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Some Questions About Second-Grade Fluid Equations

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In this talk, we will present some results obtained in [1], [3], [4], [5], [6], on the flow of an incompressible non-Newtonian fluid of grade two in $\Omega \times (0,\infty) \subseteq \mathbb{R}^3 \times (0,\infty)$:

$$(1) \quad \begin{cases} \partial_t(\mathbf{u} - \alpha\Delta\mathbf{u}) - \mu\Delta\mathbf{u} + \mathrm{rot}(\mathbf{u} - \alpha\Delta\mathbf{u}) \times \mathbf{u} + \nabla p = \mathbf{0}, \\ \mathrm{div}\,\mathbf{u} = 0, \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases}$$

Here, $\mathbf{u}$ and p denote the fluid velocity and pressure, respectively. Furthermore, $\mu > 0$ represents the kinematic viscosity and $\alpha > 0$ is a parameter characterizing the fluid's non-Newtonian behavior.

Next, let us consider the natural norm

$$\|\mathbf{u}(t)\|^2_{H^1_\alpha(\mathbb{R}^3)^3} := \|\mathbf{u}(t)\|^2_{L^2(\mathbb{R}^3)^3} + \alpha\|\nabla\mathbf{u}(t)\|^2_{L^2(\mathbb{R}^3)^3}$$

and the following function space:

$$V_2(\mathbb{R}^3)^3 := \{\mathbf{u} : \mathbf{u} \in H^1(\mathbb{R}^3)^3, \mathrm{curl}(\mathbf{u} - \alpha\Delta\mathbf{u}) \in L^2(\mathbb{R}^3)^3\}.$$

We also define, for $\mathbf{u}_0 \in L^2(\mathbb{R}^n)^n$ and $r \in \left(-\frac{n}{2}, \infty\right)$, the upper decay indicator of $\mathbf{u}_0$ by

$$P_r(\mathbf{u}_0)_+ := \limsup_{\rho\to 0^+} \rho^{-2r-n} \int_{B_\rho} |\widehat{\mathbf{u}_0}(\xi)|^2\, d\xi,$$

where $B_\rho := \{\xi \in \mathbb{R}^n : |\xi| \le \rho\}$. Moreover, we define the upper decay character of $\mathbf{u}_0 \in L^2(\mathbb{R}^n)^n$ by

$$r^*_+(\mathbf{u}_0) := \sup\{r \in \mathbb{R} : P_r(\mathbf{u}_0)_+ < \infty\}.$$

In relation to the article [4], our main results are as follows:

THEOREM 1 *Let $\mathbf{u}_0 \in V_2(\mathbb{R}^3)^3$ and suppose that $r^*_+(\mathbf{u}_0) = r^*_+ \in \left(-\frac{3}{2}, \infty\right)$. Additionally, assume that $\|\mathbf{u}_0\|_{V_2(\mathbb{R}^3)^3} < \epsilon$ for a sufficiently small $\epsilon > 0$. Then, for any weak solution $\mathbf{u}$ to (1), the following estimate holds:*

$$\|\mathbf{u}(t)\|^2_{H^1_\alpha(\mathbb{R}^3)^3} \le C\,(t+1)^{-\min\left\{\frac{3}{2}+r^*_+,\,\frac{5}{2}\right\}}, \qquad \forall\, t \ge 0,$$

where the constant $C > 0$ depends only on $\|\mathbf{u}_0\|_{V_2(\mathbb{R}^3)^3}, \alpha, r^*_+$, and μ.

We also compare the evolution of solutions $\mathbf{u}(t)$ to (1) with the solutions $\overline{\mathbf{u}}(t)$ of the linear system associated, which is the following pseudo-parabolic equation in $\mathbb{R}^3 \times (0,\infty)$:

$$(2) \quad \begin{cases} \partial_t(\overline{\mathbf{u}} - \alpha\Delta\overline{\mathbf{u}}) - \mu\Delta\overline{\mathbf{u}} = \mathbf{0}, \\ \mathrm{div}\,\overline{\mathbf{u}} = 0, \\ \overline{\mathbf{u}}(0) = \mathbf{u}_0. \end{cases}$$

THEOREM 2 *Let $\mathbf{u}_0 \in H^4(\mathbb{R}^3)^3$ with $\mathrm{div}\,\mathbf{u}_0 = 0$, and suppose that $\mathbf{u}_0$ is small in $V_2(\mathbb{R}^3)^3$, as in Theorem 1. Let $\mathbf{u}$ be a weak solution to (1), and let $\overline{\mathbf{u}}$ be the solution to the linear part (2) with the same initial data $\mathbf{u}_0 \in H^4(\mathbb{R}^3)^3$. Then, for $r^*_+(\mathbf{u}_0) = r^*_+$, with $-\frac{3}{2} < r^*_+ < \infty$, we have*

$$\|\mathbf{u}(t) - \overline{\mathbf{u}}(t)\|^2_{H^1_\alpha(\mathbb{R}^3)^3} \le C\,(t+1)^{-\min\left\{\frac{5}{2}+\frac{3}{2}r^*_+,\,\frac{5}{2}\right\}}, \qquad \forall\, t \ge 0,$$

i.e., the solution $\mathbf{u}$ of (1) is asymptotically equivalent to the solution $\overline{\mathbf{u}}$ of the pseudo-parabolic equation (2) with the same data.

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