Once upon a time: Kisco and Manolo PDEs and Control 2025 (PKM60)

D. B. (CNRS/Université Savoie Mont Blanc)
email: didier.bresch@univ-smb.fr

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In the abstract, I explained my first time in Sevilla:
– Invited by Enrique and Rosa.
– Really impressed by the number of preprints in Enrique's office with clear and deep comments on.

$\Longrightarrow$ Three key words in my mind:

- Stress tensor effects,
- Incompressibility/compressibility,
- Micro-meso-macro $\Longrightarrow$ Geophysical applications in mind.

Starting point of collaborations with Kisco & al: hydrostatic condition, boundary layers, vanishing depth, capillarity effects.......
Thanks to several Action Intégrée Picasso projects

When I came for the first time in Sevilla:
– Really impressed by the duo Kisco-Manolo:
Complementary with great complicity.
– Efficient and carying tandem both in math research
and in more earthly life.

Really nice idea to organize a joint-conference for these two persons.

Before math, let us start with a Burger Quizz (found on the web):
Kisco-Manolo quizz to test the tandem!!

Cada doble respuesta correcta del duo
$\Longrightarrow$ les hara ganar un regalo a cada uno.

From Catherine (my wife) and myself

The test in French of course !
I will only give today the first regalo....
(because it could help me :-)) and will give the other ones later-on

Je suis un mec pas très cool dans James Bond. Je m'appelle ?

- 1) Le nombre
- 2) Le chiffre
- 3) La virgule
- 4) La barre de fraction

Complétez cette citation de Georges Cantor :
"L'essence des mathématiques c'est....."

- 1) La liberté
- 2) L'égalité
- 3) La fraternité
- 4) Grave frais de ouf Askip

Comment appelle-t-on un polygône à 1000 côtés ?

- 1) Un Mexicagone
- 2) Un Chiliagone
- 3) Un Brésiliagone
- 4) Un Andalousiagone

Quel est le solide le plus proche d'un ballon de foot ?

- 1) Un dodécaèdre
- 2) Un icosaèdre tronqué
- 3) Un triacontraèdre tronqué
- 4) Un ballondefootaèdre tout simplement.

Quel type de nombres n'existe pas ?

- 1) Les nombres têtus
- 2) Les nombres tordus
- 3) Les nombres rigolos
- 4) Les nombres bizarres

Lequel de ces personnages célèbres a donné son nom à un théorème en maths

- 1) Charlemagne
- 2) Napoléon
- 3) Louis XIV
- 4) Felipe VI

Parmi ces nombres, lesquels ne sont pas premiers ?

- 1) Les nombres cousins
- 2) Les nombres sexys
- 3) Les nombres vampires
- 4) Les nombres jumeaux

Prendre du poids, on voit pas çà ?

- 1) En théorie du contrôle
- 2) En limite champ moyen dans système de particules
- 3) En méca des fluides pour les endroits où çà se passe mal.
- 4) À un moment en avion pour un vol parabolique

Introduire un problème dual, on voit pas çà ?

- 1) En théorie du contrôle.
- 2) En limite champ moyen dans système de particules.
- 3) En méca des fluides pour des estimations magiques.
- 4) Si on est raisonnable.

The goal of the talk is exactly
a story on weights and duality for fluid mechanics / mean field limit.

Joint works with

- P.–E. Jabin (Pennsylvania State University),
- M. Duerinckx (Université Libre Bruxelles).
- J. Soler (Granada University).
- Z. Wang (Pekin University)

- I– Introduction: Mean field limit and singular Kernels: Our story starting from compressible NS methods
- I– The introduction of weights
  - I–1) –A modulated free energy for first order system
  - I-2) –A weighted L^p quantity to treat second order systems
- II– A new framework (special test functions)
  - A Duality approach for mean-field limit

Goal of the first part:

Well adapted dynamical weights for mean-field limit
to get result with low regularity

Topics:

- Mean Field limits (with singular kernels)
  - First order systems
  - Second order systems

"Idea" coming from Compressible Navier-Stokes equations:
$\Longrightarrow$ Global existence à la Leray
Here large difference in the procedure and weights to be chosen.

Goal of the second part:

Develop a duality method for mean field limit.

Consider periodic boundary conditions.

