi). \end{cases}$$

Then (Hc) is null controllable iff there exists $C > 0$ such that v any solution to (Adj) satisfies

$$\int_\Omega |v(0,x)|^2 dx \leq C \int_0^T \int_\omega |v(t,x)|^2 dx dt.$$

Proof: functional analysis.

Proof.

Define

$$P : h \in L^2(Q_T) \longmapsto Pv = y(T; h, 0) \in H,$$

and

$$G : y_0 \in H \longmapsto G(y_0) = y(T; 0, y_0) \in H.$$

Control to zero $\Longleftrightarrow \boxed{R(G) \subset R(P)} \Longleftrightarrow$ there exists $C > 0$ such that

$$\|G^*(\varphi_T)\|_H \leq C\|P^*(\varphi_T)\|_{L^2(Q_T)}, \quad \forall \varphi_T \in H.$$

$$G^*(\varphi_T) = v(\cdot, 0) \quad \text{y} \quad P^*(\varphi_T) = v1_\omega$$

v solution to the adjoint system. □

Consequence of the closed graph theorem. See e.g. Tucsnak-Weiss: Observation and Control for Operator Semigroups (Birkhäuser, 2009)

Minimization

Given $y^0 \in L^2(0,\pi)$ we define

$$J(v^T) = \frac{1}{2}\int_0^T\int_\omega |v|^2 dxdt + \int_0^T y^0(x)v(0,x)dx.$$

Observability inequality $\Rightarrow$ existence of a minimum $\hat{v}^T$

$$\begin{cases} y_t - \Delta y = \hat{v}\chi_\omega & \hat{v}_t + \Delta\hat{v} = 0 \quad (t,x) \in \Omega_T, \\ y(t,0) = y(t,\pi) = 0, & \hat{v}(t,0) = \hat{v}(t,\pi) = 0, \quad t \in (0,T), \\ y(0,x) = y^0, \boxed{y(T) = 0} & \hat{v}(T,x) = \hat{v}^T, \quad x \in \Omega. \end{cases}$$

Carleman inequalities



Carleman inequalities are weighted inequalities which relate a differential operator with the local weighted norm of the solution.

Carleman inequalities



Carleman inequalities are weighted inequalities which relate a differential operator with the local weighted norm of the solution.

Carleman in 1939, introduced energy estimates with exponential weights to show uniqueness of solutions to PDE's with smooth coefficients on subsets of $\mathbb{R}^2$.

Carleman inequalities

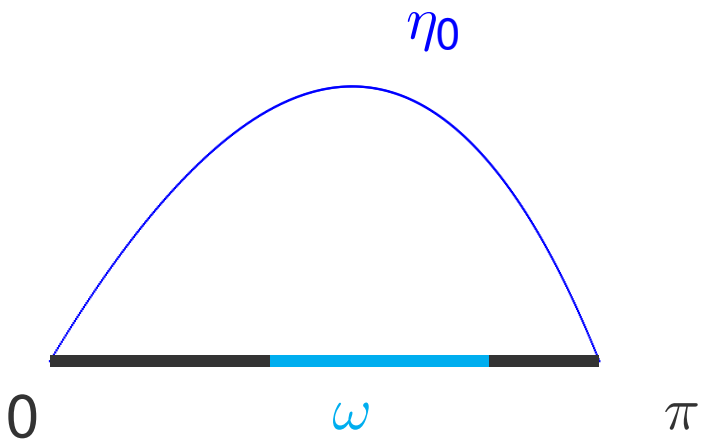


Carleman inequalities are weighted inequalities which relate a differential operator with the local weighted norm of the solution.

Carleman in 1939, introduced energy estimates with exponential weights to show uniqueness of solutions to PDE's with smooth coefficients on subsets of $\mathbb{R}^2$.

Nowadays this kind of inequalities have been generalized and are a very useful technique for inverse and control problems.

Kind of function



Kind of function



Kind of function

Theorem (Fursikov-Imanuvilov)

Let $\Omega \subset \mathbb{R}^n$ be a open and bounded set of class C^2. Let $\omega \subset \Omega$ be a non empty open set and B_δ an open ball centered at $x_0 \in \omega$ with $B_\delta \subset \omega$. Then there exists $\eta_0 \in C^\infty(\overline{\Omega_T})$ such that $\eta_0(x) > 0$ in Ω, $\frac{\partial \eta_0}{\partial \nu} < 0$ on $\partial\Omega$, $|\nabla \eta_0| > 0$ in $\Omega \backslash B_\delta$.

Carleman: weighted inequality

Given η_0 as before, we define

$$\alpha(x,t) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))} - e^{2\lambda\|\eta_0\|_\infty}}{t(T-t)}, \qquad (1)$$

$$\xi(x,t) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))}}{t(T-t)}, \qquad \rho(x,t) = e^{\alpha(x,t)},$$

with $\lambda > 0$.

