Geometric inverse problem of determining multidimensional domains

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joint work with

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AGENCIA ESTATAL DE INVESTIGACIÓN

Outline

1. Introduction
   - General ideas
   - Our background
   - Purpose of the new work
   - Related works
2. Determining multidimensional domains
   - Setting up the problem (same coefficients)
   - Main results (same coefficients)
   - Conclusions and other cases (same coefficients)
   - Setting up the problem (different coefficients)
   - Main results (different coefficients)
3. Work in progress and open problems

Outline

General ideas

- **Geometric inverse problems**:
  - One interest: find causes for an observed effect $\rightarrow$ **identification** or **reconstruction**.
  - Great development $\rightarrow$ relevant for **applications**: elastography and medical imaging, seismology, fluid mechanics, traffic models, finances...
  - Why study uniqueness?
    - **Well-posed** in the sense of Hadamard (1902): existence, uniqueness and stability.
    - If one of those conditions is **not satisfied** $\Rightarrow$ problem is **ill-posed**.
    - Majority of IP are **not well-posed**.

Our background

- “Uniqueness and numerical reconstruction for inverse problems dealing with interval size search”,
- “Some Inverse Problems for the Burgers Equation and Related Systems”:
    - (2021) [Apraiz, Cheng, Doubova, Fernández-Cara, Yamamoto].
    - **1D heat, wave, Burgers** and **related equations**.
    - **Goal**: find the size of the spatial interval from some appropriate boundary observations.
    - **Uniqueness** sensitive to boundary or initial data.

Purpose of the new work

- Geometric inverse problems $\longrightarrow$ linear **parabolic** systems (unknown initial data and coefficients) with non-homogeneous part $f(x,t)$ satisfying some specific assumptions.
- Goals:
  - **Identify a subdomain** within a multidimensional set $\Omega \subset \mathbb{R}^d$ ($d \geq 2$).
  - Establish **uniqueness results** through observations on a part of the boundary or in an interior domain.
- Derive **information** about the **initial data**.
- Main tools: **unique continuation**, time **analyticity** of the solutions and **semigroup theory**.

Some related works

- Detecting cavities by electrostatic boundary measurements (2002) [Alessandrini, Morassi, Roset].
- Identification of inmersed obstacle via boundary measurements (2005) [Alvarez, Conca, Friz, Kavian, Ortega].
- A geometric inverse problem for the Boussinesq system (2006) [Doubova, Fernández-Cara, González-Burgos, Ortega].
- Introduction to Inverse Problems for Evolution Equations: Stability and Uniqueness by Carleman Estimates (2025) [Yamamoto].

Outline

Setting up the problem (same coefficients) I

- $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) and $D_1, D_2 \subset\subset \Omega$.
- For $k = 1, 2$,

$$\begin{cases} \partial_t u_k + \mathcal{A} u_k = f(x,t) & \text{in } (\Omega \setminus \overline{D}_k) \times (0,T), \\ u_k = 0 & \text{on } \partial D_k \times (0,T), \\ u_k = g(x,t) & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{1}$$

where $f \not\equiv 0$ is an external source, $g \not\equiv 0$ is a boundary input in $(0,T)$ and

$$\mathcal{A}v(x) := -\sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v(x)) + \sum_{j=1}^{d} b_j(x)\partial_j v(x) + c(x)v(x).$$

Setting up the problem (same coefficients) II

- **Assume**: $a_{ij} = a_{ji}$ in $C^1(\overline{\Omega})$, b_j and c in $L^\infty(\Omega)$, $f \in L^2(\Omega \times (0, T))$, $g \in L^2(0, T; H^{3/2}(\partial\Omega))$, $c(x) \geq c_0 > 0$ in Ω for a constant c_0 sufficiently large, and $\exists\, \alpha > 0$:

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^d, \quad \text{a.e. in } \Omega. \tag{2}$$

- **Observe**: initial values of u_1 and u_2 are not specified.

Setting up the problem (same coefficients) III

- For $k = 1, 2$, **linear operators** $A_k : \mathcal{D}(A_k) \to L^2(\Omega \setminus \overline{D}_k)$, with

  $$\mathcal{D}(A_k) := \{v \in H_0^1(\Omega \setminus \overline{D}_k) : \ \mathcal{A} v \in L^2(\Omega \setminus \overline{D}_k)\}$$

  and

  $$(A_k v)(x) := \mathcal{A} v(x) \ \text{ a.e. in } \Omega \setminus \overline{D}_k$$

  for all $v \in \mathcal{D}(A_k)$.