A) Compressible Navier-Stokes equations:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

with velocity field u such that

$$u \in L^2_t W^{1,2}_x$$

and

$$\operatorname{div} u \approx \mathcal{A} p(\rho) \in L^q_{t,x} \text{ with } p > 2.$$

where

- $\mathcal{A}$ zero order non-local operator
- $s \mapsto p(s)$ a given function.

Compressible framework for weak regularity namely

$$\operatorname{div} u \text{ not necessarly } L^1_t L^\infty_x.$$

A) Compressible Navier-Stokes equations;

Goal: Quantitative estimates (Nonlinear weak stability)
$\Longrightarrow$ To get compactness in space
for global existence of weak solutions.

Tool:

$$\|\rho\|_{p,\theta} = \sup_{h\le 1/2} |\log h|^{-\theta} \int_{\Omega^2} \frac{|\rho(t,x)-\rho(t,y)|^p}{(h+|x-y|)^d} dx\, dy$$

Using that for $s>0$, $0<\theta<1$ and $p\in[1,+\infty)$ we have

$$W^{s,p}\subset W^p_{\log,\theta}\subset L^p$$

which are compact.

Déf: $W^p_{\log,\theta}=\{u\in L^p : \|u\|_{p,\theta}<+\infty\}$.

1) Let us look at the propagation of the information

$$\int_{\Omega^2} \frac{|\rho_n(t,x)-\rho_n(t,y)|}{(h+|x-y|)^d}(w_n(t,x)w_n(t,y))\,dxdy$$

with w_n solution of

$$\partial_t w_n + u_n\cdot\nabla w_n + \lambda P_n w_n = 0,$$

where $w_n|_{t=0} = 1$ and with P_n a positive penalization associated to (ρ_n, u_n) to be chosen and λ a large enough parameter to be chosen.

We look at the propagation of

$$R_h = \int_\Omega K_h(x-y)|\rho_n(t,x) - \rho_n(t,y)|w_n(t,x)w_n(t,y)\,dxdy.$$

Calculating the time derivative of R_h, we get

$$\frac{d}{dt}R(t) = \int_{\Omega^2} \nabla K_h(x-y)\cdot(u_n(t,x)-u_n(t,y))|\rho_n(t,x)-\rho_n(t,y)|w_n(t,x)w$$

$$-\int_{\Omega^2} K_h(x-y)(\mathrm{div}\, u_n(t,x)-\mathrm{div}\, u_n(t,y))\rho_n(t,x)s_n w_n(t,x)w_n(t,y)$$

$$+2\int_{\Omega^2} K_h(x-y)|\rho_n(t,x)-\rho_n(t,y)|x$$

$$\big(\partial_t w_n(t,x) + u_n(t,x)\cdot\nabla_x w_n(t,x) + \mathrm{div_x}\, u_n(t,x)w_n(t,x)\big)w_n(t,y)$$

where s_n is the sign of $\rho_n(t,x) - \rho_n(t,y)$.

Choose $P_n \geq 0$ in terms of the unkonwns appropriately
To conclude with a nice Gronwall Lemma.

For instance $P_n = M|\nabla u_n| + |\mathrm{div}\, u_n| + p(\rho_n)$
where $M(f)$ the maximal function of f.

2) We must show some properties on the weigths w_n i.e.

$$0 \le w_n \le c_n \le 1, \qquad \int_\Omega \rho_n |\log w_n|^q < +\infty$$

with $q > 0$ to hope to get rid of the weights at the end
$\Longrightarrow$ use the compact embedding given in the previous slide.

Remark:
Vacuum state $\Longrightarrow$ Necessity to consider
$w_n(t,x) + w_n(t,y)$ instead of $w_n(t,x)w_n(t,y)$.
$+$ Square functions instead of Maximal functions $+$ averaging in h
Use translation properties of the operators.

B) Mean Field limits

1) First order:
Liouville of forward Kolmogorov equation (general kernels)

$$\partial_t \rho_N + \sum_{i=1}^{N} \mathrm{div}_{x_i}(\rho_N \frac{1}{N-1} \sum_{j\neq i}^{N} K(x_i - x_j)) = \sum_{i=1}^{N} \sigma_N \Delta_{x_i} \rho_N$$

2) Second order:
Linear advection-diffusion equation (repulsive)

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N + \sum_{i=1}^{N} \frac{1}{N-1} \sum_{j\neq i}^{N} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \frac{\sigma^2}{2} \Delta_{v_i} f_N$$

with

$$K = \nabla V.$$

$\mathrm{div}\, K$ not necessarly L^∞.