Key fact $\lim_{s\to 0^+, T^-} \rho^{-1} = 0.$

Now..... Carleman inequality

$$\begin{cases} v_t + \Delta v = F_0 & \text{in } \Omega_T, \\ v = 0 & \text{on } \Sigma, \\ v(T,x) = v_T(x) & \text{in } \Omega, \end{cases}$$

with $v_T \in L^2(\Omega)$, $F_0 \in L^2(\Omega_T)$,

Now..... Carleman inequality

$$\begin{cases} v_t + \Delta v = F_0 & \text{in } \Omega_T, \\ v = 0 & \text{on } \Sigma, \\ v(T,x) = v_T(x) & \text{in } \Omega, \end{cases}$$

with $v_T \in L^2(\Omega)$, $F_0 \in L^2(\Omega_T)$,

Theorem

$\Omega \subset \mathbb{R}^n$ smooth. There exists constants s_0, λ_0 and C such that, for every $s \geq s_0$ and $\lambda \geq \lambda_0$, the solutions satisfy

$$\iint_{\Omega_T} \rho^{-2s}(s\lambda^2\xi|\nabla v|^2 + s^3\lambda^4\xi^3 v^2) \leq C\left(s^3\lambda^4 \int_0^T\int_\omega \rho^{-2s}\xi^3|v|^2 + \iint_{\Omega_T} \rho^{-2s}|F_0|^2\right).$$

Observability inequality from Carleman

For every $s \geq s_0(T + T^2)$. We got lower and upper bounds:

$$\rho^{-2s}(x,t)\xi^3(x,t) \geq e^{-2C(1+1/T)}\frac{1}{T^6} \quad \text{in} \quad \boxed{\Omega \times (T/4, 3T/4),}$$

$$\rho^{-2s}(x,t)\xi^3(x,t) \leq M \quad \text{in} \quad \Omega \times (0,T).$$

Then,

$$\iint\limits_{\Omega\times(T/4,3T/4)} |v|^2 \leq C \iint\limits_{\omega\times(0,T)} |v|^2$$

Classical energy estimates give:

$$\|v(0)\|^2_{L^2(\Omega)} \leq \frac{2}{T} \iint\limits_{\Omega\times(T/4,3T/4)} |v|^2$$

GAME THEORY CONCEPTS

- Concept from game theory (Gabriel Cramer 1728, Daniel Bernouilli 1738).

- Concept from game theory (Gabriel Cramer 1728, Daniel Bernouilli 1738).
- The leader (first player) does a movement. The follower (second player) reacts trying to win or optimize the response to the leader movement.

- Concept from game theory (Gabriel Cramer 1728, Daniel Bernouilli 1738).
- The leader (first player) does a movement. The follower (second player) reacts trying to win or optimize the response to the leader movement.
- Historical papers due to J. von Neumann and O. Morgenstern (1943) and Nash (1950). [What is now known as Nash equilibria is due to Cournot (1838)].

ON THE THEORY OF GAMES OF STRATEGY[1]

John von Neumann

[A translation by Mrs. Sonya Bargmann of "Zur Theorie der Gesellschaftsspiele," Mathematische Annalen 100 (1928), pp. 295-320.]

INTRODUCTION

1. The present paper is concerned with the following question:

> n players $S_1, S_2, \ldots, S_n$ are playing a given game of strategy, Γ. How must one of the participants, S_m, play in order to achieve a most advantageous result?

The problem is well known, and there is hardly a situation in daily life into which this problem does not enter. Yet, the meaning of this question is not unambiguous. For, as soon as $n > 1$ (i.e., Γ is a game of strategy in the proper sense), the fate of each player depends not only on his own actions but also on those of the others, and their behavior is motivated by the same selfish interests as the behavior of the first player. We feel that the situation is inherently circular.

Hence we must first endeavor to find a clear formulation of the question. What, exactly, is a game of strategy? A great many different things come under this heading, anything from roulette to chess, from baccarat to bridge. And after all, any event - given the external conditions and the participants in the situation (provided the latter are acting of their own free will) - may be regarded as a game of strategy if one looks at the effect it has on the participants.[2] What element do all these things have in common?

EQUILIBRIUM POINTS IN N-PERSON GAMES

By John F. Nash, Jr.*

Princeton University

Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, **8**, 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

Instituto de Matemáticas



- Several concepts.

- Several concepts.
- Two players (controls) one leader, one follower. Called Stackelberg strategy.

- Several concepts.
- Two players (controls) one leader, one follower. Called Stackelberg strategy.
- Several players: Nash equilibria, Pareto strategy etc.....

- Several concepts.
- Two players (controls) one leader, one follower. Called Stackelberg strategy.
- Several players: Nash equilibria, Pareto strategy etc.....
- We concentrate on Stackelberg strategy.

HIERARCHIC CONTROL

Hierarchic control [Lions94]

We consider the heat equation

$$\begin{cases} y_t - \Delta y = h\chi_\omega + v\chi_{\mathcal{O}} & \text{in } Q = \Omega \times (0,T) \\ y = 0 & \text{on } \Sigma \\ y(0,x) = y^0(x) & \text{in } \Omega \end{cases} \qquad \text{(Cal)}$$

h v

Two different objectives:

- OPTIMAL CONTROL: $y \approx y_d$ in $\mathcal{O}_d \times (0,T)$

$$\min_v \frac{1}{2} \iint_{\mathcal{O}_d \times (0,T)} |y - y_d|^2 + \frac{\beta}{2} \iint_{\mathcal{O} \times (0,T)} |v|^2, \quad \beta > 0.$$

- CONTROLLABILITY find h such that $y(T) = 0$.