- **Conormal derivative** associated to the coefficients a_{ij}:

  $$\frac{\partial v}{\partial \nu_A} := \sum_{i,j=1}^{d} a_{ij} \partial_i v \, \nu_j$$

  ($\nu = \nu(x)$ **outward unit normal vector** at points $x \in \partial\Omega$).

Setting up the problem (same coefficients) IV

Question Q1

- $\gamma \subset \partial\Omega$ nonempty and open.
- u_1 and u_2 **weak solutions** to (1) corresponding to D_1 and D_2 ($\forall k \in \{1, 2\}$, $u_k \in L^2(0, T; H^1_0(\Omega \setminus \overline{D}_k))$ and satisfy (1) in the distributional sense).

$$\text{Does } \frac{\partial u_1}{\partial \nu_A} = \frac{\partial u_2}{\partial \nu_A} \text{ on } \gamma \times (0, T) \Longrightarrow D_1 = D_2?$$

Main results (same coefficients) I

- **Assumptions** for f :
  - $$f, \partial_t f, \dots, \partial_t^m f \in L^2(\Omega \times (0,T)) \text{ for some } m \geq 0 \tag{3}$$
  - $$\begin{cases} \exists\, t_0, t_1, t_2 \text{ with } 0 < t_0 < t_1 < t_2 \leq T \text{ such that} \\ \partial_t^m f(x,t) = \begin{cases} a_1 f_0(x) + r_1(x,t) & \text{for } t_0 < t < t_1, \\ a_2 f_0(x) + r_2(x,t) & \text{for } t_1 < t < t_2, \end{cases} \\ \text{where } r_1 : (t_0, t_1] \to L^2(\Omega) \text{ is analytic, } r_2 \in L^2(\Omega \times (t_1, t_2)), \\ a_1, a_2 \in \mathbb{R},\ f_0 \in L^2(\Omega) \text{ and } a_1 f_0(x) \not\equiv a_2 f_0(x). \end{cases} \tag{4}$$
  - **Assumptions** on r_1: $\exists\, \varepsilon > 0$ such that r_1 can be extended to an analytical function in $(t_0, t_1 + \varepsilon)$.
  - $f_0 \not\equiv 0$ and $a_1 \neq a_2$.
- **Assumptions** for g:
  - It's independent of t and $g \in H^{3/2}(\partial\Omega)$.

Main results (same coefficients) II

Theorem (1 - Answer to Q1)

Let u_1 and u_2 be solutions to (1) respectively corresponding to the simply connected open sets D_1 and D_2. Suppose that f satisfies (3), (4) and moreover the functions f_0, r_1 and r_2 in (4) satisfy

$$\begin{cases} f_0(x) = 0 \text{ in } D_1 \cup D_2,\ r_1(x,t) = 0 \text{ in } (D_1 \cup D_2) \times (t_0, t_1) \\ \text{and } r_2(x,t) = 0 \text{ in } (D_1 \cup D_2) \times (t_1, t_2). \end{cases} \tag{5}$$

Then the answer to **Q1** *is yes. Moreover, $u_1(\cdot\,,0) = u_2(\cdot\,,0)$ in $\Omega \setminus (\overline{D_1 \cup D_2})$.*

Conclusions and other cases (same coefficients) I

- **Answers** for **Q1** in **other cases**:

1. $f(x,t) = f_0(x)\mu(t)$, where

$$\begin{cases} f_0 \in L^2(\Omega),\ f_0(x) = 0 \text{ in } D_1 \cup D_2 \text{ and } f_0 \not\equiv 0, \\ \mu \text{ is piecewise polynomial and } \mu \notin C^m([0,T]) \text{ for some } m \geq 0. \end{cases}$$

$\Longrightarrow$ **uniqueness** for **Q1**.

2. $f \equiv 0$ and $g \not\equiv 0$.
   - $g(x,t) = g_0(x)\mu(t)$ for all $(x,t) \in \partial\Omega \times (0,T)$, where $g_0 \in H^{3/2}(\partial\Omega)$ and

$$\mu(t) = \begin{cases} a_1 t, & \text{if } 0 < t < t_1, \\ a_2(t - t_1) + a_1 t_1, & \text{if } t_1 < t < T \end{cases}$$

for some $a_1, a_2 \in \mathbb{R}$ with $a_1 \neq a_2$ and some t_1 with $0 < t_1 < T$.

$\Longrightarrow$ **uniqueness** for **Q1**.