Goal: Quantitative estimates or at least uniform bounds
$\Longrightarrow$ To get mean field limit justification

Tools:
Appropriate weights: modification of pure Gaussian

- 1) Modulated free energy (First order system).
- 2) L^p estimate (Second order system).

Our ideas:

- Choose a norm used with "regular" kernel K (truncation)
- Introduce dynamical weights
  to cancel quantities containing $\mathrm{div} K$
- Study the effect of the weights
  in terms of control with respect to N
- Go back to the original norm and conclude
  using Lemma comparing kernel and regularized kernel.

I.1) A modulated free energy for first order systems

Consider N particles, identical and interacting two by two through the kernel K. For $X_i(t) \in \Pi^d$ the position of the i-th particle,

$$d\,X_i = \frac{1}{N-1} \sum_{j \neq i} K(X_i - X_j)\,dt + \sqrt{2\sigma}\,dW_i,$$

with mean field scaling and N independent Brownian motions W_i where

$K = -\nabla V$ Gradient flow

Main question: Behavior of the system as $N \to \infty$.
For simplicity in the talk: σ is fixed but $\sigma = \sigma_N$ is also of interest.

The most classical cases:

- Poisson law with $d = 2$ (Patlak-Keller-Segel)

$$V(x) = \lambda \log|x| + \underline{perturbation},$$

- Coulomb law: For $n \geq 2$:

$$V(x) = d\lambda/|x|^{d-2} + \underline{perturbation} \text{ if } d \geq 3,$$

$$V(x) = -\lambda \log|x| + \underline{perturbation} \text{ it } d = 2.$$

Motivations and kernels covered by modulated free energy:

- Attractive Kernel :
  The Patlak-Keller-Segel.
- Repulsive Kernel :
  More general kernel than Riesz potential, Coulomb potential.

Focus on the joint law $\rho_N(t, x_1, \cdots, x_N)$ of the process $(X_1, \cdot, X_N)$ which solves the Liouville or forward Kolmogorov equation

$$\partial_t \rho_N + \sum_{i=1}^N \mathrm{div}_{x_i}(\rho_N \frac{1}{N-1} \sum_{j\neq i}^N K(x_i - x_j)) = \sigma \sum_{i=1}^N \Delta_{x_i} \rho_N$$

$$\rho_N|_{t=0} = \rho_N^0 \text{ such that } \int_{\Pi^{dN}} \rho_N^0 = 1.$$

Two previous approaches

Two key results appeared recently

- In Jabin-Wang, new estimates relative entropy of joint law

$$\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N$$

where $\bar\rho_N = \bar\rho^{\otimes N} = \Pi_{i=1}^N \bar\rho(t,x_i)$ and ρ_N the joint law of the process $(X_1,\cdots,X_N)$ which satisfies the Liouville equation

$$\partial_t\rho_N + \sum_{i=1}^N \mathrm{div}_{x_i}\left(\rho_N\frac{1}{N}\sum_{j\neq i}^N K(x_i-x_j)\right) = \sigma\sum_{i=1}^N \Delta_{x_i}\rho_N.$$

They give optimal rates of convergence in $\frac{1}{\sqrt{N}}$ provided that $K,\ \mathrm{div}\,K \in W^{-1,\infty}$.

$\longrightarrow$ very well – 2d Navier-Stokes because $\mathrm{div}K = 0$.

$\longrightarrow$ very poorly – gradient flows: Log-Lipschtiz Kernels.

See L. St Raymond, Bourbaki 70ème année, 2017–2018, no 1143.

Remark/idea: $\mathrm{div}\, K$ in relative entropy propagation is a bad term!

