Workshop on PDEs and Control 2025 (PKM-60)
September 3-5, 2025, Sevilla, Spain

Attractors for a class of impulsive systems

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The theory of impulsive systems describes the evolution of processes where the continuous dynamics are interrupted by abrupt changes of state.

Definition 1 A *semiflow* on X (denoted by (X, π)) is a family of maps $\{\pi(t) \colon t \in \mathbb{R}_+\}$ acting from X to X such that $\pi(0) = I$, $\pi(t+s) = \pi(t)\pi(s)$ for all $t, s \in \mathbb{R}_+$, and $\mathbb{R}_+ \times X \ni (t,x) \mapsto \pi(t)x \in X$ is continuous.

Definition 2 Given a semiflow (X, π), a nonempty closed subset $M \subset X$ is called an *impulsive set* if for each $x \in M$ there exists $\epsilon_x > 0$ such that $\bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \emptyset$.

Definition 3 An *impulsive dynamical system* (X, π, M, I) consists of a semiflow (X, π), an impulsive set $M \subset X$ and a continuous function $I \colon M \to X$ called impulsive function.

The *impact function* associated to (X, π, M, I) is given by

$$\phi(x) = \begin{cases} s, & \text{if } \quad \pi(s)x \in M \text{ and } \pi(t)x \notin M, \ 0 < t < s, \\ \infty, & \text{if } \quad \pi(t)x \notin M \text{ for all } \ t > 0. \end{cases}$$

The *impulsive positive trajectory* of $x \in X$ in (X, π, M, I) is a map $\tilde{\pi}(\cdot)x \colon J_x \to X$ defined on some interval $J_x \subseteq \mathbb{R}_+$ containing 0, given inductively by the following way: if $\phi(x) = \infty$ then $\tilde{\pi}(t)x = \pi(t)x$ for all $t \in \mathbb{R}_+$. But, if $\phi(x) < \infty$ then we set $x = x_0^+$ and we define $\tilde{\pi}(\cdot)x$ on $[0, \phi(x_0^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & \text{if} \quad 0 \leqslant t < \phi(x_0^+), \\ I(\pi(\phi(x_0^+))x_0^+), & \text{if} \quad t = \phi(x_0^+). \end{cases}$$

Now, set $s_0 = \phi(x_0^+)$, $x_1 = \pi(s_0)x_0^+$ and $x_1^+ = I(\pi(s_0)x_0^+)$. Since $s_0 < \infty$, the previous process can go on, but now starting at x_1^+. If $\phi(x_1^+) = \infty$ then we define $\tilde{\pi}(t)x = \pi(t - s_0)x_1^+$ for all $t \geq s_0$. But, if $s_1 = \phi(x_1^+) < \infty$ i.e., $x_2 = \pi(s_1)x_1^+ \in M$ then we define $\tilde{\pi}(\cdot)x$ on $[s_0, s_0 + s_1]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t - s_0)x_1^+, & \text{if} \quad s_0 \leqslant t < s_0 + s_1, \\ I(x_2), & \text{if} \quad t = s_0 + s_1. \end{cases}$$

Here, we denote $x_2^+ = I(x_2)$. This process ends after a finite number of steps if $\phi(x_n^+) = \infty$ for some $n \in \mathbb{N}$, or it may proceed indefinitely, if $\phi(x_n^+) < \infty$ for all $n \in \mathbb{N}$ and, in this case, $\tilde{\pi}(\cdot)x$ is defined in $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$. We shall assume that $T(x) = \infty$ for all $x \in X$.

Definition 4 A nonempty set $\tilde{\mathcal{A}} \subset X$ is called a *global attractor* for (X, π, M, I) if $\tilde{\mathcal{A}}$ is pre-compact and $\tilde{\mathcal{A}} = \overline{\tilde{\mathcal{A}}} \backslash M$, $\tilde{\mathcal{A}}$ is $\tilde{\pi}$-invariant ($\tilde{\pi}(t)A = A$ for all $t \in \mathbb{R}_+$), and $\mathrm{d_H}(\tilde{\pi}(t)B, \tilde{\mathcal{A}}) \overset{n \to \infty}{\longrightarrow} 0$ for every bounded set $B \subset X$, where $\mathrm{d_H}$ is the Hausdorff semidistance.

Let $\hat{X} = \{x \in I(M) \colon \phi(x_k^+) < \infty \text{ for all } k \in \mathbb{N}\}$ and $g \colon \hat{X} \to \hat{X}$ be given by $g(x) = I(\pi(\phi(x))x)$. The system $(\hat{X}, g)$ defines a discrete dynamical system on $\hat{X}$ associated with the impulsive dynamical system (X, π, M, I).

Definition 5 A set $\hat{\mathcal{A}} \subset \hat{X}$ is called a *discrete global attractor* for $(\hat{X}, g)$ if $\hat{\mathcal{A}}$ is compact, $\hat{\mathcal{A}}$ is g-invariant ($g(\hat{B}) = \hat{B}$), and $\mathrm{d_H}(g^n(\hat{B}), \hat{\mathcal{A}}) \overset{n \to \infty}{\longrightarrow} 0$ for every bounded set $\hat{B} \subset \hat{X}$.

In this work, we establish sufficient conditions for the existence of global attractors for the systems (X, π, M, I) and $(\hat{X}, g)$. Furthermore, we investigate the relationship between these attractors. An application involving a nonlinear reaction-diffusion initial boundary value problem is also presented.

Acknowledgements

This research was partially supported by FAPESP # 2020/14075-6 and CNPq # 310540/2019-4.

References

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