Heat equation

- The leader control h wants to achieve a controllability objective.
- The follower control v wants to achieve, given h, a minimal of a given functional, an optimization target.

How to solve the problem?

- Step 1: Fix h and obtain

$$\min_{v} \quad \frac{1}{2}\iint_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 + \frac{\beta}{2}\iint_{\mathcal{O}\times(0,T)} |v|^2.$$

The functional is continuous, strictly convex and coercive so there is a unique minimizer characterized by

$$\boxed{v = -\tfrac{1}{\beta} p\chi_{\mathcal{O}}}$$

$$\begin{cases} -p_t - \Delta p = (y-y_d)\chi_{\mathcal{O}_d} & \text{in } Q \\ p(T,x) = 0 \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Sigma. \end{cases} \tag{Seg}$$

How to solve the problem? (Cont.)

- Step 2:

$$\begin{cases} y_t - \Delta y = h\chi_\omega - \frac{1}{\beta} p\chi_{\mathcal{O}} & \text{in } Q \\ -p_t - \Delta p = (y - y_d)\chi_{\mathcal{O}_d} & \text{in } Q \\ y(0,x) = y^0(x),\ p(T,x) = 0, \quad y = p = 0 & \text{on } \Sigma. \end{cases} \tag{L-S}$$

In [Lions94] is proved that (L-S) is APPROXIMATELLY CONTROLLABLE – $\|y(T)\|_{L^2(\Omega)} \leq \varepsilon$.

Araruna et al. (2015) proved when $\omega \cap \mathcal{O}_d \neq \emptyset$ and

$$\boxed{\iint_Q \rho^2 |y_d|^2 < \infty} \quad \text{for } \rho \to \infty \text{ as } t \to T,$$

then (L-S) is NULL CONTROLLABLE.

How to solve the problem? (Cont.)

Prove the observability inequality

$$\|\varphi(0)\|^2_{L^2(\Omega)} + \iint_Q \rho^{-2}|\theta|^2 \leq C \iint_{\omega\times(0,T)} |\varphi|^2$$

for the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = \theta\chi_{\mathcal{O}_d} & \text{in } Q \\ \theta_t - \Delta\theta = -\frac{1}{\beta}\varphi\chi_{\mathcal{O}} & \text{in } Q \\ \varphi(T,x) = \varphi_T,\ \theta(0,x) = 0, \quad \varphi = \theta = 0 & \text{on } \Sigma. \end{cases} \tag{2}$$

Important fact: $\theta(0) = 0$ and β LARGE .

- Step 3: We define $h = \hat{\varphi}\chi_\omega$ where $\hat{\varphi}$ is the solution to (2) with

$$J(\hat{\varphi}_T) = \min_{\varphi_T \in L^2(\Omega)} \frac{1}{2}\iint_{\omega\times(0,T)} |\varphi|^2 + \int_\Omega y^0\varphi(0) - \iint_{\mathcal{O}_d\times(0,T)} y_d\,\theta$$

Works related with hierarchic control

- Heat and wave equations: Lions, 1994.
- Ocean circulation models: Díaz-Lions 1997, Díaz (2002)
- Stokes: Guillen-González et al., 2013.
- Moving Domains (wave equation): IP de Jesus, 2014, 2015.
- Moving domains (Parabolic equations) Approximate control: Límaco, J.; Clark, H. R.; Medeiros, L. A. , 2009.
- Linear and semilinear parabolic equations: Araruna, Fernández-Cara, Santos, 2015 – Control to trajectories.
- Micropolar fluids: Araruna, F. D.; de Menezes, S. D. B.; Rojas-Medar, M. A. 2014 – approximate controllability, control in both equations.
- Coupled parabolic equations: Hernández-Santamaría; DeT, Pozniak (2016)

Null controllability of the 2D Boussinesq system

E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov and J.-P. Puel, SIAM J. Control Optim., 45 (2006), 146–173.

The 2D Boussinesq system: one scalar control

Let $\Omega \subset \mathbb{R}^2$ be smooth, $Q = (0,T) \times \Omega$, and $\omega \subset \Omega$ open.

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla \pi = \theta e_2 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = f 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } (0,T) \times \partial\Omega, \\ (y, \theta)|_{t=0} = (y^0, \theta^0). \end{cases}$$

Null controllability

Find $f \in L^2((0,T) \times \omega)$ such that $\boxed{y(T) = 0, \quad \theta(T) = 0.}$

Only the temperature is directly controlled; velocity is influenced through the buoyancy force θe_2.

Linearization and the cascade mechanism

Around the zero trajectory, the linearized equations are

$$\begin{cases} y_t - \Delta y + \nabla \pi = \theta e_2, & \nabla \cdot y = 0, \\ \theta_t - \Delta \theta = f 1_\omega, \\ y|_{\partial\Omega} = \theta|_{\partial\Omega} = 0. \end{cases}$$

For the adjoint variables (φ, q, ψ):

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla q = 0, & \nabla \cdot \varphi = 0, \\ -\psi_t - \Delta \psi = \varphi_2, \\ \varphi|_{\partial\Omega} = \psi|_{\partial\Omega} = 0. \end{cases}$$

The key question: can an observation of ψ on ω recover both adjoint components? A SINGLE SCALAR OBSERVATION MUST DETECT THE VELOCITY THROUGH THE COUPLING φ_2.

