Minimality of sequences and control for parabolic systems

Solved and open problems

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Minimality and control for parabolic systems

The aim of this talk is to link the controllability of parabolic problems to the minimality of sequences in a Hilbert space. We will start from the contributions of Manuel González-Burgos (and his collaborators on the boundary control of parabolic systems) to arrive at to recent results obtained with Manuel González-Burgos, Morgan Morancey and Luz de Teresa. Here is the outline :

1. Control of parabolic systems: Method of Moments and biorthogonal families;
2. Minimal sequences in Hilbert spaces;
    1. Application 1 : Carleman inequalities and minimal sequences;
    2. Application 2 : Spectral inequality and minimal sequences;
    3. Some extensions and open problems for the union of minimal sequences

The beginning of our collaboration

In the early $2000's$, we were working on the *internal* control of two coupled parabolic equations:

$$\begin{cases} \partial_t y - \big(D\Delta y + A\big)\, y = 1_\omega B v & \text{in } \Omega \times (0,T), \\ y(\cdot,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases}$$

where

$$\omega \subset \Omega, \quad D, A \in \mathcal{M}_2(\mathbb{R}), \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and we wanted to understand the results on nonlinear control for the scalar heat equation. We then invited Enriqué to Besançon to explain it to us. He told us that in Seville there were also people working on the same subject, Manolo and Chari (and we later learned that Lucero had already obtained results on this problem). So we applied for an exchange program between our two universities.

The beginning of our collaboration

We organized a meeting in Besançon (in which some of this room participated). Manolo asked us if we knew how to handle the boundary control of two parabolic equations:

$$\begin{cases} \partial_t y - \big(D\Delta y + A\big)\, y = 0 & \text{in } \Omega \times (0,T), \\ y(\cdot,t) = 1_\Gamma B v & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases}$$

where $\Gamma \subset \partial\Omega$. Our answer was: of course, we do as in the scalar case to go from internal control to boundary control. Of course, our answer was wrong and Manolo proved it to us on the board.

We then tried to use Carleman's inequalities to address this question of boundary controllability, but without success! And now we know why!

The beginning of our collaboration

Years passed and Manolo, in collaboration with Enriqué and Lucero, was able to obtain the first boundary control result for two coupled parabolic equations. To do this, they extracted themselves from the "Carleman inequalities", which at the time were dominant in parabolic control, and brought the method of moments back into fashion. This method was used for the control of a heat equation in 1-dimensional space by Fattorini and Russell in the 1970*s*. Then it was "forgotten".
Based on this, Manolo, Lucero and we decided to work on the general case:

$$\begin{cases} \partial_t y - \left(D\partial_{xx}^2 + A\right) y = 0 & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

where $T > 0$ is a given time, $D, A \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{m,n}(\mathbb{R})$, $n, m \in \mathbb{N}$, $m \leq n$.

Boundary control of coupled parabolic equations: an example

$$\begin{cases} \partial_t y - \left(D\partial_{xx}^2 + A\right) y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

where $T > 0$ is a given time,

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \text{ (with } d > 0), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The case $d = 1$ had already been studied in:

E. Fernandez Cara, M. González-Burgos, L. de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. **259** (2010), no. 7, pp. 1720–1758.

The method of moments

- Let φ be a solution of the adjoint problem:

$$\begin{cases} -\partial_t \varphi - \left(D\partial_{xx}^2 + A^*\right)\varphi = 0, & \text{in } Q = (0,\pi)\times(0,T), \\ \varphi(0,\cdot) = \varphi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \varphi(\cdot,T) = \varphi^0 \ \in H_0^1(0,\pi)^2. \end{cases}$$

- If y is a solution of the direct problem, then

$$\left\langle y(T), \varphi^0 \right\rangle = \left\langle y^0, \varphi(0) \right\rangle + \int_0^T v(t) B^* D \partial_x \varphi(0,t)\, dt$$

Thus $y(T) = 0$ if, and only if, there exists $v \in L^2(0,T)$ such that

$$\int_0^T v(t) B^* D \partial_x \varphi(0,t)\, dt = -\left\langle y^0, \varphi(0) \right\rangle, \ \forall \varphi^0 \in H_0^1(0,\pi)^2.$$

The method of moments

Material at our disposal

- **The spectrum**

$$\sigma\left(D\partial_{xx}^2 + A^*\right) = \cup_{k\geq 1}\left\{-k^2, -dk^2\right\} := \cup_{k\geq 1}\left\{-\lambda_{k,1}, -\lambda_{k,2}\right\}$$