Similar quantity $\mathrm{div}\, u$ appears for compressible Navier-Stokes eqs. D.B., P.–E. Jabin (published *Ann. Math.* 2018) introduced weights satisfying PDE related to the unknowns in the quantity encoding the low regularity to cancel bad terms and prove some quantitative estimates on the density leading to compactness.

$$
\begin{aligned}
&\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^n)\log\frac{\rho_N(t,X^N)}{\overline{\rho}_N(t,X^n)}\,dX^N(t)\\
&\leq \frac{1}{N}\int_{\Pi^{dN}} \rho_N(0,X^N)\log\frac{\rho_N(0,X^N)}{\overline{\rho}_N(0,X^n)}\,dX^N(0)\\
&-\frac{\sigma_N}{N}\int_0^t\int_{\Pi^{d\,N}} d\rho_N\left|\nabla\log\frac{\rho_N}{\overline{\rho}_N}\right|^2\\
&-\frac{1}{N^2}\sum_{i,j=1}^N\int_0^t\int_{\Pi^{dN}} \rho_N(K(x_i-x_j)-K\star_x\overline{\rho}(x_i))\cdot\nabla_{x_i}\log\overline{\rho}_N dX^N ds\\
&-\frac{1}{N^2}\sum_{i,j=1}^N\int_0^t\int_{\Pi^{dN}} \rho_N(\mathrm{div}K(x_i-x_j)-\mathrm{div}K\star_x\overline{\rho}(x_i))dX^N ds
\end{aligned}
$$

where we call $\overline{\rho}_N(t,X^N)=\Pi_{i=1}^N\overline{\rho}(t,x_i)$.

Modify the relative entropy here to cancel $\operatorname{div} K$ and conclude ?

This was the starting point question with P.–E. Jabin and Z. Wang.

- Duerinckx-Serfaty and Serfaty (for $\sigma = 0$):
  Modulated potential energy

$$\frac{1}{2}\int_{\Pi^{2d}\cap\{x\neq y\}} V(x-y)(\mu_N(dx)-\bar{\rho}(x)dx)(\mu_N(dy)-\bar{\rho}(y)dy)$$

(with $\mu_N = \left(\sum_{i=1}^N \delta(x - X_i(t))\right)/N$ the empirical measure) allows to deal with repulsive singular Riesz potential of the type

$$V = \frac{C}{|x|^\alpha} \text{for } C > 0 \text{ and } \alpha < d$$

$\longrightarrow$ Works beyond Poisson kernel.
$\longrightarrow$ Does not work
for stochastic systems with attractive potentials.
$\longrightarrow$ Use the explicit formula of the kernel
allowing to reformulate the energy
in terms of potential or extension representation
(for the fractional laplacian) by Caffarelli-Silvestre

The Gibbs equilibria

Denote by G_N the Gibbs equilibrium of the system, and by $G_{\bar\rho_N}$ the corresponding distribution where the exact field is replaced by the mean field limit according to the law $\bar\rho$,

$$G_N(t, X^N) = \exp\Big(- \frac{1}{2N\sigma} \sum_{i\neq j} V(x_i - x_j)\Big),$$

$$G_{\bar\rho_N}(t, X^N) = \exp\Big(- \frac{1}{\sigma} \sum_{i=1}^{N} V \star \bar\rho(x_i) + \frac{N}{2\sigma} \int_{\Pi^d} V \star \bar\rho\, \bar\rho\Big),$$

$$G_{\bar\rho}(t, x) = \exp\Big(- \frac{1}{\sigma} V \star \bar\rho(x) + \frac{1}{2\,\sigma} \int_{\Pi^d} V \star \bar\rho\, \bar\rho\Big).$$

Our method uses the modified relative entropy

$$E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\left(\frac{\rho_N(t,X^N)}{G_N(X^N)}\,\frac{G_{\bar\rho_N}(t,X^N)}{\bar\rho_N(t,X^N)}\right)dX^N.$$

A modified free energy

One may also write

$$E_N\left(\frac{\rho_N}{G_N} \mid \frac{\bar\rho_N}{G_{\bar\rho_N}}\right) = \mathcal{H}_N(\rho_N|\bar\rho_N) + \mathcal{K}_N(G_N|G_{\bar\rho_N}),$$

where

$$\mathcal{H}_N(\rho_N|\bar\rho_N) = \frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\Big(\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\Big)\,dX^N$$

is exactly the relative entropy introduced in Jabin-Wang and

$$\mathcal{K}_N(G_N|G_{\bar\rho_N}) = -\frac{1}{N}\int_{\Pi^{dN}} \rho_N(t,X^N)\log\Big(\frac{G_N(t,X^N)}{G_{\bar\rho_N}(t,X^N)}\Big)\,dX^N$$

is expectation of modulated potential energy in Serfaty, Duerinckx multiplied by $1/\sigma$.