Why the pressure makes observability difficult

- In the Stokes adjoint, ∇q is a nonlocal constraint enforcing $\nabla \cdot \varphi = 0$. The two velocity components do not satisfy independent heat equations.
- A scalar Carleman estimate for ψ does not directly estimate the full vector φ: the coupling gives access first to φ_2.
- Taking divergence yields $\Delta q = 0$ for the homogeneous Stokes adjoint, but the pressure cannot simply be set to zero: its boundary behaviour matters.
- In dimension two, the divergence-free condition and the vorticity $\operatorname{curl} \varphi = \partial_1 \varphi_2 - \partial_2 \varphi_1$ provide additional structure; eliminating pressure by taking curl also introduces boundary difficulties.

Analytical task

Establish a *coupled Stokes–heat observability inequality* from the temperature observation, with suitable geometry and boundary conditions.

The nonlinear problem: a local result

The nonlinear terms are

$$(y \cdot \nabla)y, \qquad y \cdot \nabla\theta.$$

- A linear observability estimate is the starting point, but does not by itself imply nonlinear null controllability.
- Work in weighted spaces adapted to the final condition; prove suitable estimates for the linearized control map.
- Use an inverse-mapping or fixed-point argument to handle the quadratic nonlinearities.
- The resulting controllability statement is local: for sufficiently small initial data (in the relevant regularity class), a temperature control can steer (y, θ) to $(0, 0)$ at time T, *provided the required linear observability estimate holds*.

Two distinct difficulties: pressure and nonlinearity

Transition to hierarchic control

We now add a second temperature control, with a different objective, and study the resulting leader–follower problem.

Already at the linear level: the Stokes pressure

For the adjoint Stokes–heat cascade,

$$-\varphi_t - \Delta\varphi + \nabla q = 0, \quad \nabla\cdot\varphi = 0, \qquad -\psi_t - \Delta\psi = \varphi_2.$$

The scalar observation of ψ first reveals φ_2, not the whole velocity φ. The pressure q couples the velocity components through incompressibility and the boundary conditions: a componentwise heat estimate is insufficient.

At the nonlinear level: transport and convection

The terms $(y \cdot \nabla)y$ and $y \cdot \nabla\theta$ introduce products of unknowns and possible derivative loss. One needs a *weighted* linear controllability estimate with enough regularity to control these products.

From linear observability to local nonlinear control

1. **Linear stage:** prove an observability inequality for the *coupled* Stokes–heat adjoint, including the unobserved velocity component; the pressure is handled using the divergence-free structure and suitable Carleman estimates.
2. **Functional setting:** obtain a right inverse of the linearized controlled operator in weighted spaces that encode the vanishing final state.
3. **Nonlinear stage:** treat the quadratic convection terms by an inverse-mapping (or fixed-point) argument, requiring small data in appropriate norms.

Boussinesq system

T. Takahashi, LDET, Y. Wu-Zhang

$N = 2$

$\Omega \subset \mathbb{R}^2$ and $T > 0$. Let ω and $\mathcal{O}$ be two nonempty open subsets of Ω with

$$\omega \cap \mathcal{O} = \emptyset.$$

We consider the Boussinesq system with two controls f and $f^\star$ acting on the temperature:

$$\begin{cases} \partial_t y - \Delta y + \nabla \pi_y + (y \cdot \nabla)\, y = \theta e_2 & \text{in } (0,T) \times \Omega, \\ \partial_t \theta - \Delta \theta + y \cdot \nabla \theta = f 1_\omega + f^\star 1_{\mathcal{O}} & \text{in } (0,T) \times \Omega, \\ \nabla \cdot y = 0 & \text{in } (0,T) \times \Omega, \\ y = 0, \quad \theta = 0 & \text{on } (0,T) \times \partial\Omega, \\ y(0,\cdot) = y^0, \quad \theta(0,\cdot) = \theta^0 & \text{in } \Omega. \end{cases} \tag{3}$$

In the above system $y = y(t,x) \in \mathbb{R}^2$, $\pi_y = \pi_y(t,x) \in \mathbb{R}$ and $\theta = \theta(t,x) \in \mathbb{R}$ are respectively the velocity, the pressure and the temperature of the fluid, whereas (e_1, e_2) is the canonical basis of $\mathbb{R}^2$.

Objectives

The functions f and $f^\star$ are two controls that have two different objectives:

1. the aim of $f^\star$ is that (y, θ) remains close to a given target $(y^\star, \theta^\star) \in L^2(0, T; L^2(\Omega)^3)$;
2. the aim of f is that the state is at rest at the final time T: $(y, \theta)(T, \cdot) = 0$ in Ω.