Conclusions and other cases (same coefficients) II

3. $f \equiv 0$ and $g \not\equiv 0$ and

$$u_k(x,0) = 0 \ \text{ in } \Omega \setminus \overline{D_k} \text{ for } k = 1,2. \tag{6}$$

$\Longrightarrow$ **uniqueness** for **Q1**.

4. $f \equiv 0$ and $g \not\equiv 0$ and (6) is not satisfied $\Longrightarrow$ **uniqueness** can **fail** for **Q1** (Counterexample).

5. (6), $f \not\equiv 0$ and $g \equiv 0 \Longrightarrow$ **uniqueness** can **fail** for **Q1**

6. (6), $g \equiv 0$ and $f(x,t) = f_0(x)\mu(t)$ with a smooth $\mu \Longrightarrow$ **uniqueness** for **Q1**:

Proposition

Let us assume that $f(x,t) = f_0(x)\mu(t)$ *a.e. with*

$$\begin{cases} f_0 \in L^2(\Omega), \ \ \textit{Supp}\, f_0 \subset \Omega \setminus \overline{(D_1 \cup D_2)} \ \ \textit{and} \ \ f_0 \not\equiv 0, \\ \mu \in C^1([0,T]) \ \ \textit{and} \ \ \mu \not\equiv 0 \end{cases}$$

and $g \equiv 0$. *Also, let us assume that* (6) *holds. Then, the answer to* **Q1** *is yes.*

Setting up the problem (different coefficients) I

- $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ and $D_1, D_2 \subset\subset \Omega$.
- For $k = 1, 2$,

$$\mathcal{A}^k v(x) := -\sum_{i,j=1}^{d} \partial_i (a_{ij}^k(x) \partial_j v(x)) + \sum_{j=1}^{d} b_j^k(x) \partial_j v(x) + c^k(x) v(x)$$

- **Assume:** $a_{ij}^k = a_{ji}^k \in C^1(\overline{\Omega})$, $b_j^k, c^k \in L^\infty(\Omega)$ given for $k = 1, 2$, with the a_{ij}^k satisfying (2) and the c^k satisfying $c^k(x) \geq c_0 > 0$ a.e. in Ω for c_0 sufficiently large.
- **Operators** $P_k : \mathcal{D}(P_k) \to L^2(\Omega \setminus \overline{D}_k)$ as before:

$$\mathcal{D}(P_k) := \{ v \in H_0^1(\Omega \setminus \overline{D}_k) : \ \mathcal{A}^k v \in L^2(\Omega \setminus \overline{D}_k) \}$$

and

$$(P_k v)(x) := \mathcal{A}^k v(x) \ \text{ a.e. in } \Omega \setminus \overline{D}_k, \ \forall\, v \in \mathcal{D}(P_k).$$

Setting up the problem (different coefficients) II

Question Q2

- $\omega \subset\subset \Omega \setminus (\overline{D_1 \cup D_2})$ nonempty and open.
- u_k a **weak solution** to

$$\begin{cases} \partial_t u_k + \mathcal{A}^k u_k = f(x,t) & \text{in } (\Omega \setminus \overline{D}_k) \times (0,T), \\ u_k = 0 & \text{on } \partial D_k \times (0,T), \\ u_k = g(x,t) & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{7}$$

for $k = 1, 2$.

Does $u_1 = u_2$ in $\omega \times (0,T) \Longrightarrow D_1 = D_2$?

Main results (different coefficients) I

Theorem (2 - Answer to **Q2**)

Let ω, D_1 and D_2 be as above and let u_k be a weak solution to (7) for $k = 1, 2$. Assume that f satisfies (3), (4) and moreover the functions f_0, r_1 and r_2 in (4) satisfy

$$\begin{cases} f_0(x) = 0 \text{ in } D_1 \cup D_2 \cup \omega,\ r_1(x,t) = 0 \text{ in } (D_1 \cup D_2 \cup \omega) \times (t_0, t_1) \\ \text{and } r_2(x,t) = 0 \text{ in } (D_1 \cup D_2 \cup \omega) \times (t_1, t_2). \end{cases}$$

Also, assume that

$$P_1 P_2 v = P_2 P_1 v \quad \forall v \in C_0^\infty(\Omega \setminus (\overline{D_1 \cup D_2})).$$

*Then the answer to **Q2** is yes.*

Outline

Work in progress and open problems I

1. **Work in progress**: **Reconstruction, same coefficients**:

Let $\gamma \subset \partial\Omega$ be a nonempty open subboundary and u a **weak solution** to

$$\begin{cases} \partial_t u + \mathcal{A}u = f(x,t) & \text{in } (\Omega \setminus \overline{D}) \times (0,T), \\ u = 0 & \text{on } \partial(\Omega \setminus \overline{D}) \times (0,T), \end{cases}$$

for some nonempty simply connected open set $D \subset\subset \Omega$. **Assume** that

$$\frac{\partial u}{\partial \nu_A} = \beta \text{ on } \gamma \times (0,T).$$

Can we find D (and $u|_{t=0}$) from f and β?