$\longrightarrow$ E_N is a modulated free energy for the system.

The time evolution of E_N

The modulated free energy E_N has the right algebraic structure with for any V even that

$$E_N\left(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) \le E_N\left(\frac{\rho_N}{G_N}\,|\,\frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(0)$$
$$-\frac{\sigma}{N}\int_0^t\int_{\Pi^{d\,N}} d\rho_N\left|\nabla\log\frac{\rho_N}{\bar\rho_N}-\nabla\log\frac{G_N}{G_{\bar\rho_N}}\right|^2$$
$$-\frac{1}{2}\int_0^t\int_{\Pi^{dN}}\int_{\Pi^{2\,d}\cap\{x\neq y\}}\nabla V(x-y)\cdot\left(\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(x)-\nabla\log\frac{\bar\rho}{G_{\bar\rho}}(y)\right)$$
$$(d\mu_N-d\bar\rho)^{\otimes 2}d\rho_N,$$

where $\mu_N=\frac{1}{N}\sum_{i=1}^N\delta(x-x_i)$ is the empirical measure.

The main points of the proof

The previous simple expression leaves two main points in the proof

- Bound the right-hand side in terms of E_N.
- Show that E_N is almost positive or more specifically that for some constant C

$$E_N\left(\frac{\rho_N}{G_N} \,|\, \frac{\bar\rho_N}{G_{\bar\rho_N}}\right)(t) \geq \frac{1}{C}\,\mathcal{H}_N\left(\rho_N \,|\, \bar\rho_N\right)(t) - \frac{C}{N^\theta}.$$

Remark. See recent papers by A. Chodron de Courcel, M. Rosenzweig, S. Serfaty.

I-2) A weighted Lq quantity

Ssecond order systems of Mean field limit with singular kernels

D. B., P.–E. Jabin, J. Soler. A new approach to the mean-field limit of Vlasov-Fokker-Plank equations. Analysis & PDE vol. 18, No. 4, 2025.

- Strongly uses the diffusion in velocity
- Bound weighted L^p norms of the marginals.
- Mean-field limit for very singular interaction kernels including repulsive Poisson interactions in $2D$.
- Quantitative estimates for a general kernel in L^2

Second order particle system:

$$\frac{d}{dt}X_i(t) = V_i(t), \quad X_i(t=0) = X_i^0,$$
$$dV_i(t) = \frac{1}{N-1}\sum_{j\neq i} K(X_i - X_j)\,dt + \sigma\,dW_i,\; V_i(t=0) = V_i^0, \tag{1}$$

where the W_i are N independent Wiener process, and where K is a pairwise interaction kernel deriving from a potential

$$K = -\nabla\phi$$

for a positive, even potential ϕ. For simplicity we take the positions X_i on the torus Π^d, while the velocities lie in $\mathbb{R}^d$.

Goal: Derive the mean-field limit for (1)
A kinetic, Vlasov-Fokker-Planck equation
Posed on the limiting 1-particle density $f(t, x, v)$

$$\partial_t f + v \cdot \nabla_x f + (K \star_x \rho) \cdot \nabla_v f = \frac{\sigma^2}{2} \Delta_v f \quad \text{with} \quad \rho = \int_{\mathbb{R}^d} f dv.$$

The mean-field of the Vlasov-Poisson equation has remained a long-standing open problem due to the difficulty in general of handling singular kernels for second-order model.

Known results:

In one dimension: Singular kernels.

Higher space dimensions: Regularized and cutted-off Kernels ; convergence results of discrete approximation for multidimensional Vlasov-Poisson systems with mollified mass and charge density.

