Attractors for a class of impulsive systems

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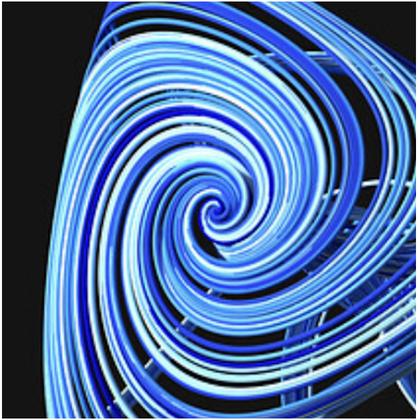
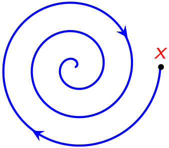
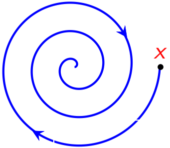
It is well known that to understand the asymptotic behavior of evolution equations the notion of attractor plays a central role.



Attractor: a compact set which satisfies an invariance property and that attracts (in some sense) a class of subsets of the phase space in which the equation is stated.



- $\begin{array}{rcl} \pi\colon X \times \mathbb{R}_+ & \to & X \\ (x,t) & \mapsto & \pi(x,t) \end{array}$ flow
- $\mathcal{A}$ attractor



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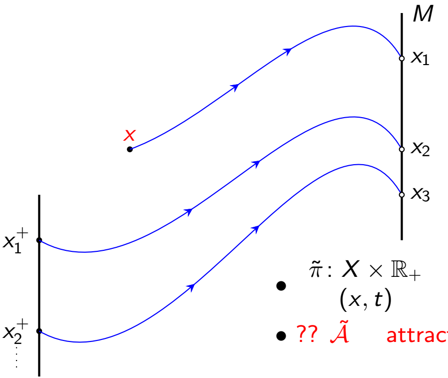

- $$\begin{array}{rcl} \tilde{\pi} : X \times \mathbb{R}_+ & \to & X \\ (x,t) & \mapsto & \tilde{\pi}(x,t) \end{array}$$ flow
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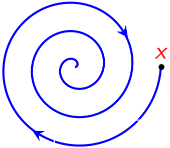

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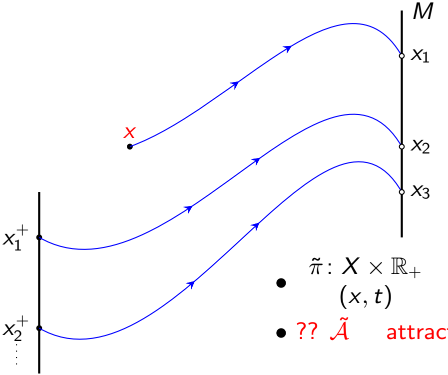

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Does $\{x_1^+, x_2^+, x_3^+, \ldots\}$ generate a discrete dynamical system? Attractor $\hat{\mathcal{A}}$?

If the attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$, $\hat{\mathcal{A}}$ exist, are they related?

E. M. Bonotto, M. C. Bortolan, A. N. Carvalho, and R. Czaja, Global attractors for impulsive dynamical systems - a precompact approach, J. Differential Equations, 259 (2015), 2602–2625.

E. M. Bonotto, and J. M. Uzal, Global attractors for a class of discrete dynamical systems.J. Dynamics and Differential Equations, (2024). https://doi.org/10.1007/s10884-024-10356-9

Summary

- (X, d) is a complete metric space.

Definition 1.1

A **continuous semigroup** in X is a family of maps $\{\pi(t) \colon t \geq 0\} \subset C(X)$, indexed on $\mathbb{R}_+$, satisfying:

- $\pi(0) = I$;
- $\pi(t+s) = \pi(t)\pi(s)$ for all $t, s \in \mathbb{R}_+$;
- the map $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x$ is continuous.

For each $x \in X$, consider the map $\pi_x \colon \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(t)x$ called the **trajectory** of x.