Minimization objective

Optimal control problem for $f^\star$ where we fix

$$\boxed{f \in L^2(0,T;L^2(\omega)), \quad y^0 \in \mathbb{V}, \quad \theta^0 \in H_0^1(\Omega).}$$

$$J(f^\star) := \frac{1}{2} \iint_{(0,T)\times\mathcal{O}^\star} \left[|y(f^\star) - y^\star|^2 + |\theta(f^\star) - \theta^\star|^2 \right] dx\, dt + \frac{1}{2} \iint_{(0,T)\times\mathcal{O}} \frac{|f^\star|^2}{\mu} dx\, dt \tag{4}$$

μ is a weight function satisfying

$$\boxed{\mu \in C^0([0,T];\mathbb{R}_+), \quad \mu(0) = \mu(T) = 0, \quad \mu > 0 \text{ in } (0,T),}$$

Lemma

Assume (53) *and that* $(y^\star, \theta^\star) \in L^2(0,T;L^2(\Omega)^3)$. *There exists a sufficiently small constant* $C > 0$ *such that if* $\|\mu\|_{L^\infty(0,T)} \leq C$, *then the optimal control problem admits a unique solution given by*

$$\boxed{f^\star := -\mu\sigma,}$$

where (y, θ, u, σ) *is the solution of the coupled system*

$$\begin{cases} \partial_t y - \Delta y + \nabla \pi_y + (y \cdot \nabla)\, y = \theta e_2 \\ \partial_t \theta - \Delta \theta + y \cdot \nabla \theta = f 1_\omega - \mu\sigma 1_{\mathcal{O}} \\ -\partial_t u - \Delta u + \nabla \pi_u + (\nabla y)^\top u - (y \cdot \nabla)\, u + \sigma \nabla \theta = (y - y^\star)\, 1_{\mathcal{O}^\star} \\ -\partial_t \sigma - \Delta \sigma - y \cdot \nabla \sigma = u_2 + (\theta - \theta^\star)\, 1_{\mathcal{O}^\star} \\ \nabla \cdot y = \nabla \cdot u = 0 \\ y = u = 0, \quad \theta = \sigma = 0 \\ y(0,\cdot) = y^0, \quad \theta(0,\cdot) = \theta^0, \quad u(T,\cdot) = 0, \quad \sigma(T,\cdot) = 0. \end{cases}$$

FOR THE HIERARCHICAL BOUSSINESQ PROBLEM

The leader must control a *forward–backward coupled system*: the follower's adjoint variables add further coupling. Thus its observability estimate is more involved than that for the single-control Boussinesq system.

Theorem

Assume that $\omega \cap \mathcal{O} = \emptyset$ *and that* $\omega \cap \mathcal{O}^\star \neq \emptyset$. *There exists a sufficiently small constant* $C > 0$ *and* $\rho_1, \rho_2 \in C^0([0,T])$ *such that* $\rho_1(0) = \rho_1(T) = \rho_2(T) = 0$, $\rho_1 > 0$ *and* $\rho_2 > 0$ *in* $[0,T)$ *with the following property: if*

$$\frac{\mu}{\rho_1} \in L^\infty(0,T), \quad \frac{(y^\star, \theta^\star)}{\rho_2} \in L^2(0,T; L^2(\Omega)^3), \quad (y^0, \theta^0) \in \mathbb{V} \times H_0^1(\Omega), \tag{5}$$

with

$$\left\| \frac{(y^\star, \theta^\star)}{\rho_2} \right\|_{L^2(0,T;L^2(\Omega)^3)} + \left\| (y^0, \theta^0) \right\|_{H^1(\Omega)^3} \leq C,$$

then, there exist a control $f \in L^2(0,T; L^2(\omega))$ *and a solution*

$$(y, \theta, u, \sigma)$$

of (5) *satisfying* $\boxed{y(T,\cdot) = 0, \quad \theta(T,\cdot) = 0 \quad \text{in } \Omega.}$

Change of roles

B. Calsavara, E. Fernández-Cara, LdeT, J.A. Villa

$$\begin{cases} y_t - \Delta y + a(x,t)y = f1_{\mathcal{O}} + v1_{\omega} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(0,\cdot) = y_0 & \text{in} \quad \Omega \end{cases} \tag{6}$$

Objectifs: We consider the secondary functional

$$S(v;f) := \frac{1}{2}\iint_Q \rho^2 |y|^2 + \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \rho_0^2 |f|^2, \tag{7}$$

where ρ and ρ_0 are appropriate weights that blow up as $t \to T^-$ and the main functional

$$P(v;f) := \frac{\alpha}{2}\iint_{\mathcal{O}_d\times(0,T)} |y - y_d|^2 + \frac{\mu}{2}\iint_{\omega\times(0,T)} \rho_0^2 |v|^2, \tag{8}$$

where α and μ are positive constants and $y_d = y_d(x,t)$ is a given function.

