Eigenvalue bounds for asymmetric shear flows[1]

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[1] Joint work with J. Carvalho

Parallel flows of incompressible fluids

Navier Stokes equations - incompressible fluids

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \dfrac{1}{R}\Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Reynolds number: $R > 1$

Unknowns

Velocity $\mathbf{u} : \mathbb{R} \times [0,1] \times [0,\infty) \longrightarrow \mathbb{R}^2$
Pressure $p : \mathbb{R} \times [0,1] \times [0,\infty) \longrightarrow \mathbb{R}$

Parallel flow

$$\mathbf{U}(x,y) = (U(y), 0)$$

Equations for perturbations (Linearized)

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \dfrac{1}{R}\Delta\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

Componentwise: $(\mathbf{u} = (u, v),\ \mathbf{U} = (U(y), 0))$

$$\begin{cases} u_t + Uu_x + vU' + p_x = \dfrac{1}{R}\Delta u \\ v_t + Uv_x + p_y = \dfrac{1}{R}\Delta v \\ u_x + v_y = 0 \end{cases}$$

Derive 1st w.r.t. y, 2nd w.r.t. x, take difference, use 3rd, obtain

$$(v_x - u_y)_t + U(v_x - u_y)_x - vU'' = \frac{1}{R}\Delta(v_x - u_y)$$

Streamfunction:

$$\begin{cases} \psi_x &= v \\ \psi_y &= -u \end{cases}$$

Equation:

$$\Delta\psi_t + U\Delta\psi_x - \psi_x U'' = \frac{1}{R}\Delta^2\psi$$

Laplace in t: ,

$$s\Delta\widetilde{\psi} + U\Delta\widetilde{\psi}_x - \widetilde{\psi}_x U'' = \frac{1}{R}\Delta^2\widetilde{\psi}$$

Fourier in x: $\left(D := \frac{d}{dy}\right)$

$$s(D^2 - k^2)\widehat{\psi} + ikU(D^2 - k^2)\widehat{\psi} - ik\widehat{\psi}U'' = \frac{1}{R}(D^2 - k^2)^2\widehat{\psi}$$

Transformed equation:

$$[(s+ikU)(D^2-k^2)-ikU'']\widehat{\psi}=\frac{1}{R}(D^2-k^2)^2\widehat{\psi} \qquad (1)$$

Stability: $Re(s)<0$.
Physicists & engineers: Perturbations

$$\psi = \widetilde{\psi}(y)e^{i\alpha(x-ct)}$$

Classical Orr-Sommerfeld equation:

$$i\alpha\left[(U-c)(D^2-\alpha^2)-U''\right]\widetilde{\psi} = \frac{1}{R}(D^2-\alpha^2)^2\widetilde{\psi}$$

Obs. Basically eq. (1) with $\alpha=k$ and $c=\dfrac{is}{k}$

Orr-Sommerfeld equation

$$i\alpha\left[(U-c)(D^2-\alpha^2)-U''\right]\widetilde{\psi} \;=\; \frac{1}{R}(D^2-\alpha^2)^2\widetilde{\psi}$$

- Stability depends on the imaginary part of $c = c_r + ic_i$:
  $c_i > 0 \Rightarrow$ perturbations grow in time(instability);
  $c_i < 0 \Rightarrow$ perturbações decay in time(stability).

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- Aim: To find bounds for c_i; parameter values such that $c_i < 0$?

Orr-Sommerfeld equation

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- Aim: To find bounds for c_i; parameter values such that $c_i < 0$?
- Remark: Choosing perturbations of the above form is essentially the same as taking the Laplace-Fourier transform of the equations. In this case, stability if $\mathrm{Re}\, s < 0$, where s is the variable of the Laplace transform.

Joseph, 1968

One has

$$c_i \leq \frac{q}{2\alpha} - \frac{\pi^2 + \alpha^2}{\alpha R}$$

where $q := \max_{y \in [0,1]} |U'(y)|$.