- Lipschitz interactions K: McKean 67 and Sznitman 91 for the stochastic setting and by Braun-Hepp 77, Dobrushin 79 in the deterministic setting. Still important to further understand the framework.
  See for example Golse 16, Golse-Mouhot-Ricci 13, Hauray-Mischler 14, Mischler-Mouhot 13...
- Mild singularities $K(x) << |x|^{-1}$:
  Handled in Hauray-Jabin 09 and 15.
- Truncated kernels (essential for numerics):
  Boers-Pickl 16, Lazarovici-Pickl 17, Pickl 19 and in Huang-Liu-Pickl with diffusion.
- Singularity not at the origin:
  Carrillo-Choi-Hauray-Salem 18 for swarming models

- Deriving Vlasov-Poisson and Vlasov-Poisson-Fokker-Planck in dimension 1 seems to be more accessible: Hauray-Salem 19, Guillin–Le Bris–Monmarché 23.
- Derivations of fluid equations or first order macroscopic systems directly from second order models are also known, see Duerinckx–Serfaty 20 and Han-Kwan–Iacobelli 21.
- Marginals also play a key role in understanding fluctuations, Lacker 21, and corrections to the mean-field limit as in Duerinckx–Saint-Raymond 21.
- Deriving collisions models is even harder, also relies on controlling the marginals (without diffusion!). See Lanford 75 and more recently Gallagher–Saint-Raymond–Texier 14, Bodineau–Gallagher–Saint-Raymond 17, Bodineau–Gallagher–Saint-Raymond–Simonella 20 or Pulvirenti–Saffirio–Simonella 14, Pulvirenti–Simonella 17

Main tool:

Uses BBGKY hierarchy equations for k point marginals: $f_{k,N}$
as introduced in Lacker's paper
(definition given later on in the duallity approach).

Consider the weighted quantity

$$X_k(t) = \int |f_{k,N}|^q e^{\lambda(t) e_k}, \qquad \lambda(t) = \frac{1}{\Lambda(1+t))}$$

with Λ a parameter and e_k the dynamical weight to be chosen.

The dynamical weight $w = \exp(\lambda(t)e_k)$ is linked to a reduced energy e_k namely an energy of reduced k particles

$$e_k(x_1, v_1, \cdots, x_k, v_k) = \sum_{i \le k}(1 + |v_i|^2) + \frac{1}{N}\sum_{i,j \le k}\phi(x_i - x_j)$$

Observing that

$$L_k e_k = 0$$

where

$$L_k = \sum_{i \le k} v_i \cdot \nabla_{x_i} + \frac{1}{N}\sum_{i \le k}\sum_{j \le k} K(x_i - x_j) \cdot \nabla_{v_i}$$

Let $d = 2$ and consider the Poisson kernel $K = -\nabla\phi$ with associated potential $\phi(x) = -\ln|x|$. Then the convergence leading to the Vlasov-Poisson-Fokker-Planck system holds.

More generally, result holds for $K \in L^p(\Pi^d)$ with $p > 1$.

Important remarks:

- The BBGKY hierarchy eqs lose one derivative in the velocity variable, but it can be absorbed into the diffusion term
- The result is a not a short time result as in Lacker's paper.

A duality approach for first AND second order systems.
with or without brownian motion quantity

Duality method developped for transport type equations.
Duality method seems never used for mean field purposes.

$$\begin{cases} dX_{i,N} = V_{i,N}dt, \\ dV_{i,N} = \dfrac{1}{N-1}\displaystyle\sum_{j:j\neq i}^{N} K(X_{i,N} - X_{j,N})\,dt + \sqrt{2\alpha}\,dB_{i,N}, \end{cases} \tag{2}$$

where $\{B_{i,N}\}_{1\leq i\leq N}$ are N independent Brownian motions and where the temperature $0 \leq \alpha < \infty$ is a fixed parameter (possibly vanishing).

Switching to a statistical perspective, we consider a probability density F_N on the N-particle phase space with $(\mathbb{D})^N := (\Omega \times \mathbb{R}^d)^N$, and Newton's equations then formally lead to the following Liouville equation,

$$\partial_t F_N + \sum_{i=1}^{N} \Big(v_i \cdot \nabla_{x_i} F_N + \frac{1}{N-1} \sum_{j: j \neq i}^{N} K(x_i - x_j) \cdot \nabla_{v_i} F_N \Big) = \alpha \sum_{i=1}^{N} \Delta_{v_i} F_N. \quad (3)$$

We shall assume for simplicity that at initial time $t = 0$ particles are f°-chaotic in the sense of

$$F_N|_{t=0} = (f^\circ)^{\otimes N}, \tag{4}$$

for some $f^\circ \in \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D})$.