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Definition 1.2

An **impulsive dynamical system (IDS)**

$$(X, \pi, M, I)$$

consists of:

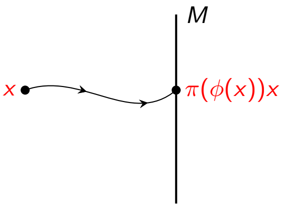
- a continuous semigroup $\{\pi(t) \colon t \geq 0\}$ in X;
- a non-empty closed subset M of X **(called the impulsive set)** such that

$$\text{for every } x \in M, \text{ there is } \epsilon_x > 0 \text{ such that } \bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \emptyset;$$

- a continuous function $I \colon M \to X$ **(called the impulse function)**.

The **impact function** $\phi\colon X \to (0,\infty]$ is defined by

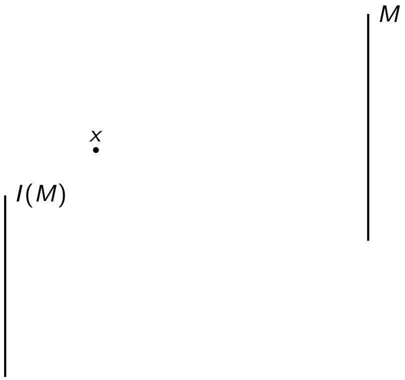
$$\phi(x) = \begin{cases} s, & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ \infty, & \text{if } \pi(t)x \notin M \text{ for all } t > 0. \end{cases}$$



Impulsive semitrajectory of $x \in X$

$$\begin{array}{rccc} \tilde{\pi}_x : & [0, T(x)) & \to & X \\ & t & \mapsto & \tilde{\pi}_x(t) \end{array}$$

Impulsive semitrajectory



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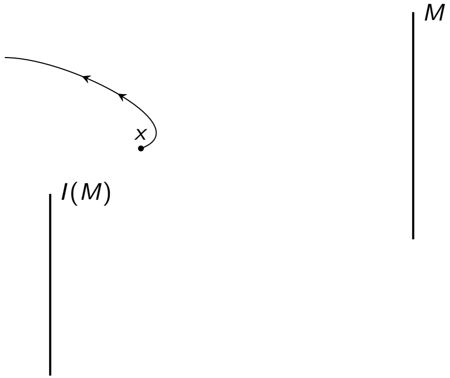

$$\phi(x) = \infty \quad \text{and} \quad \tilde{\pi}_x(t) = \pi(t)x \quad \text{for all} \quad t \geq 0.$$

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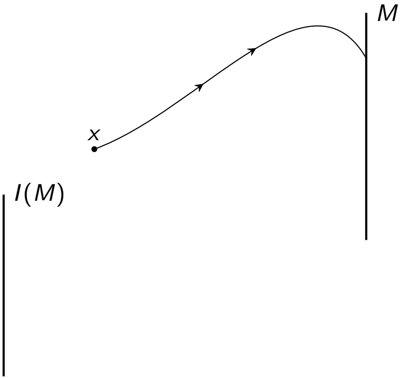

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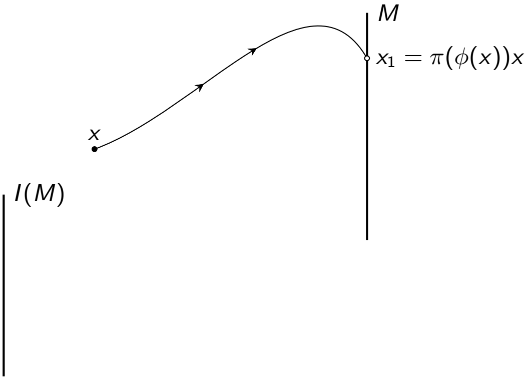
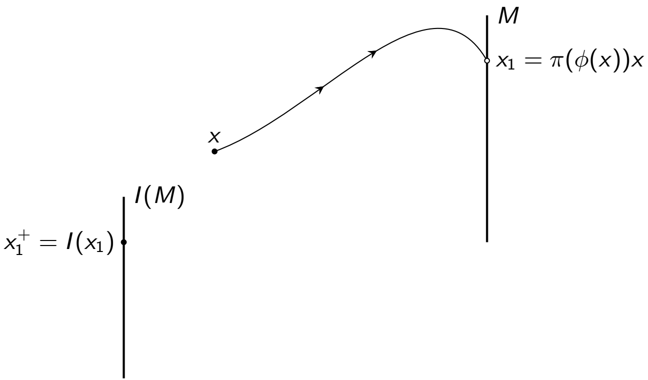
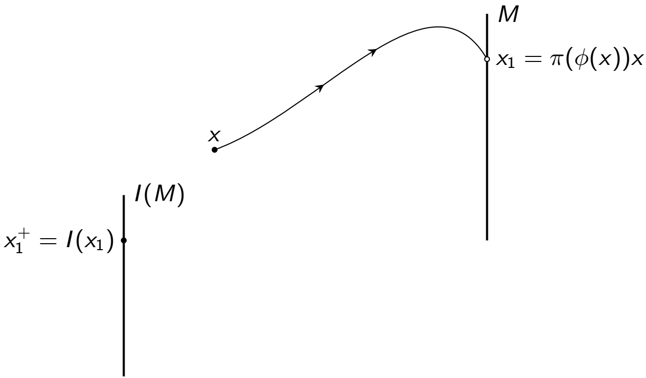