Work in progress and open problems II

- **Reformulation** of the reconstruction problem:

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\left\| \dfrac{\partial u}{\partial \nu_A} - \beta \right\|_X^2 \\ \text{Subject to } D \in \mathcal{B},\ u_0 \in L^2(\Omega \setminus \overline{D}),\ u \ \text{ solves (8)}, \end{cases}$$

where β is given, the **admissible class of subdomains** $\mathcal{B}$ and the **Hilbert space** X are appropriately chosen and:

$$\begin{cases} \partial_t u + \mathcal{A}u = f(x,t) & \text{in } \Omega \setminus \overline{D} \times (0,T), \\ u = 0 & \text{on } \partial(\Omega \setminus \overline{D}) \times (0,T), \\ u|_{t=0} = u_0 & \text{in } \Omega \setminus \overline{D}. \end{cases} \tag{8}$$

- Numerical resolution $\longrightarrow$ **method of fundamental solutions.**
- **Useful**: "Some new results for geometric inverse problems with the method of fundamental solutions" (2021) [Carvalho, Doubova, Fernández-Cara, Rocha de Faria].

Work in progress and open problems III

2. **Similar results for other equations**:

   2.1 **Quasi-Stokes system** (linear parabolic): for $k = 1, 2$, (u_k, p_k) solutions to

$$\begin{cases} \partial_t u_k - \nu_0 \Delta u_k + (a \cdot \nabla) u_k + (u_k \cdot \nabla) b + \nabla p_k = f(x,t) & \text{in } (\Omega \setminus \overline{D}_k) \times (0,T), \\ \nabla \cdot u_k = 0 & \text{in } (\Omega \setminus \overline{D}_k) \times (0,T), \\ u_k = 0 & \text{on } \partial(\Omega \setminus \overline{D}_k) \times (0,T). \end{cases}$$

- $\nu_0 > 0$, $a, b \in L^\infty(\Omega)^d$, and the components of f satisfy (3)–(5).
- **Notation**:

$$\sigma(u,p) := -p\,\mathrm{Id.} + 2\nu_0\, e(u), \quad \text{where} \quad e(u) := \frac{1}{2}(\nabla u + (\nabla u)^t)$$

- **Assume**

$$\sigma(u_1, p_1) \cdot \nu = \sigma(u_2, p_2) \cdot \nu \quad \text{on} \quad \gamma \times (0,T).$$

$\Longrightarrow D_1 = D_2$.

Work in progress and open problems IV

2.2 **Linearized Boussinesq systems**: for $k = 1, 2$ (u_k, p_k, θ_k) satisfies

$$\begin{cases} \partial_t u_k - \nu_0 \Delta u_k + (a\cdot\nabla) u_k + (u_k\cdot\nabla) b + \nabla p_k = \theta_k g + f(x,t) & \text{in } (\Omega\setminus\overline{D}_k)\times(0,T), \\ \nabla\cdot u_k = 0 & \text{in } (\Omega\setminus\overline{D}_k)\times(0,T), \\ \partial_t\theta_k - \kappa_0\Delta\theta_k + a\cdot\nabla\theta_k + u_k\cdot\nabla c = 0 & \text{in } (\Omega\setminus\overline{D}_k)\times(0,T), \\ u_k = 0, \quad \theta_k = 0 & \text{on } \partial(\Omega\setminus\overline{D}_k)\times(0,T), \end{cases}$$

- $\nu_0 > 0$, $g \in \mathbb{R}^d$, $\kappa_0 > 0$, $a, b \in L^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$.
- **Assume**

$$\sigma(u_1, p_1)\cdot\nu = \sigma(u_2, p_2)\cdot\nu \text{ and } \frac{\partial\theta_1}{\partial\nu} = \frac{\partial\theta_2}{\partial\nu} \text{ on } \gamma\times(0,T),$$

$$\Longrightarrow D_1 = D_2.$$

☞ **Our work:** *"Uniqueness in determining multidimensional domains with unknown initial data"*. J. Apraiz, A. Doubova, E. Fernández-Cara, M. Yamamoto. Inverse Problems (2025).