Moreover, no amplified disturbances($c_i > 0$) exist if

$$\alpha R q \quad < \quad f(\alpha) = \max\{M_1, M_2\}$$

where

$$\begin{aligned} M_1 &= (4,73)^2 \pi + 2^{\frac{3}{2}} \alpha^3, \\ M_2 &= (4,73)^2 \pi + 2\alpha^2 \pi. \end{aligned}$$

Particular case: Plane Couette flow

$$
\begin{cases}
u_t + (u \cdot \nabla)u + \nabla p = \frac{1}{R}\Delta u \\
\nabla \cdot u = 0 \\
u(x,0,t) = (0,0) \\
u(x,1,t) = (1,0) \\
u(x,y,t) = u(x+1,y,t) \\
u(x,y,0) = f(x,y)
\end{cases}
$$

Stationary solution

$U(x,y) = (y,0)$, $P = constant$
$U(x,y) = (y,0)$: Couette flow

Romanov, 1973: $c_i \leq -\delta < 0, \forall R$. $\delta \sim \mathcal{O}(R^{-1})$.

One may use bounds for the resolvent of the linear operator associated to quantify the domain of stability (using the full nonlinear equation);

- Kreiss, H.O., Lorenz, J., Acta Numerica, 1998 (resolvent method)
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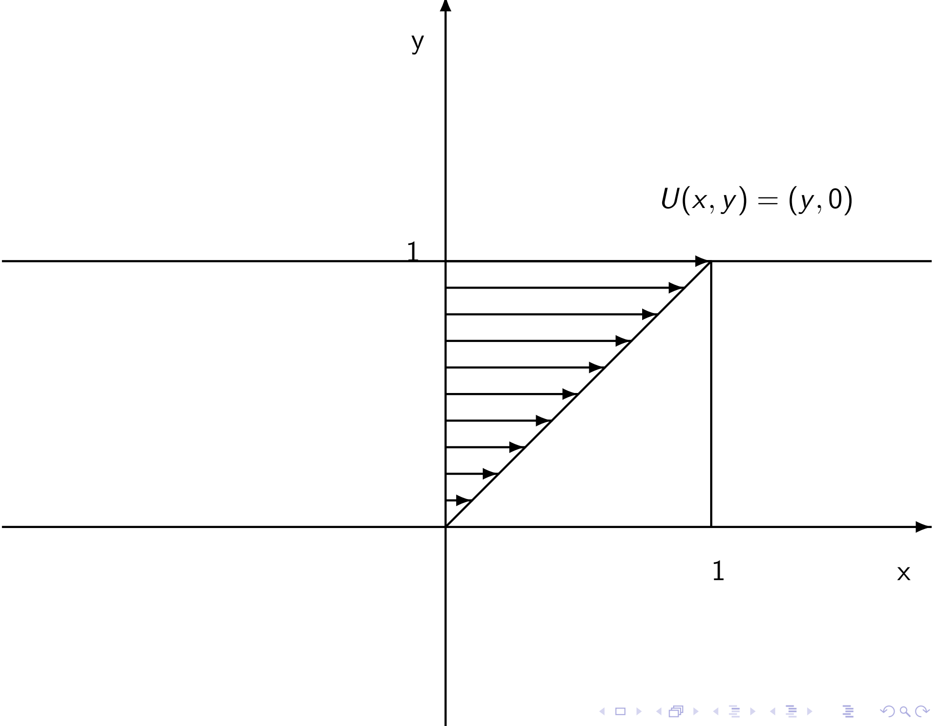
- PBS, SIAM J. Appl. Math., 2005 (resolvent bounds)
- PBS, Nonlinear Analysis: Real World Applications, 2012 (quantification)



y
$U(x, y) = (y, 0)$
1
1
x

Asymmetric incompressible fluids

System

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = (\nu + \nu_r)\Delta \mathbf{u} + 2\nu_r \operatorname{curl} \mathbf{w}, \\ \mathbf{w}_t + (\mathbf{u} \cdot \nabla)\mathbf{w} + 4\nu_r \mathbf{w} = (c_a + c_d)\Delta \mathbf{w} + (c_0 + c_d - c_a)\nabla \operatorname{div} \mathbf{w} \\ \qquad\qquad\qquad\qquad\qquad\qquad + 2\nu_r \operatorname{curl} \mathbf{u}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases}$$

Positive constants ν, ν_r, c_0, c_a, c_d, related with viscosity, satisfying $c_0 + c_d > c_a$.