M
$x_1 = \pi(\phi(x))x$
x
$I(M)$
$x_1^+ = I(x_1)$



$$\phi(x) < \infty \quad \text{and} \quad \tilde{\pi}_x(t) = \begin{cases} \pi(t)x & \text{if} \quad 0 \leq t < \phi(x) \\ x_1^+ & \text{if} \quad t = \phi(x). \end{cases}$$

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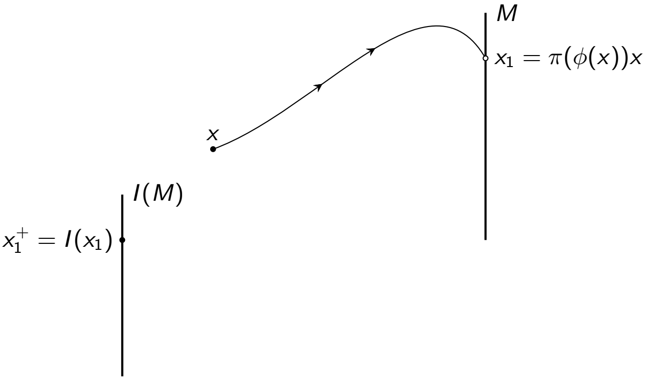

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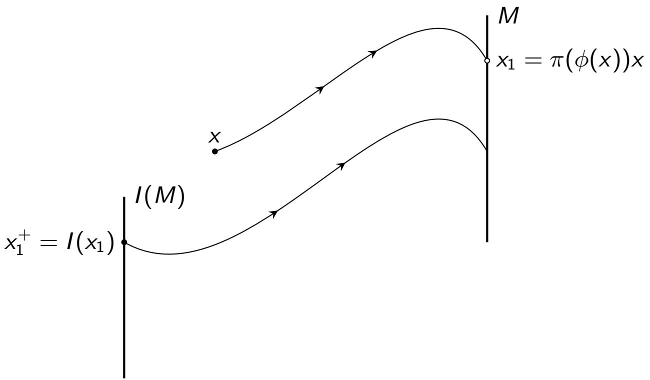

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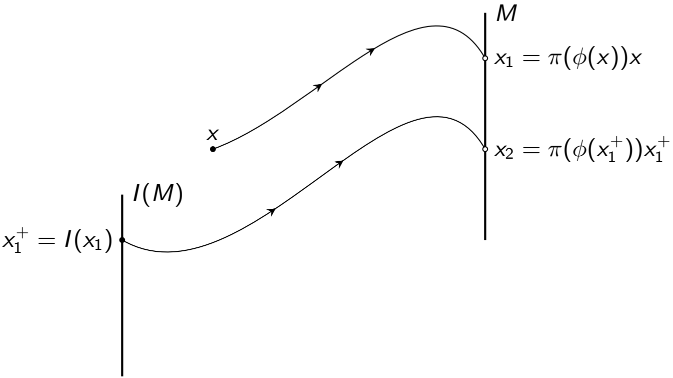