- $V_{k,1}$ and $V_{k,2}$: eigenvectors of the matrix $\left(-k^2 D + A^*\right)$.
- **Eigenfunctions of** $\left(D\partial_{xx}^2 + A^*\right)$: $\Phi_{k,i} = V_{k,i}\sin(kx)$.
- $\left(\Phi_{k,i}\right)$ is a (Riesz) basis of $H_0^1(0,\pi)^2$

The method of moments

- $$y(T) = 0 \Leftrightarrow \left\langle y(T), \Phi_{k,i} \right\rangle = 0, \quad \forall k \geq 1,\, i = 1,2$$
- Choosing $\varphi^0 = \Phi_{k,i}$, we have $\varphi\,(x,t) = e^{-\lambda_{k,i}(T-t)}\Phi_{k,i}$ and
$$\varphi(x,0) = e^{-\lambda_{k,i}T}\Phi_{k,i}(x), \quad \partial_x\varphi(0,t) = ke^{-\lambda_{k,i}(T-t)}V_{k,i}$$
- Thus $y\,(T) = 0$ if, and only if, there exists $v \in L^2(0,T)$ such that
$$kB^*DV_{k,i}\int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\left\langle y^0, \Phi_{k,i}\right\rangle, \ \forall\,(k,\ i)$$

$$\begin{cases} \partial_t y - \left(D\partial_{xx}^2 + A\right)y = 0 & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

Approximate controllability : a necessary condition (I)

- $$kB^*DV_{k,i}\int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -e^{-\lambda_{k,i}T}\left\langle y^0,\Phi_{k,i}\right\rangle,\ \forall\,(k,\ i)$$
- A necessary condition : $B^*DV_{k,i} \neq 0$ for all $k \geq 1, i = 1,2$.
- Recall

$$B^* = (0,1), \quad V_{k,1} = \begin{pmatrix} 1 \\ \frac{1}{(d-1)k^2} \end{pmatrix}, \quad V_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \forall k \geq 1.$$

So, here

$$B^*DV_{k,i} \neq 0, \quad \forall k \geq 1, i = 1,2.$$

Approximate controllability : a necessary condition (II)

If $\lambda_{k,1} = \lambda_{j,2} = \lambda$, then:

$$\begin{cases} kB^*DV_{k,1}\int_0^T v(T-t)e^{-\lambda t}\,dt = -e^{-\lambda T}\left\langle y^0, \Phi_{k,1}\right\rangle \\ \\ jB^*DV_{j,2}\int_0^T v(T-t)e^{-\lambda t}\,dt = -e^{-\lambda T}\left\langle y^0, \Phi_{j,2}\right\rangle \end{cases}$$

So it is necessary to have $\lambda_{k,1} \neq \lambda_{j,2}$ and this is equivalent to:

$$k^2 \neq dj^2, \quad \forall k \neq j \geq 1 \Leftrightarrow \sqrt{d} \notin Q$$

So, in the sequel, we will assume :

$$\sqrt{d} \notin \mathbb{Q}$$

The method of moments

Summarizing

Let $m_{k,i} = -\frac{\left\langle y^0, \Phi_{k,i} \right\rangle}{kB^* DV_{k,i}} e^{-\lambda_{k,i} T}$

$$\exists\, ?\, v \in L^2(0,T) \;:\; \int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = m_{k,i},\ \forall\, k \geq 1,\ i = 1,2$$

Biorthognal family

Following a remark of Morgan Morancey, by choosing

$$y_0 = kB^*DV_{k,i}e^{\lambda_{k,i}t}\psi_{k,i},$$

where $(\Psi_{k,i})$ is the associated biorthogonal family to $(\Phi_{k,i}))$, the null controllability implies the existence of $\left(q_{k,i}\right)$ in $L^2\left(0,T\right)$ such that:

$$\int_0^T e^{-\lambda_{k,i}t}q_{l,j}(t)\,dt = \begin{cases} 1, & \text{if } (k,i) = \left(l,j\right) \\ 0, & \text{if } (k,i) \neq \left(l,j\right) \end{cases}$$

This family is called a biorthogonal family in $L^2(0,T)$ to $\left(e^{-\lambda_{k,i}t}\right)$. Using :

L. SCHWARTZ, *Étude des Sommes d'Exponentielles*, 2ième éd., Publications de l'Institut de Mathématiques de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris, 1959.

- $\sum\limits_{k\geq 1,\, i=1,2} \frac{1}{|\lambda_{k,i}|} < +\infty$
- $\lambda_{k,1} \neq \lambda_{j,2}$,

the family $\left(e^{-\lambda_{k,i}t}\right)$ admits a biorthogonal family in $L^2(0,T)$.