Stackelberg strategy

The following spaces will be used:

$$\mathcal{U} := \{\, v : \rho_0 v \in L^2(\omega \times (0,T)) \,\},$$

$$\mathcal{Y} := \{\, y : \rho y \in L^2(Q) \,\}, \quad \mathcal{F} := \{\, f : \rho_0 f \in L^2(\mathcal{O} \times (0,T)) \,\}.$$

Stackelberg strategy

The control process can be described as follows:

1. We associate to each leader $v \in \mathcal{U}$ the unique solution $f[v]$ to the following extremal problem:

$$\begin{cases} \text{Minimize } S(v; f) \\ \text{Subject to } f \in \mathcal{F} \end{cases} \tag{9}$$

2. Then, we look for admissible controls $\hat{v} \in \mathcal{U}$ satisfying

$$P(\hat{v}; f[\hat{v}]) = \min_{v} P(v; f[v]) \tag{10}$$

Weight functions from Carleman inequalities

Weight functions ρ and ρ_0.

Let $\eta_0 = \eta_0(x)$ be a function satisfying

$$\begin{cases} \eta_0 \in C^2(\overline{\Omega}),\ \eta_0 > 0 \text{ in } \Omega,\ \eta_0 = 0 \text{ on } \partial\Omega, \\ |\nabla \eta_0| > 0 \text{ in } \overline{\Omega} \setminus \omega. \end{cases}$$

Function η_0 given by Fursikov-Imanuvilov 1996. Then, we introduce the weight functions

$$\sigma(x,t) := \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)}, \quad \xi(x,t) := \frac{e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)},$$

$$\rho := e^{s\sigma}, \quad \rho_0 := (s\xi)^{-3/2}\lambda^{-2}\rho, \quad \rho_1 := (s\xi)^{-1}\lambda^{-1}\rho, \quad \rho_2 := s\xi\,\rho,$$

where $s, \lambda > 0$ are large enough and $\ell \in C^\infty([0,T])$ satisfies $\ell(t) \geq T/2$ in $[0, T/2]$ and $\ell(t) = T - t$ in $[T/2, T]$.

Main Result

Theorem

Let us consider the linear system (6)*, where* $a \in L^\infty(Q)$ *and* $y_0 \in L^2(\Omega)$*.*

1. *For every* $v \in \mathcal{U}$*, there exists exactly one solution* $f[v]$ *minimizing* $S(v, f)$*.*
2. *Let us set* $\tilde{P}(v) := P(v; f[v])$*. Then there exists exactly one minimizer* $\hat{v}$ *of* $\tilde{P}$ *in* $\mathcal{U}$ *and, consequently, one associated follower* $f[\hat{v}]$ *.*

Idea of the proof

Let us introduce the notation

$$L_a y := y_t - \Delta y + a y, \quad L_a^* p := -p_t - \Delta p + a p$$

and let the space $\mathcal{P}_0$ be given by

$$\mathcal{P}_0 := \{\, p \in C^2(\overline{Q}) : p = 0 \ \text{ on } \ \Sigma \,\}.$$

We define the symmetric bilinear form on $\mathcal{P}_0$

$$m(a; p, p') := \iint_Q \left(\rho^{-2} L_a^* p \, L_a^* p' + 1_{\mathcal{O}} \rho_0^{-2} p \, p' \right)$$

that is in fact a scalar product. In fact, (Carleman Inequality)

$$\iint_Q \left[\rho_2^{-2} (|p_t|^2 + |\Delta p|^2) + \rho_1^{-2} |\nabla p|^2 + \rho_0^{-2} |p|^2 \right] \leq C_0 \, m(a; p, p).$$

Idea of the proof (Cont.)

Define

$$\mathcal{P} = \overline{\mathcal{P}_0}$$

Theorem

Let v be given in U. Then, there exists exactly one solution $f[v] \in L^2(\mathcal{O} \times (0,T))$ minimizing S. Furthermore, one has

$$f[v] = -\rho_0^{-2} p\big|_{\mathcal{O}\times(0,T)}, \quad y = \rho^{-2} L_a^* p,$$

where p is the unique solution to the linear problem

$$\begin{cases} m(a; p, p') = \displaystyle\iint_{\omega\times(0,T)} v\, p' + \int_\Omega y_0(x)\, p'(x,0)\, dx \\ \forall p' \in \mathcal{P}, \quad p \in \mathcal{P}. \end{cases} \tag{11}$$

In fact we use Lax-Milgram Theorem to prove that $p \in \mathcal{P}$ and one has

$$\begin{cases} L_a(\rho^{-2} L_a^* p) + \rho^{-2} p 1_{\mathcal{O}\times(0,T)} = v 1_{\omega\times(0,T)} & \text{in} \quad Q, \\ p = 0, \quad \rho^{-2} L_a^* p = 0 & \text{on} \quad \Sigma, \\ \rho^{-2} L_a^* p\big|_{t=0} = y_0, \quad \rho^{-2} L_a^* p\big|_{t=T} = 0 & \text{in} \quad \Omega. \end{cases} \tag{12}$$

That is

$$f[v] = \rho^{-2} p 1_{\mathcal{O}\times(0,T)}, \qquad y = \rho^{-2} L_a^* p$$

Theorem

The real-valued function $v \mapsto P(v; f[v])$ *is well defined,* C^1 *and strictly convex on* $\mathcal{U}$ *and satisfies*

$$P(v; f[v]) \to +\infty \quad \text{as} \quad v \in \mathcal{U},\ \|v\|_{\mathcal{U}} \to +\infty.$$

Consequently, there exists exactly one control $v \in \mathcal{U}$ *minimizing* P.