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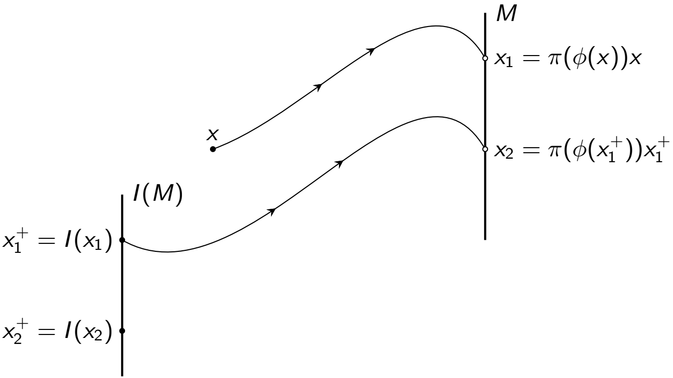

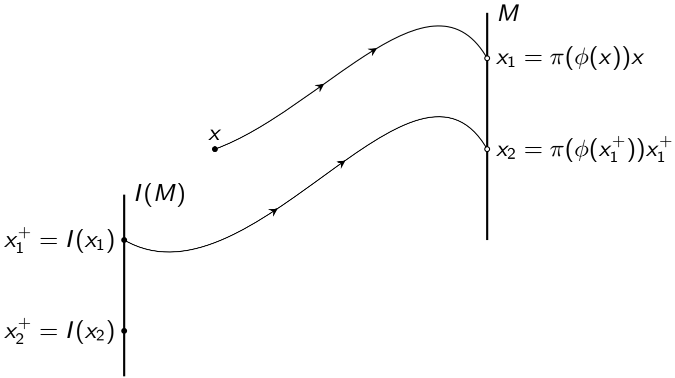

$$\tilde{\pi}_x(t) = \begin{cases} \pi(t - \phi(x))x_1^+ & \phi(x) \le t < \phi(x) + \phi(x_1^+) \\ x_2^+ & t = \phi(x) + \phi(x_1^+). \end{cases}$$

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Proposition 1.3

Let (X, π, M, I) be an IDS. Then:

(*i*) $\tilde{\pi}(0)x = x$ *for all* $x \in X$;

(*ii*) $\tilde{\pi}(t)\tilde{\pi}(s)x = \tilde{\pi}(t+s)x$ *for all* $x \in X$ *and* $t, s \geq 0$

Behavior of $\tilde{\pi}$

Question: Is it possible to obtain the convergence property $\tilde{\pi}(\phi(x))x_n \to \tilde{\pi}(\phi(x))x$?

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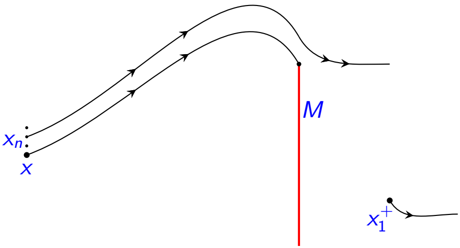

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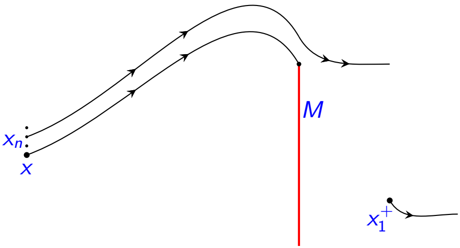

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Lemma 1.4

Let $x \in X \setminus M$ and $t \geq 0$. Assume that $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a sequence such that $x_n \overset{n\to\infty}{\longrightarrow} x$. Then there is a sequence $\{\beta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\beta_n \overset{n\to\infty}{\longrightarrow} 0$ and

$$\tilde{\pi}(t+\beta_n)x_n \overset{n\to\infty}{\longrightarrow} \tilde{\pi}(t)x.$$

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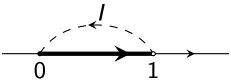
In fact, let $(\mathbb{R}, \pi, M, I)$ be an IDS given by $\pi(t)x = x + t$, for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $M = \{1\}$ and $I(1) = 0$.



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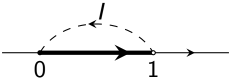

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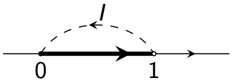

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Let $B \subset X$ be a positively $\tilde{\pi}$-invariant set. Then $\overline{B} \setminus M$ is positively $\tilde{\pi}$-invariant.