Unknowns

Velocity $\mathbf{u} : \mathbb{R} \times [0,1] \times [0,\infty) \longrightarrow \mathbb{R}^2$
Angular velocity $\mathbf{w} : \mathbb{R} \times [0,1] \times [0,\infty) \longrightarrow \mathbb{R}^2$
Pressure $p : \mathbb{R} \times [0,1] \times [0,\infty) \longrightarrow \mathbb{R}$

Shear flows

$$\mathbf{U} = (U(y), 0, 0), \mathbf{W} = (0, 0, W(y)), P = constante, y \in (0, 1),$$

Perturbations

$$\mathbf{u} = (u(x, y), v(x, y), 0), \;\; \mathbf{w} = (0, 0, w(x, y)).$$

Linearized equations for perturbations

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{U} + (\mathbf{U} \cdot \nabla)\mathbf{u} + \nabla p = (\nu + \nu_r)\Delta\mathbf{u} + 2\nu_r \mathrm{curl}\, \mathbf{w}$$

$$\begin{aligned} \mathbf{w}_t + (\mathbf{U} \cdot \nabla)\mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{W} &= 2\nu_r \mathrm{curl}\, \mathbf{u} - 4\nu_r \mathbf{w} \\ &\quad + (c_0 + c_d - c_a)\nabla \mathrm{div}\, \mathbf{w} + (c_a + c_d)\Delta\mathbf{w} \end{aligned}$$

Dimensionless parameters: $R_\mu, \; R_k, \; R_\nu, \; R_\gamma, \; R_0$.

Streamfunction: ψ such that

$$\begin{aligned} \psi_x &= v \\ \psi_y &= -u. \end{aligned}$$

Perturbations

$$\begin{aligned} \psi &= \widetilde{\psi}(y) e^{i\alpha(x-ct)}, \\ w &= \widetilde{w}(y) e^{i\alpha(x-ct)}, \end{aligned}$$

Equations (dimensionless): $\left(D := \dfrac{d}{dy}\right)$

$$\begin{aligned} i\alpha \left[(U-c)(D^2-\alpha^2) - U''\right] \widetilde{\psi} &= \left(\frac{1}{R_\mu} + \frac{1}{2R_k}\right)(D^2-\alpha^2)^2 \widetilde{\psi} \\ &\quad + \frac{R_0}{R_k}(D^2-\alpha^2)\widetilde{w} \end{aligned}$$

$$i\alpha \left[(U-c)\widetilde{w} - W'\widetilde{\psi}\right] = \frac{1}{R_\gamma}(D^2-\alpha^2)\widetilde{w} - \frac{2R_0}{R_\nu}\widetilde{w} + \frac{1}{R_\nu}(D^2-\alpha^2)\widetilde{\psi}$$

Eigenvalue bounds

$R_1 = \min\left\{\frac{1}{R_\mu}, \frac{1}{2R_k}, \frac{R_0}{R_\nu}\right\}$, $R_2 = \max\left\{\frac{R_0}{2R_k}, \frac{1}{2R_\nu}\right\}$, $R_1 > R_2 \Rightarrow$

$$c_i \leq \frac{q_1 + q_2}{2\alpha} - \frac{\pi^2 + \alpha^2}{\alpha R},$$

where $\frac{1}{R} = \min\left\{R_1 - R_2, \frac{1}{R_\mu} + \frac{1}{2R_k}, \frac{1}{R_\gamma}\right\}.$,

$$q_1 := \max_{y\in[0,1]}|U'(y)|,\ q_2 := \max_{y\in[0,1]}|W'(y)|.$$

Moreover, perturbations decay if

$$\alpha R q_1 < f(\alpha) := max\{M_1, M_2\}$$

$$\alpha R q_2 < g(\alpha) := max\{N_1, N_2\},$$

where

$$M_1 = (4.73)^2\pi + 2\alpha^2\pi, \quad M_2 = (4.73)^2\pi + 2^{\frac{3}{2}}\alpha^3,$$
$$N_1 = 2(4.73)^2\pi + 2\alpha^3, \quad N_2 = 2(4.73)^2\pi + 2^{\frac{3}{2}}\alpha\pi.$$