Characterization

The unique solution $\hat{v}$ satisfies, together with the associated $f[\hat{v}]$, the associated state $\hat{y}$ and the additional variables $\hat{p}$ and $\hat{\phi}$, the following optimality system: $f[\hat{v}] = -\rho_0^{-2}\hat{p}\big|_{\mathcal{O}\times(0,T)}, \quad \hat{y} = \rho^{-2}L_a^*\hat{p},$

Characterization

$$\begin{cases} \hat{y}_t - \Delta \hat{y} + a(x,t)\hat{y} = f[\hat{v}]1_{\mathcal{O}} + \hat{v}1_{\omega} & \text{in} \quad Q, \\ \hat{y} = 0 & \text{on} \quad \Sigma, \\ \hat{y}(0,\cdot) = y_0 & \text{in} \quad \Omega, \end{cases}$$

$$f[\hat{v}] = -\rho_0^{-2}\hat{p}\big|_{\mathcal{O}\times(0,T)}, \quad \hat{y} = \rho^{-2}L_a^*\hat{p},$$

$$\begin{cases} m(a;\hat{p},p') = \displaystyle\iint_{\omega\times(0,T)} \hat{v}\,p' + \int_{\Omega} y_0(x)\,p'(x,0)\,dx \\ \forall p' \in \mathcal{P}, \quad \hat{p} \in \mathcal{P}, \end{cases} \tag{13}$$

$$\hat{v} = -\frac{1}{\mu}\rho_0^{-2}(\hat{\psi} + \hat{\phi})\big|_{\mathcal{O}\times(0,T)},$$

$$\begin{cases} m(a;p',\hat{\psi}) = -\displaystyle\iint_{\mathcal{O}\times(0,T)} \rho_0^{-2}\hat{\phi}\,p' \\ \forall p' \in \mathcal{P}, \quad \hat{\psi} \in \mathcal{P}, \end{cases}$$

Stackelberg–Nash strategies

Araruna et al., ESAIM: COCV 21 (2015), 835–856.

Stackelberg–Nash: one leader, two followers

Let $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$.

$$\begin{cases} y_t - \Delta y + a(x,t)y = f\,1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,\cdot) = y_0 & \text{in } \Omega. \end{cases}$$

- Leader $f \in L^2(\mathcal{O} \times (0,T))$: seeks exact controllability to a prescribed trajectory.
- Followers $v^i \in L^2(\mathcal{O}_i \times (0,T))$, $i = 1,2$: optimize their own costs, with f fixed.

$$J_i(f; v^1, v^2) = \frac{\alpha_i}{2} \int_0^T \int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 + \frac{\mu_i}{2} \int_0^T \int_{\mathcal{O}_i} |v^i|^2, \quad i = 1,2.$$

What is a Nash equilibrium?

Fix the leader control f. A pair $(\hat{v}^1, \hat{v}^2)$ is a Nash equilibrium if neither follower can improve its cost by changing its own control alone:

$$J_1(f; \hat{v}^1, \hat{v}^2) \le J_1(f; w^1, \hat{v}^2) \qquad \forall w^1 \in L^2(\mathcal{O}_1 \times (0,T)),$$
$$J_2(f; \hat{v}^1, \hat{v}^2) \le J_2(f; \hat{v}^1, w^2) \qquad \forall w^2 \in L^2(\mathcal{O}_2 \times (0,T)).$$

For the **linear state equation**, each cost is convex with respect to its own follower control. Thus the Nash conditions are equivalent to

$$D_{v^1} J_1(f; \hat{v}^1, \hat{v}^2)[w^1] = 0, \qquad D_{v^2} J_2(f; \hat{v}^1, \hat{v}^2)[w^2] = 0$$

for all admissible directions w^1, w^2. Nash equilibrium does not mean joint minimization of $J_1 + J_2$.

Characterization through adjoint states

For fixed f, let y be the state corresponding to a Nash equilibrium. Introduce φ^i $(i = 1, 2)$ solving

$$\begin{cases} -\varphi^i_t - \Delta\varphi^i + a(x,t)\varphi^i = \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} & \text{in } Q, \\ \varphi^i = 0 & \text{on } \Sigma, \\ \varphi^i(T,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

Integration by parts in the first-order Nash conditions yields

$$\int_0^T \int_{\mathcal{O}_i} (\varphi^i + \mu_i v^i) w^i = 0 \quad \forall w^i, \qquad \boxed{v^i = -\frac{1}{\mu_i}\varphi^i\big|_{\mathcal{O}_i \times (0,T)}}.$$

Hence the state and the two adjoints form a coupled forward–backward system:

$$y_t - \Delta y + ay = f1_{\mathcal{O}} - \sum_{i=1}^{2} \frac{1}{\mu_i}\varphi^i 1_{\mathcal{O}_i}.$$

Existence of Nash equilibria and the leader's goal

- For a fixed f, the first-order conditions give a linear system $\mathcal{L}(v^1, v^2) = \Psi$ on $L^2(\mathcal{O}_1 \times (0,T)) \times L^2(\mathcal{O}_2 \times (0,T))$.
- If the penalties μ_i are sufficiently large, $\mathcal{L}$ is coercive. Lax–Milgram gives a **unique Nash equilibrium** for each f (Proposition 2.1).
- The leader then minimizes

$$\frac{1}{2}\int_0^T \int_{\mathcal{O}} |f|^2 \qquad \text{subject to} \qquad y(T,\cdot) = \bar{y}(T,\cdot),$$

where $\bar{y}$ is a prescribed uncontrolled trajectory.