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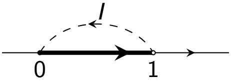

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Summary

Definition 2.1

A subset $\mathcal{A} \subset X$ will be called a **global attractor** *for the semigroup $\{\pi(t) : t \geq 0\}$ if it satisfies the following conditions:*

(i) $\mathcal{A}$ *is* **compact***;*

(ii) $\mathcal{A}$ *is* $\pi-$**invariant** $(\pi(t)\mathcal{A} = \mathcal{A}$ *for all* $t \in \mathbb{R}_+)$*;*

(iii) $\mathcal{A}$ $\pi-$**attracts** *bounded subsets of* X $\left(\lim_{t\to\infty} \mathrm{d}_H(\pi(t)B, \mathcal{A}) = 0, \ \forall B \in \mathcal{B}(X)\right)$.

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Definition 2.2

A subset $\tilde{\mathcal{A}} \subset X$ *will be called a* **global attractor** *for the IDS* (X, π, M, I) *if it satisfies the following conditions:*

(i) $\tilde{\mathcal{A}}$ *is* **precompact** *and* $\tilde{\mathcal{A}} = \overline{\tilde{\mathcal{A}}} \setminus M$*;*

(ii) $\tilde{\mathcal{A}}$ *is* $\tilde{\pi}-$**invariant** $(\tilde{\pi}(t)\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$ *for all* $t \in \mathbb{R}_+)$*;*

(iii) $\tilde{\mathcal{A}}$ $\tilde{\pi}-$**attracts** *bounded subsets of* X $\left(\lim_{t\to\infty} \mathrm{d}_H(\tilde{\pi}(t)B, \tilde{\mathcal{A}}) = 0, \ \forall B \in \mathcal{B}(X)\right)$.

- If $\tilde{\mathcal{A}}$ exists, then it is uniquely determined.
- If $M = \emptyset$, then the definitions coincide.

Definition 2.3

An IDS (X, π, M, I) is called **asymptotically compact**, if given a set $B \in \mathcal{B}(X)$, a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ with $t_n \to \infty$, and a sequence $\{x_n\}_{n \in \mathbb{N}} \subset B$, then the sequence $\{\tilde{\pi}(t_n)x_n\}_{n \in \mathbb{N}}$ possesses a convergent subsequence in X.

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An IDS (X, π, M, I) is called **dissipative**, if there exists a set $B_0 \in \mathcal{B}(X)$, called absorbing set, such that for every $B \in \mathcal{B}(X)$ there exists a time $T_B \geq 0$ such that $\tilde{\pi}(t)B \subset B_0$ for all $t \geq T_B$.

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Theorem 2.5

An IDS (X, π, M, I) admits a global attractor $\tilde{\mathcal{A}}$ if and only if it is asymptotically compact and dissipative.

$$\tilde{\mathcal{A}} = \tilde{\omega}(B_0)\backslash M,$$

where $\tilde{\omega}(B_0) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \tilde{\pi}(\tau)B_0}$.

Summary

Assume that:

(H1) ϕ is continuous on $X \setminus M$;

(H2) there exists $z \in I(M)$ such that $\phi(z_k^+) < \infty$ for all $k \in \mathbb{N}$;

(H3) $I(M) \cap M = \emptyset$.

A discrete dynamical system

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Consider the set

$$\hat{X} = \{x \in I(M) \colon \phi(x_k^+) < \infty \text{ for all } k \in \mathbb{N}\}$$

and the map $g \colon \hat{X} \to \hat{X}$ given by

$$g(x) = I(\pi(\phi(x))x). \tag{1}$$

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$$g(x) = I(\pi(\phi(x))x). \tag{1}$$

- Condition (H2) $\Longrightarrow$ $\hat{X} \neq \emptyset$.

A discrete dynamical system

Assume that:

(H1) ϕ is continuous on $X \setminus M$;

(H2) there exists $z \in I(M)$ such that $\phi(z_k^+) < \infty$ for all $k \in \mathbb{N}$;

(H3) $I(M) \cap M = \emptyset$.