Region of linear stability

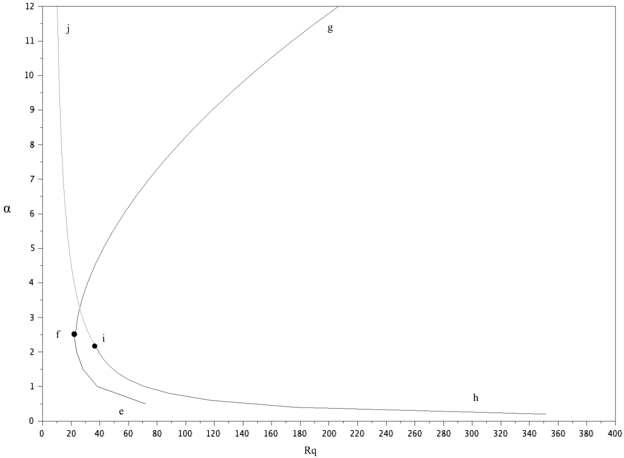


Figura: Linear stability bounds for the Orr-Sommerfeld equation for micropolar fluids. The region of certain linear stability lies to the left of the curve **ej**: **ef** is the graph of $\frac{M_2(\alpha)}{\alpha}$, **fg** is the graph of $\frac{M_1(\alpha)}{\alpha}$, **hi** is the graph of $\frac{N_2(\alpha)}{\alpha}$ and **ij** is the graph of $\frac{N_1(\alpha)}{\alpha}$.

Idea of the proof

Inner product of the first equation by $\widetilde{\psi}$, the second by $\widetilde{w}$, take real part, obtain (dropping the $\widetilde{}$)

$$
\begin{aligned}
&\alpha c_i(||\psi'||^2+\alpha^2||\psi||^2+||\omega||^2)\\
&+\Big(\frac{1}{R_\mu}+\frac{1}{2R_k}\Big)(||\psi''||^2+2\alpha^2||\psi'||^2+\alpha^4||\psi||^2)\\
&+\frac{1}{R_\gamma}||\omega'||^2+\left(\frac{\alpha^2}{R_\gamma}+\frac{2R_0}{R_\nu}\right)||\omega||^2\\
&\le Q+\overline{Q}+\left(\frac{R_0}{2R_k}+\frac{1}{2R_\nu}\right)(\|\psi''\|^2+\alpha^4\|\psi\|^2+2\|\omega\|^2).
\end{aligned}
$$

where

$$Q=\frac{i}{2}\int_0^1(U'\psi\overline{\psi'}-W'\psi\overline{\omega})\,dy.$$

Therefore, if

$$\frac{1}{R_\mu} + \frac{1}{2R_k} > \frac{R_0}{2R_k} + \frac{1}{2R_\nu}, \quad \frac{2R_0}{R_\nu} > \frac{R_0}{R_k} + \frac{1}{R_\nu}$$

which is assured, for example, by requiring

$$R_1 := \min\left\{\frac{1}{R_\mu}, \frac{1}{2R_k}, \frac{R_0}{R_\nu}\right\} > R_2 := \max\left\{\frac{R_0}{2R_k}, \frac{1}{2R_\nu}\right\}$$

one has

$$\begin{aligned}
&\alpha c_i(||\psi'||^2 + \alpha^2||\psi||^2 + ||\omega||^2) \\
&+(R_1 - R_2)||\psi''||^2 + \left(\frac{1}{R_\mu} + \frac{1}{2R_k}\right)2\alpha^2||\psi'||^2 + (R_1 - R_2)\alpha^4||\psi||^2 \\
&+ \frac{1}{R_\gamma}||\omega'||^2 + \left(\frac{\alpha^2}{R_\gamma} + R_1 - R_2\right)||\omega||^2 \leq Q + \overline{Q}.
\end{aligned}$$

Taking

$$\frac{1}{R} := \min\left\{R_1 - R_2, \frac{1}{R_\mu} + \frac{1}{2R_k}, \frac{1}{R_\gamma}\right\}$$

$$\begin{aligned}&\alpha c_i(||\psi'||^2 + \alpha^2||\psi||^2 + ||\omega||^2)\\ &+ \frac{1}{R}\left(||\psi''||^2 + 2\alpha^2||\psi'||^2 + \alpha^4||\psi||^2 + ||\omega'||^2 + \alpha^2||\omega||^2 + \|\omega\|^2\right)\\ &\leq Q + \overline{Q} = 2Re\left(\frac{i}{2}\int_0^1 (U'\psi\overline{\psi'} - W'\psi\overline{\omega})\, dy\right).\end{aligned}$$

$$\Rightarrow c_i \leq \frac{q_1 + q_2}{2\alpha} - \frac{\pi^2 + \alpha^2}{\alpha R},$$

