WORKSHOP ON PDEs AND CONTROL 2025 (PKM-60)
September 3-5, 2025, Sevilla, Spain

Observability inequality for the Grushin equation on a multi-dimensional domain

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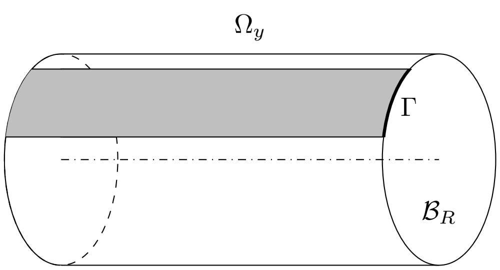
Introduction and main result

Let $\mathcal{B}_R$ denote the open ball of radius $R > 0$ of $\mathbb{R}^{d_1}$, and Ω_y denote a non-empty bounded open set of $\mathbb{R}^{d_2}$ with $d_1, d_2 > 0$. For $T > 0$, we consider u a solution of the Grushin equation posed in $(0,T) \times \mathcal{B}_R \times \Omega_y$, i.e. satisfying

$$(G)\quad \begin{cases} \partial_t u - \Delta_x u - ||x||^2 \Delta_y u = 0 \text{ in } (0,T)\times\mathcal{B}_R\times\Omega_y, \\ u = 0, \text{ on } (0,T)\times\partial(\mathcal{B}_R\times\Omega_y). \end{cases}$$

It is a degenerate parabolic equation where the diffusion coefficient in y vanishes at the center of the ball.



For $\Gamma \subset \partial\mathcal{B}_R$, we say that the Grushin equation is observable through $\Gamma \times \Omega_y$ at time T if there exists a constant C such that any solution of (G) satisfies

$$(O_T)\quad \int_{\mathcal{B}_R\times\Omega_y} |\nabla u(T)|^2 dx dy \le C \int_0^T \int_{\Omega_y} \int_\Gamma \left|\frac{\partial u}{\partial n}\right|^2 d\sigma(x) dy dt,$$

THEOREM 1 *Let Γ be a non-empty open subset of $\partial\mathcal{B}_R$, and $T^* = \frac{R^2}{2d_1}$, then (O_T) holds for all $T > T^*$, and (O_T) does not hold for all $T < T^*$.*

Remarks: The minimal time T^* appears because of the degeneracy at $x = 0$. Indeed if there is no degeneracy, the system reduces to the heat equation which is observable at any $T > 0$.

If $\Gamma = \partial\mathcal{B}_R$, this result is known, see [1]. The goal of this work is to generalize this result if Γ is a non-empty open subset of $\partial\mathcal{B}_R$. Since the negative result on $\partial\mathcal{B}_R$ implies the negative result on Γ, it remains to prove that (O_T) holds for $T > T^*$.

Strategy of proof

We denote $(\lambda_p, \phi_p)_{p\in\mathbb{N}}$ the sequence of eigenvalues and eigenfunctions of the Dirichlet-Laplacian on Ω_y. As it is done in [1] we decompose the equation in the basis $(\phi_p)_{p\in\mathbb{N}}$. Then $u_p := \langle u, \phi_p\rangle$ satisfies the harmonic-heat equation

$$(H_p)\quad \begin{cases} \partial_t u_p - \Delta u_p + \lambda_p^2 ||x||^2 u_p = 0 \text{ in } (0,T)\times\mathcal{B}_R, \\ u_p = 0, \text{ on } (0,T)\times\partial\mathcal{B}_R. \end{cases}$$

The strategy is to prove that (H_p) is observable through Γ uniformly in p. Here are the main steps to get this observability inequality.

- From [2] we already know that (H_p) is observable through $\partial\mathcal{B}_R$ at any time $T > 0$. However, we need to refine this result to get an observability constant which is explicit in T and λ_p.
- We apply a Lebeau-Robbiano type strategy to obtain the observability through Γ. To this end we adapt Miller's proof [3], tracking the dependance in λ_p in the estimates.
- We deduce that (H_p) is observable through Γ uniformly in p at time T, if $T > T^*$.

Once we have the observability inequality for (H_p) with the constant uniform in p, we obtain (O_T) by summing in p.

References

[1] K. Beauchard, P. Cannarsa, M. Yamamoto, *Inverse source problem and null controllability for multidimensional parabolic operators of Grushin type*, Inverse Problem, 2014. https://doi.org/10.1088/0266-5611/30/2/025006

[2] K. Beauchard, J. Dardé, S. Ervedoza, *Minimal time issues for the observability of Grushin-type equations*, Annales de l'Institut Fourier, 2020. https://doi.org/10.5802/aif.3313

[3] L. Miller, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*. Discrete and Continuous Dynamical Systems - B, 2010. https://doi.org/10.3934/dcdsb.2010.14.1465

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