Consider the set

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Definition 3.1

A subset $\hat{B} \subset \hat{X}$ is said to be:

(*i*) **positively g-invariant** w.r.t. $(\hat{X}, g)$, if $g(\hat{B}) \subset \hat{B}$;

(*ii*) **negatively g-invariant** w.r.t. $(\hat{X}, g)$, if $g(\hat{B}) \supset \hat{B}$;

(*iii*) **g-invariant** if it is both positively and negatively g-invariant w.r.t. $(\hat{X}, g)$.

Definition 3.2

The **omega limit set** of a subset $\hat{B} \subset \hat{X}$ is given by

$$\hat{\omega}(\hat{B}) = \{x \in \hat{X} \colon \text{ there exist sequences } \{x_k\}_{k\in\mathbb{N}} \subset \hat{B} \text{ and } \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$$
$$\text{with } \; n_k \to \infty \; \text{ such that } g^{n_k}(x_k) \to x\}.$$

Let $\mathcal{B}(\hat{X})$ denote the set of all bounded subsets from $\hat{X}$.

Definition 3.3

A set $\hat{\mathcal{A}} \subset \hat{X}$ is called a **discrete global attractor** for $(\hat{X}, g)$ if:

(i) $\hat{\mathcal{A}}$ is compact;

(ii) $\hat{\mathcal{A}}$ is g-invariant;

(iii) $\mathrm{d_H}(g^n(\hat{B}), \hat{\mathcal{A}}) \to 0$ for every $\hat{B} \in \mathcal{B}(\hat{X})$.

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Theorem 3.4

Assume that (X, π, M, I) satisfies conditions (H1)-(H3) and $(\hat{X}, g)$ is asymptotically compact and bounded dissipative with absorbing set $\hat{B}_0$. Then $(\hat{X}, g)$ has a discrete global attractor $\hat{\mathcal{A}}$ given by $\hat{\mathcal{A}} = \hat{\omega}(\hat{B}_0)$.

Attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$

- semigroup $\{\pi(t)\colon t \geq 0\}$ with global attractor $\mathcal{A}$.

- IDS (X, π, M, I) with global attractor $\tilde{\mathcal{A}}$.

- DDS $(\hat{X}, g)$ with discrete global attractor $\hat{\mathcal{A}}$.

Example 1

Consider the system of differential equations

$$\begin{cases} x' = -x, \\ y' = -y, \end{cases}$$

in $X = \mathbb{R}^2$. In this simple example, $\mathcal{A} = \{(0,0)\}$.

(a) If $M = \bigcup_{n\in\mathbb{N}} M_n$ with $M_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = n^2\}$, $n = 1, 2, \ldots$, and $I(x,y) = (x(1+\frac{1}{2n}), y(1+\frac{1}{2n}))$ for $(x,y) \in M_n$, $n = 1, 2, \ldots$, then the systems (X, π, M, I) and $(\hat{X}, g)$ do not admit global attractors.

(b) If $M = \bigcup_{n\in\mathbb{N}} M_n$ with $M_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = n^2\}$, $n = 1, 2, \ldots$, and $I(x,y) = (\frac{x}{2n}, \frac{y}{2n})$ for all $(x,y) \in M_n$, $n = 1, 2, \ldots$. Then $\tilde{\mathcal{A}} = \mathcal{A}$ and $\hat{X} = \emptyset$.

(c) If $M = \mathbb{R} \times \{1\}$ and $I(x,1) = (\tan^{-1}(x), 2)$, $x \in \mathbb{R}$. Then $\hat{\mathcal{A}} = \{(0,2)\}$ and $\tilde{\mathcal{A}} = \{(0,y) : 1 < y \leq 2\} \cup \{(0,0)\}$. Note that $\hat{X} = I(M) = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \{2\}$. Moreover,

$$\tilde{\mathcal{A}} = \pi([0, ln2))\hat{\mathcal{A}} \cup \mathcal{A}.$$

Attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$

- $I(M) \cap M = \emptyset$

Proposition 4.1

Assume that $\{\pi(t)\colon t \geq 0\}$ *has a global attractor* $\mathcal{A}$ *with* $\mathcal{A} \cap M = \emptyset$ *and* (X, π, M, I) *has a global attractor* $\tilde{\mathcal{A}}$. *Then* $\mathcal{A} \subset \tilde{\mathcal{A}}$.

