WORKSHOP ON PDES AND CONTROL 2025 (PKM-60)
September 3-5, 2025, Sevilla, Spain

On a model of flows in a deformable porous solid with small strain and density depending material modulus

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Consider the following model of incompressible slow flows in a deformable solid in $\mathbb{R}^d$, with an implicit constitutive relation for the Cauchy stress tensor $\mathbf{T}_s$ of the solid:

$$\boldsymbol{\epsilon}_s = E_{1s}(1+\lambda_2 \mathrm{tr}(\boldsymbol{\epsilon}_s))\mathbf{T}_s + E_{2s}(1+\lambda_3 \mathrm{tr}(\boldsymbol{\epsilon}_s))\mathrm{tr}(\mathbf{T}_s)\mathbf{I}, \tag{1}$$

the balance of linear momentum for the solid taking into account the interaction with the fluid through the parameter α:

$$\mathbf{div}(\mathbf{T}_s) + \alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s) = \mathbf{0}, \tag{2}$$

and the flow equation for the fluid taking into account the interaction with the solid:

$$\begin{aligned} \alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s) - \mu_f \Delta\, \mathbf{v}_f + \nabla\, p_f &= -\varrho_f \partial_t \mathbf{v}_f, \\ \mathrm{div}\, \mathbf{v}_f &= 0. \end{aligned} \tag{3}$$

Here $\mathrm{tr}(\mathbf{T}_s)$ is the trace of the tensor $\mathbf{T}_s$, $\boldsymbol{\epsilon}_s$ is the symmetric gradient tensor of the solid's displacement $\mathbf{u}_s$, μ_f and ϱ_f are the fluid's viscosity and density and $E_{1s} > 0$ and $E_{2s} < 0$ are elasticity parameters. The system (1)-(2)-(3) is supplemented with initial and boundary conditions.

The model for the solid is an example taken from [1] for small strain, namely

$$\|\boldsymbol{\epsilon}_s\| \leq \delta \ll 1 \tag{4}$$

where $\|\cdot\|$ is the Frobenius norm. In addition a linearized dependence on the density yields the factors $(1+\lambda_2 \mathrm{tr}(\boldsymbol{\epsilon}_s))$ and $(1+\lambda_3 \mathrm{tr}(\boldsymbol{\epsilon}_s))$, where λ_2 and λ_3 are also assumed to be small. The resulting relation (1) for $\boldsymbol{\epsilon}_s$ remains nonlinear without compactness nor monotonicity property.

We can take advantage of (4) and suitably truncate $\mathrm{tr}(\boldsymbol{\epsilon}_s)$, i.e., replace $\mathrm{tr}(\boldsymbol{\epsilon}_s)$ by $T_{\tilde{\delta}}\mathrm{tr}(\boldsymbol{\epsilon}_s)$, where T_k is the standard truncation operator at height k and $\tilde{\delta} = \sqrt{d}\delta$. This allows to obtain the following expression for $\mathbf{T}_s$:

$$\mathbf{T}_s = \frac{1}{E_{1s}(1+\lambda_2 T_{\tilde{\delta}} \mathrm{div}\, \mathbf{u}_s)}\Big(\boldsymbol{\epsilon}_s - E_{2s}(1+\lambda_3 T_{\tilde{\delta}} \mathrm{div}\, \mathbf{u}_s)\frac{\mathrm{div}\, \mathbf{u}_s}{F(\mathbf{u}_s)}\mathbf{I}\Big)$$

where

$$F(\mathbf{u}_s) = E_{1s}(1+\lambda_2 T_{\tilde{\delta}} \mathrm{div}\, \mathbf{u}_s) + dE_{2s}(1+\lambda_3 T_{\tilde{\delta}} \mathrm{div}\, \mathbf{u}_s).$$

This new formulation does not change the problem as long as (4) holds. In particular, it does not remedy the lack of compactness and monotonicity but permits to derive some a priori estimates. However, owing that λ_2 and λ_3 are small, the new formulation can now be viewed as a small perturbation of the fully linear model (i.e., with $\lambda_2 = \lambda_3 = 0$) which is itself well-posed. Existence of an exact solution can be obtained by an implicit function argument inspired by [2, 3]. In contrast, the derived a priori estimates allow to directly carry the error analysis of some standard finite element methods without invoking the implicit function theorem.

Acknowledgements

AB is partially supported by the NSF Grant DMS-2409807. DG acknowledges the support of the Natural Science and Engineering Research Council (NSERC), grant RGPIN-2021-04311.

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