Exact controllability to trajectories

Assume $\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d$, $\mathcal{O}_d \cap \mathcal{O} \neq \emptyset$, μ_i large, and $\int_{\mathcal{O}_d \times (0,T)} \hat{\rho}^2 |y_{i,d} - \bar{y}|^2 < \infty$ for a suitable weight $\hat{\rho}(t) \to \infty$ as $t \to T$. Then there exist a leader f and associated Nash followers with $y(T,\cdot) = \bar{y}(T,\cdot)$.

Additional problems

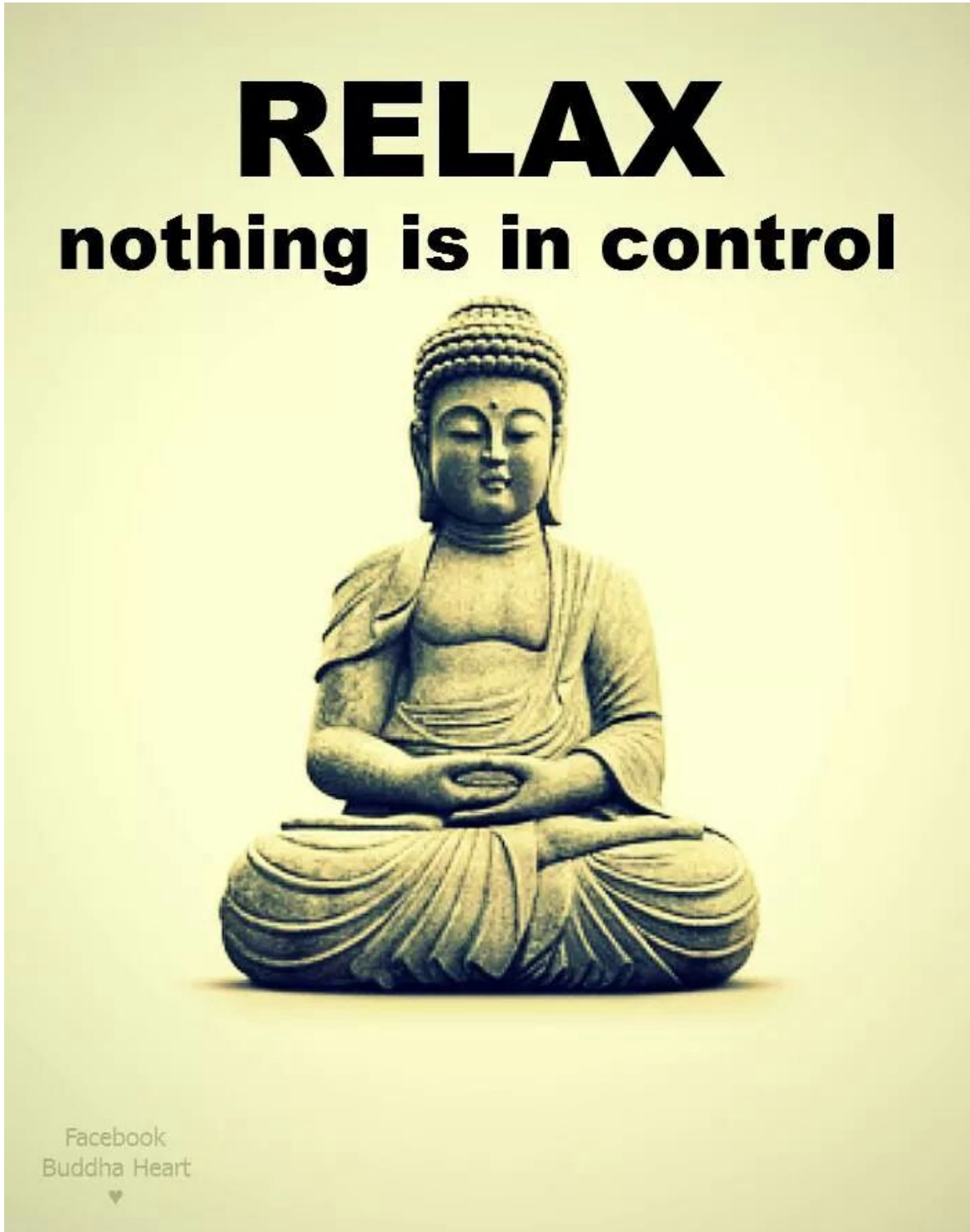
- Semilinear equation
- Multiobjective: Pareto Equilibria
- Distributed control leader + boundary control follower
- Both controls on the boundary.
- Wave equation.
- Other equations.

¡Gracias!

Thank you!

Merci!

Obrigada!


RELAX
nothing is in control
Facebook
Buddha Heart
♥