The method of moments

Biorthognal family

- Let v given by:

$$v(T-t) = \sum_{k\geq 1, i=1,2} m_{k,i} q_{k,i}(t), \quad m_{k,i} = -\frac{\langle y^0, \Phi_{k,i} \rangle}{kB^* DV_{k,i}} e^{-\lambda_{k,i} T}$$

- The question is:

$$\sum_{k\geq 1} m_{k,i} q_{k,i}(t) \in L^2(0,T)?$$

- But this question itself amounts to:

$$\left\| q_{k,i} \right\|_{L^2(0,T)} \underset{k\to\infty}{\sim} ?$$

Here appears the index of condensation of

$$\Lambda := (\lambda_{k,1}, \lambda_{k,2}) = \{k^2, dk^2\}_{k\geq 1}$$

The method of moments

The family $(q_{k,i})$ can be constructed (L. Schwartz, ...) such that

$$\forall \varepsilon > 0, \quad \left\|q_{k,i}\right\|_{L^2(0,T)} \leq C_\varepsilon e^{\lambda_{k,i}(\varepsilon + c(\Lambda))}$$

where $c(\Lambda) \in [0, +\infty]$ is the index of condensation of the sequence

$$\Lambda := (\lambda_{k,1}, \lambda_{k,2}) = \{k^2, dk^2\}_{k\geq 1}.$$

The method of moments

Recall that we want to solve

$$\int_0^T v(T-t)e^{-\lambda_{k,i}t}\,dt = -\frac{\langle y^0, \Phi_{k,i}\rangle}{kB^*DV_{k,i}}e^{-\lambda_{k,i}T} = m_{k,i}, \forall\,(k,\ i)$$

- $v(T-t) = \sum_{k\geq 1} m_{k,i}q_{k,i}(t) \in L^2\,(0,T)$?
- $$|m_{k,i}|\,\|q_{k,i}\|_{L^2(0,T)} \leq C_\varepsilon \frac{\left|\langle y^0, \Phi_{k,i}\rangle_{H^{-1},H_0^1}\right|}{k\,|B^*DV_{k,i}|}e^{-\lambda_{k,i}(T-\varepsilon-c(\Lambda))}$$
- One can check that $k\,|B^*DV_{k,i}| \geq C_\varepsilon e^{-\varepsilon\lambda_{k,i}}$
- Then
$$T > c\,(\Lambda) \Rightarrow v(T-t) = \sum_{k\geq 1} m_{k,i}q_{k,i}(t) \in L^2\,(0,T)\,.$$

Index of condensation : some background

- The *index of condensation* of a sequence $\Lambda = (\lambda_k) \subset \mathbb{C}$ is a real number $c(\Lambda) \in [0, +\infty]$ associated with this sequence and which "measures" the condensation at infinity.
- This notion has been introduced by V.l. Bernstein in 1933 for real sequences:

V. BERNSTEIN, *Leçons sur les progrès récents de la théorie des séries de Dirichlet*, Paris 1933.

- Extended by J. R. Shackell in 1967 for complex sequences:

J. R. SHACKELL, *Overconvergence of Dirichlet series with complex exponents*, J. Anal. Math. 22 (1969) 135-170 .

Index of condensation : some background

Let $\Lambda = (\lambda_k) \subset \mathbb{C}$ be a sequence with pairwise distinct elements and:

$$\exists \delta > 0 : \Re(\lambda_k) \geq \delta |\lambda_k| > 0, \; \forall k \geq 1,$$

$$\lim_{k\to\infty} \frac{k}{|\lambda_k|} = d \in [0, \infty[$$

and the interpolating function:

$$E(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \; E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j\neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)$$

Index of condensation : some background

Definition

The index of condensation of Λ is:

$$c(\Lambda) = \limsup_{k \to \infty} \frac{-\ln |E'(\lambda_k)|}{\Re \lambda_k} \in [0, +\infty].$$

Index of condensation : some background

Zero index of condensation

- **Gap condition 1:**

$$\exists \rho > 0 \; : \; |\lambda_k - \lambda_l| \geq \rho \, |k - l| \Rightarrow c\,(\Lambda) = 0.$$

In particular, for $\lambda_k = k^\alpha$ with $\alpha \geq 1$,

$$|\lambda_k - \lambda_l| = |k^\alpha - l^\alpha| \geq |k - l|$$

$$\Lambda = (k^\alpha)_{k \geq 1} \; (\alpha \geq 1) \Rightarrow c(\Lambda) = 0$$

- **Gap condition 2: If**
$\exists \rho > 0, \; \exists k_1 \geq 1 \,, \left(\forall k \geq k_1, \; \forall \ell \neq k, \;\; |\lambda_k - \lambda_\ell| \geq \rho \sqrt{|\lambda_k|} \right)$ then $c\,(\Lambda) = 0.$

