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Exact controllability to zero for general linear parabolic equations

I. Gayte* and I. Marín-Gayte†

This paper studies the existence and characterization of partially distributed controls that drive the solution of a linear parabolic problem with general diffusion coefficients to zero in a fixed time T. Specifically, given $\Omega \subset \mathbb{R}^N$ a bounded open set, whose boundary is $C^{0,1}$, $\omega \subset \Omega$ an open set and 1_ω the characteristic function on ω, $\Sigma = \partial\Omega \times (0, T)$ and a matrix $A \in L^\infty(\Omega \times (0, T))^{N\times N}$ satisfying

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(t,x)\xi_i\xi_j \leq \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^N,$$

we prove that there exists $\hat{u} \in L^2(\Omega \times (0, T))$ such that the solution of the linear parabolic problem

$$(1) \qquad \begin{cases} \hat{y}_t - \nabla\cdot(A\nabla\hat{y}) = \hat{u}1_\omega \text{ in } \Omega \times (0, T) \\ \hat{y}|_\Sigma = 0 \\ \hat{y}(0) = y_0 \text{ in } \Omega, \end{cases}$$

verifies

$$\hat{y}(T) = 0 \text{ in } \Omega.$$

The result represents a novelty in control theory for two reasons: first, because the elliptic operator given by the matrix A does not have regular coefficients. And the second reason is that domains of class $C^{0,1}$ are sufficient.

In [4] we proved the following theorem:

THEOREM 1 *Let be $u \in L^2(\omega \times (0,T))$, $u \geq c > 0$ in $\omega \times (0,T)$ and zero outside of ω, $y_0 \in L^2(\Omega)$, $y_0 \geq 0$. Then, there exists $v^* \in L^2(\omega \times (0,T))$, $0 \leq v^* \leq u$, $\|\Psi_{v^*}(T)\| < \|y(T)\|$ such that*

$$\hat{u} = \frac{\|y(T)\|v^* - \|\Psi_{v^*}(T)\|u}{\|y(T)\| - \|\Psi_{v^*}(T)\|}$$

is a control in ω for the initial data y_0.

The solution of a problem like (1) with right-hand side v^* is denoted by Ψ_{v^*} and $\|\cdot\|$ is the norm in $L^2(\Omega)$.

When the initial data is any function $y_0 \in L^2(\Omega)$, we apply this theorem with y_0^+ and y_0^-.

The proof is based on a kind of maximum principle in the final time and the linearity of the equation. It does not use the standard techniques of Carleman's inequalities, (see [2], [1]), because of the diffusion coefficients are not continuous functions in general. Besides, the spatial dimension is any (see [3]).
An interesting application of control to this type of problem is the diffusion of cancer cells in a brain tumor (see [5]). The model consists of a linear parabolic problem where the diffusion coefficients are piecewise constant functions.

References

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*Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, 41004 Sevilla (SPAIN). Email: gayte@us.es
†Departamento de Métodos Cuantitativos, Universidad Loyola Andalucía, 41704 Dos Hermanas (SPAIN). Email: imgayte@uloyola.es