Wave speed bounds

$U_{max}, U_{min}, U''_{max}, U''_{min}, W'_{max}, W'_{min}$: maximum and minimum values of $U(y), U''(y)$ and $W'(y)$ for $y \in [0, 1]$. :

(a) $U''_{min} \geq 0, W'_{min} \geq 0 \Rightarrow$

$$U_{min} - \frac{W'_{max}}{2\alpha} \leq C_r \leq U_{max} + \frac{U''_{max}}{2(\pi^2 + \alpha^2)} + \frac{W'_{max}}{2\alpha}$$

(b) $U''_{min} \geq 0, W'_{min} \leq 0 \leq W'_{max} \Rightarrow$

$$U_{min} - \frac{W'_{max}}{\alpha} + \frac{W'_{min}}{2\alpha} \leq C_r \leq U_{max} + \frac{U''_{max}}{2(\pi^2 + \alpha^2)} - \frac{W'_{min}}{\alpha} + \frac{W'_{max}}{2\alpha}$$

(c) $U''_{min} \geq 0, W'_{max} \leq 0 \Rightarrow$

$$U_{min} + \frac{W'_{min}}{2\alpha} \leq C_r \leq U_{max} + \frac{U''_{max}}{2(\pi^2 + \alpha^2)} - \frac{W'_{min}}{2\alpha}$$

(d) $U''_{min} \leq 0 \leq U''_{max}, W'_{min} \geq 0 \Rightarrow$

$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{2\alpha} \leq C_r \leq U_{max} + \frac{U''_{max}}{2(\pi^2 + \alpha^2)} + \frac{W'_{max}}{2\alpha}$$

Wave speed bounds

(e) $U''_{min} \leq 0 \leq U''_{max}, W'_{min} \leq 0 \leq W'_{max} \Rightarrow$

$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{\alpha} + \frac{W'_{min}}{2\alpha} \leq C_r \leq U_{max} + \frac{U''_{max}}{2(\pi^2 + \alpha^2)} - \frac{W'_{min}}{\alpha} + \frac{W'_{max}}{2\alpha}$$

(f) $U''_{min} \leq 0 \leq U''_{max}, W'_{max} \leq 0 \Rightarrow$

$$U_{min} + \frac{U''_{min}}{2\alpha^2} + \frac{W'_{min}}{2\alpha} \leq C_r \leq U_{max} + \frac{U''_{max}}{2\alpha^2} - \frac{W'_{min}}{2\alpha}$$

(g) If $U''_{max} \leq 0, W'_{min} \geq 0 \Rightarrow$

$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{2\alpha} \leq C_r \leq U_{max} + \frac{W'_{max}}{2\alpha}$$

(h) If $U''_{max} \leq 0, W'_{min} \leq 0 \leq W'_{max} \Rightarrow$

$$U_{min} + \frac{U''_{min}}{2\alpha^2} - \frac{W'_{max}}{\alpha} + \frac{W'_{min}}{2\alpha} \leq C_r \leq U_{max} - \frac{W'_{min}}{\alpha} + \frac{W'_{max}}{2\alpha}$$

(i) $U''_{max} \leq 0, W'_{max} \leq 0 \Rightarrow$

$$U_{min} + \frac{U''_{min}}{2\alpha^2} + \frac{W'_{min}}{2\alpha} \leq C_r \leq U_{max} - \frac{W'_{min}}{2\alpha}$$

- Is a better result possible, in the direction of Romanov's result for NS? Linear stability for all R?
- If not, critical Reynolds numbers to get transition to turbulence? (As in Poiseuille for Navier-Stokes, for example?) If so, can one find it ? Study global, conditional, monotonic stability ...

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For Navier-Stokes:	Plane Poiseuille:	49,6	1000	5772
	Plane Couette:	20,7	360	∞

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