- Let $\alpha > 1$, $\beta > 0$ and $\Lambda = (\lambda_k)_{k\geq 1}$ with $\lambda_{2k} = k^{\alpha}$, $\lambda_{2k+1} = k^{\alpha} + e^{-k^{\beta}}$:

$$c(\Lambda) = \lim \frac{-\ln(e^{-k^{\beta}})}{k^{\alpha}} = \lim k^{\beta-\alpha} = \begin{cases} 0 & \beta < \alpha \\ 1 & \beta = \alpha \\ +\infty & \beta > \alpha \end{cases}$$

Note that: $\liminf_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0.$

- $\Lambda = (\lambda_k)_{k\geq 1}$ with

$$\lambda_{k^2+n} = k^2 + ne^{-k^2}, \quad n \in \{0, \cdots, 2k\}, \quad k \geq 1$$

$$c(\Lambda) = +\infty$$

The result

$$\begin{cases} \partial_t y - \left(D\partial_{xx}^2 + A\right) y = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y^0 & \text{in } (0,\pi), \end{cases}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Theorem

Let $\sqrt{d} \notin \mathbb{Q}$ and $\Lambda = \{k^2, dk^2, k \geq 1\}$. If $T > c(\Lambda)$ then the system is null-controllable at time T. If $T < c(\Lambda)$, the system is not null controllable at time T.

FAK, AB, M. GONZÁLEZ-BURGOS, L. DE TERESA, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014), no. 7, pp. 2077–2151.

$c(\Lambda)$?

Theorem

Let $\Lambda = \left(k^2,\ d\,k^2\right)$. *For any* $\delta \in [0,\infty]$, *there exists* $\sqrt{d} \in \mathbb{R}\backslash\mathbb{Q}$ *such that* $c\,(\Lambda) = \delta$

Remark

El Hadji Samb proved that for $\Lambda = \{k^2, dk^2, k \geq 1\}$,

$$c(\Lambda) = \limsup \frac{-\ln\left|k^2 - dj_k^2\right|}{k^2}$$

where j_k *is the nearest integer to* $\frac{k}{\sqrt{d}}$, *i.e:*

$$\left|\frac{k}{\sqrt{d}} - j_k\right| \leq \frac{1}{2}$$

E.H SAMB, *Boundary null-controllability of two coupled parabolic equations: simultaneous condensation of eigenvalues and eigenfunctions, ESAIM: COCV* **27** *(2021).*

The boundary control of parabolic coupled equations

Summarizing

- High frequency phenomena can prevent boundary null controllability of coupled parabolic equations:
  - This probably explains our failure to solve the problem with Carleman's inequalities
  - The method of moments seems to be the appropriate method
- The method of moments requires that the *"spectral" family* admits a biorthogonal family

Have we answered the question Manolo asked in 2002?

$$\begin{cases} \partial_t y - \left(D\Delta y + A\right) y = 0 & \text{in } \Omega \times (0,T), \\ y(\cdot,t) = 1_\Gamma B v & \text{on } \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases}$$

where $\Gamma \subset \partial\Omega$.

- Only partially: in the case where Ω is an interval (one dimensional case).

The boundary control of parabolic coupled equations

Why our result is restricted to the one dimensional case?

- The sequence $\{e^{-\lambda_n t}, n \in \mathbb{N}^*\}$ admits a biorthogonal family in $L^2(0,T)$ for any $T \in (0,\infty)$ if and only if the series $\sum_{n>0} \frac{1}{\lambda_n}$ converges
- Let $\{\lambda_n, \psi_n\}_{n\geq 1}$ the eigenelements of the Dirichlet-Laplace operator. Due to the Weyl's law, $\lambda_n \sim cn^{\frac{2}{d}}$ and therefore for $d > 1$, $\sum_{n\geq 1} \frac{1}{\lambda_n} = +\infty$.

L. SCHWARTZ, *Étude des Sommes d'Exponentielles*, 2ième éd., Publications de l'Institut de Mathématiques de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris, 1959.

Consequently, to solve the boundary control of parabolic coupled equations in the case where Ω is an open set of $\mathbb{R}^d$ with $d > 1$, it is necessary to prove that the *spectral* family $\{e^{-\lambda_n t} 1_\Gamma Bv,\ n \geq 1\}$ admits a biorthogonal family in $L^2((0,T); L^2(\Gamma))$.

