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Nitsche's prescription of Dirichlet conditions for the conforming finite element approximation of Maxwell's problem

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This work deals with the FE approximation of the following simplified problem arising in magnetostatics, often called Maxwell's problem: find a magnetic induction field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^d$ and a scalar field $p : \Omega \longrightarrow \mathbb{R}$ solution of the boundary value problem

$$\begin{aligned} \nu\nabla\times\nabla\times\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ -\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{u} &= \boldsymbol{n}\times\bar{\boldsymbol{u}} && \text{on } \Gamma, \\ p = \bar{p} &:= 0 && \text{on } \Gamma, \end{aligned}$$

where Ω is a domain of $\mathbb{R}^d$ $(d = 2,3)$, $\Gamma = \partial\Omega$, $\nu > 0$ is a physical parameter, $\bar{\boldsymbol{u}}$ is given and $\boldsymbol{f}$ is assumed to be solenoidal.

For $\bar{\boldsymbol{u}} = \boldsymbol{0}$, the problem is equivalent to the two variational equations:

$$\begin{aligned} & a(\boldsymbol{u},\boldsymbol{v}) + b(p,\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega \quad \forall \boldsymbol{v}\in V_0, \\ & b(q,\boldsymbol{u}) = 0 \quad \forall q \in Q_0, \\ & a(\boldsymbol{u},\boldsymbol{v}) := \nu(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})_\Omega, \quad b(p,\boldsymbol{v}) := (\nabla p,\boldsymbol{v})_\Omega, \\ & V_0 = H_0(\mathrm{curl},\Omega), \quad Q_0 = H_0^1(\Omega). \end{aligned}$$

Its conforming finite element approximation consists of building finite element spaces $V_{h,0} \subset V_0$ and $Q_{h,0} \subset Q_0$ and find $\boldsymbol{u}_h \in V_{h,0}$, $p_h \in Q_{h,0}$ such that

$$\begin{aligned} a(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(p_h,\boldsymbol{v}_h) &= \langle \boldsymbol{f},\boldsymbol{v}_h\rangle_\Omega && \forall \boldsymbol{v}_h \in V_{h,0}, \\ b(q_h,\boldsymbol{u}_h) &= 0 && \forall q_h \in Q_{h,0}. \end{aligned}$$

When $\bar{\boldsymbol{u}} \neq \boldsymbol{0}$, the boundary condition for $\boldsymbol{u}$ can be prescribed weakly using Nitsche's method, which can also be used to prescribe the boundary condition for p_h. If V_h and Q_h are the finite element spaces without boundary conditions, the problem consists of finding $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_{\mathrm{N}}([\boldsymbol{u}_h,p_h],[\boldsymbol{v}_h,q_h]) = L_{\mathrm{N}}([\boldsymbol{v}_h,q_h]) \quad \forall [\boldsymbol{v}_h,q_h] \in V_h \times Q_h,$$

where

$$\begin{aligned} B_{\mathrm{N}}([\boldsymbol{u}_h,p_h],[\boldsymbol{v}_h,q_h]) &:= \nu(\nabla\times\boldsymbol{v}_h,\nabla\times\boldsymbol{u}_h)_\Omega \\ &+ (\boldsymbol{v}_h,\nabla p_h)_\Omega + (\boldsymbol{u}_h,\nabla q_h)_\Omega \\ &- \nu\langle \boldsymbol{n}\times\boldsymbol{v}_h,\nabla\times\boldsymbol{u}_h\rangle_\Gamma - \langle \boldsymbol{n}\cdot\boldsymbol{u}_h,q_h\rangle_\Gamma \\ &- \nu\langle \boldsymbol{n}\times\boldsymbol{u}_h,\nabla\times\boldsymbol{v}_h\rangle_\Gamma - \langle \boldsymbol{n}\cdot\boldsymbol{v}_h,p_h\rangle_\Gamma \\ &+ N_u\frac{\nu}{h}\langle \boldsymbol{n}\times\boldsymbol{v}_h,\boldsymbol{n}\times\boldsymbol{u}_h\rangle_\Gamma - N_p\frac{L_0^2}{\nu h}(p_h,q_h)_\Gamma \\ L_{\mathrm{N}}([\boldsymbol{v}_h,q_h]) &:= \langle \boldsymbol{v}_h,\boldsymbol{f}\rangle_\Omega - \nu\langle \boldsymbol{n}\times\bar{\boldsymbol{u}},\nabla\times\boldsymbol{v}_h\rangle_\Gamma \\ &+ N_u\frac{\nu}{h}\langle \boldsymbol{n}\times\boldsymbol{v}_h,\boldsymbol{n}\times\bar{\boldsymbol{u}}\rangle_\Gamma. \end{aligned}$$

We prove that if $V_{h,0}$ and $Q_{h,0}$ satisfy the classical inf-sup condition for Maxwell's problem, then Nitsche's method yields a stable solution that is optimally convergent in the norm:

$$\begin{aligned} |||[\boldsymbol{v}_h,q_h]|||^2_{V\times Q,\mathrm{N}} &:= \nu\|\nabla\times\boldsymbol{u}\|^2_{L^2(\Omega)} + \frac{\nu}{L_0^2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} \\ &+ \frac{L_0^2}{\nu}\|\nabla p\|^2_{L^2(\Omega)} + \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{v}_h\|^2_{L^2(\Gamma)} + \frac{L_0^2}{\nu h}\|q_h\|^2_{L^2(\Gamma)}. \end{aligned}$$

where L_0 is a characteristic length of Ω.

We also prove a similar result for a stabilised finite element method presented in [1], in which spaces $V_{h,0}$ and $Q_{h,0}$ do not need to satisfy any inf-sup condition. The present work is based on [2].

References

[1] S. Badia and R. Codina. A nodal-based finite element approximation of the Maxwell problem suitable for singular solutions. *SIAM Journal on Numerical Analysis*, 50:398–417, 2012.

[2] D. Boffi, R. Codina and Ö. Türk. An analysis of Nitsche's prescription of Dirichlet conditions for the conforming finite element approximation of Maxwell's problem. Submitted.

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