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Proposition 4.2

Assume that (X, π, M, I) has a global attractor $\tilde{\mathcal{A}}$ and $(\hat{X}, g)$ has a discrete global attractor $\hat{\mathcal{A}}$. Then $\hat{\mathcal{A}} \subset \tilde{\mathcal{A}}$.

Attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$

- $I(M) \cap M = \emptyset$

Proposition 4.1

Assume that $\{\pi(t)\colon t \geq 0\}$ *has a global attractor* $\mathcal{A}$ *with* $\mathcal{A} \cap M = \emptyset$ *and* (X, π, M, I) *has a global attractor* $\tilde{\mathcal{A}}$*. Then* $\mathcal{A} \subset \tilde{\mathcal{A}}$*.*

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Assume that (X, π, M, I) *has a global attractor* $\tilde{\mathcal{A}}$ *and* $(\hat{X}, g)$ *has a discrete global attractor* $\hat{\mathcal{A}}$*. Then* $\hat{\mathcal{A}} \subset \tilde{\mathcal{A}}$*.*

Theorem 4.3

Assume that $\{\pi(t)\colon t \geq 0\}$ *admits a global attractor* $\mathcal{A}$ *with* $\mathcal{A} \cap M = \emptyset$, $(\hat{X}, g)$ *has a global attractor* $\hat{\mathcal{A}}$, (X, π, M, I) *is dissipative and* $\phi(x) < \infty$ *for all* $x \in I(M)$*. Then* (X, π, M, I) *admits a global attractor* $\tilde{\mathcal{A}}$ *given by*

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a \right).$$

Consider the nonlinear reaction-diffusion initial boundary value problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for} \quad (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{for} \quad (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & \text{for} \quad x \in \Omega, \end{cases} \tag{2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary and Δ is the Laplace operator in Ω. The nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the growth and dissipative conditions:

(a) $|f(t) - f(s)| \leq c|t-s|$, for all $t, s \in \mathbb{R}$, where $c > 0$;

(b) $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$.

For each $u_0 \in L^2(\Omega)$, $\exists!$ solution u of (2) with $u \in C([0,\infty), L^2(\Omega))$ such that the map $u_0 \mapsto u(t)$ is continuous in $L^2(\Omega)$. Thus, the map $\pi(t)\colon L^2(\Omega) \to L^2(\Omega)$ given by

$$\pi(t)u_0 = u(t)$$

defines a dynamical system $(L^2(\Omega), \pi)$ on $L^2(\Omega)$.

- The dynamical system $(L^2(\Omega), \pi)$ is dissipative with absorbing set

$$B_0 = \left\{ v \in L^2(\Omega) : \|v\|_2 \leq \frac{\rho_0 C|\Omega|}{\epsilon_0} \right\}, \quad \rho_0 > 1, \epsilon_0 > 0.$$

- $(L^2(\Omega), \pi)$ admits a global attractor $\mathcal{A}$.

- Let $r_0 > \max \left\{ 1, \frac{2\rho_0 C|\Omega|}{\epsilon_0} \right\}$. Consider the set $M = \{v \in L^2(\Omega) : \ \|v\|_2 = r_0\}$ and a continuous function $I \colon M \to L^2(\Omega)$.

- If $\|I(z)\|_2 < \|z\|_2$ for all $z \in M$ then $\tilde{\mathcal{A}} = \mathcal{A}$ and $\hat{\mathcal{A}}$ does not exist.

The operator $-\Delta$ with the Dirichlet boundary conditions admits an orthonormal complete sequence of eigenfunctions $\{e_i\}_{i=1}^{\infty}$ in $L^2(\Omega)$ with corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ satisfying $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$, $\lambda_n \to \infty$.

$$\pi(t)u_0 = u(t) = \sum_{i=1}^{\infty} \alpha_i(t) e_i.$$

where $\alpha_i(t)$ (Fourier coefficient) satisfies the ODE $\alpha_i'(t) + \lambda_i \alpha_i(t) = (f(u(t)), e_i)$, $i \in \mathbb{N}$.

- If $I(v) = v + 3r_0 e_1$ for all $v \in M$, then

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a \right).$$

Thank you for your attention!

¡Feliz cumpleaños, Kisko y Manolo!