Null-controllability and biorthogonal families: the general case

$$\begin{cases} y' + \mathcal{A}y = \mathcal{B}v \\ y(0) = y_0 \end{cases}$$

on a Hilbert space H. Assume that $\mathcal{A}$ generates a $\mathcal{C}^0-$semigroup and that $\mathcal{B}$ ensures well-posedness for any $v \in L^2(0, T; U)$ where U is the Hilbert space of controls.
We assume that the operator $\mathcal{A}^*$ has a family of eigenvalues $\Lambda \subset (0, +\infty)$ and that the family of associated eigenvectors $\{\psi_\lambda\}_{\lambda\in\Lambda}$ forms a Riesz basis of H.
Then, the control $v \in L^2(0, T; U)$ is such that $y(T) = 0$ if and only if

$$y(T) = 0 \underset{\text{span}\{\psi_\lambda\}_{\lambda\in\Lambda}=H}{\Longleftrightarrow} \langle y(T), \psi_\lambda \rangle = 0,\ \forall \lambda \in \Lambda$$

$$\Longleftrightarrow \left\langle v(T-t), e^{-\lambda t}\mathcal{B}^*\psi_\lambda \right\rangle_{L^2((0,T);U)} = -e^{-\lambda T} \langle y_0, \psi_\lambda \rangle,\ \forall \lambda \in \Lambda.$$

Null-controllability and method of moments

If the family $\mathcal{F} = \{F_\lambda = e^{-\lambda t}\mathcal{B}^*\psi_\lambda\}_{\lambda\in\Lambda}$ admits a biorthogonal family in $L^2((0,T);U)$, that is to say, there exists $\mathcal{G} := \{G_\lambda\}_{\lambda\in\Lambda} \subset L^2((0,T);U)$ such that

$$\left\langle G_\mu, F_\lambda \right\rangle_{L^2((0,T);U)} = \delta_{\lambda\mu}, \qquad \forall \lambda, \mu \in \Lambda,$$

then a formal solution of the problem of moments is

$$v(T-t,\cdot) = u(t,\cdot) = \sum_{\lambda\in\Lambda} \left(-e^{-\lambda T}\langle y_0, \psi_\lambda\rangle\right) G_\lambda$$

Null-controllability and method of moments

Moreover, as Morgan Morancey showed, the null-controllability implies the existence of a biorthogonal family to $\mathcal{F}$.

Consequently, to solve a null-control problem, it is necessary to prove the existence of a biorthogonal family to $\mathcal{F} := \{e^{-\lambda t}\mathcal{B}^*\psi_\lambda,\ \lambda \in \Lambda\}$ in $L^2((0,T);U)$. Furthermore if T is such that the series

$$\sum_{\lambda\in\Lambda} \left(-e^{-\lambda T}\langle y_0, \psi_\lambda\rangle\right) G_\lambda$$

converges in $L^2((0,T);U)$, then the null-control problem has a solution.

Main question

Does family $\mathcal{F} = \{F_\lambda = e^{-\lambda t}\mathcal{B}^*\psi_\lambda\}_{\lambda\in\Lambda}$ admit a biorthogonal family in $L^2((0,T);U)$?

To answer to the previous question one can

- Construct an *explicit* biorthogonal family.
  - This was done for the family $\{e^{-\lambda t}, \lambda \in \Lambda\}$ with the assumption

$$\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|} < +\infty$$

L. SCHWARTZ, *Étude des Sommes d'Exponentielles*, 2ième éd., Publications de l'Institut de Mathématiques de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris, 1959.

  - This was done recently under two assumptions
    1. there exist positive constants κ and θ such that

$$\mathcal{N}_\Lambda(r) \leq \kappa r^\theta, \quad \forall r \in (0, \infty)$$

    2. $\Psi = \{\psi_n\}_{n \geq 1}$ is an orthonormal basis of $L^2(\Omega)$ that satisfies:there exists $C > 0$, $\beta > 0$ such that for all $\lambda > 0$:

$$\left\| \sum_{\sqrt{\lambda_n} \leq \lambda} b_n \psi_n(x) \right\|^2_{L^2(\Omega)} \leq C e^{\beta\lambda} \left\| \sum_{\sqrt{\lambda_n} \leq \lambda} b_n \psi_n(x) \right\|^2_{L^2(\omega)}$$

F. AMMAR KHODJA, A. BENABDALLAH, M. GONZÀLEZ-BURGOS, M. MORANCEY, L. DE TERESA, *New results on biorthogonal families in cylindrical domains and controllability consequences*, (2024) arXiv:2406.05104

To answer to the previous question one can

- Use some characterizations of minimal sequences in Hilbert spaces:

This is what Cherif will develop in the second part of this presentation...

Happy birthday Kisko and Manolo!!