In the macroscopic limit $N \uparrow \infty$, we aim at an averaged description of the system, describing the evolution of the phase-space density of a typical particle, as given by first marginal

$$F_{N,1}(z) := \int_{\mathbb{D}^{N-1}} F_N(z, z_2, \ldots, z_N)\, dz_2 \ldots dz_N.$$

As is well known, formally neglecting particle correlations, we expect that $F_{N,1}$ remains close to a solution $f \in L^\infty(\mathbb{R}^+; \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D}))$ of the corresponding mean-field Vlasov equation,

$$\partial_t f + v \cdot \nabla_x f + (K * f) \cdot \nabla_v f = \alpha \Delta_v f, \qquad f|_{t=0} = f^\circ, \qquad (5)$$

where we define $K * f(x) := \int_{\mathbb{D}} K(x - x') f(x', v')\, dx' dv'$. More generally, for all $k \geq 0$, the kth marginal

$$F_{N,k}(z_1, \dots, z_k) := \int_{\mathbb{D}^{N-k}} F_N(z_1, \dots, z_N)\, dz_{k+1} \dots dz_N \qquad (6)$$

is expected to remain close to the tensor product $f^{\otimes k}$ of the mean-field Vlasov solution. This is known as propagation of chaos.

Mean-field limits for kinetic equations dimension $d \geq 2$.
Rigorous derivation of mean-field limit for Vlasov-Poisson still fully open in multi-dimension:

- The case of Lipschitz interactions $K(x)$ was handled by McKean in for the stochastic setting and by Braun and Hepp, and Dobrushin in the deterministic case.
- Mild singularities $K(x) << |x|^{-1}$ in Hauray-Jabin 09 and 15.
- Truncated kernels (essential for numerics) in Boers-Pickl 16, Lazarovici-Pickl 17, Pickl 19.
- Swarming models: Carrillo-Choi-Hauray-Salem 18 for cones of vision, Mucha-Peszek 18 for mild singular communication weights in Cucker-Smale.
- So-called Monokinetic limits with the full singularity were obtained in Duerinckx-Serfaty 20.
- Repulsive 2d Vlasov-Poisson-Fokker-Planck in Bresch-Jabin-Soler 22, with a partial result in 3d.

A Statistical description

Many ways to formulate limit of many particle systems.
Here statistical description introducing the full joint law

$F_N(t, x_1, \cdots, x_N)$ = joint law of the system $(x_1, \cdots, x_N)$ at time t

together with its various marginals

$F_{N,k}(t, x_1, \cdots, x_k)$ = joint law of the system $(x_1, \cdots, x_k)$ at time t

A simple important relation:

$$F_{N,k}(t, x_1, \cdots, x_k) = \int_{\Omega^{N-k}} F_N(t, x_1, \cdots, x_N) dx_{k+1} \cdots x_N.$$

Hypothesis (H)

- Let $0 \leq \alpha < +\infty$ and let $K \in L^2_{loc}(\Omega; \mathbb{R}^d)$, and assume for convenience $K \in L^\infty_{loc}(|x| > 1)$.
- Consider a global weak duality solution $F_N \in L^\infty_{loc}(\mathbb{R}^+; L^1(\mathbb{D}^N) \cap L^\infty(\mathbb{D}^N))$ of the Liouville equation with f°-chaotic initial data for some density $f^\circ \in \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D})$.
- Let $f \in L^\infty_{loc}(\mathbb{R}^+; \mathcal{P}(\mathbb{D}) \cap L^\infty(\mathbb{D}))$ be a bounded weak solution of the Vlasov equation with initial data f°,
- Assume that for some $T > 0$ it satisfies $K * f \in L^\infty([0, T] \times \Omega)$ and has bounded Fisher information

$$\int_0^T \Big(\int_{\mathbb{D}} |\nabla_v \log f|^2 f \Big)^{\frac{1}{2}} < \infty.$$

Theorem

Assume (H) be satisfied. The propagation of chaos holds namely: For all $k \geq 0$, the k-th marginal $F_{N,k}$ converges to $f^{\otimes k}$ as $N \to +\infty$ in the sense of distributions on $[0, T] \times \mathbb{D}^k$.

Qantitative estimates ?

Yes, if $K \in H^s_{\rm loc}(\Omega; \mathbb{R}^d)$ for some $s > 0$ and $K \in W^{1,+\infty}_{\rm loc}(|x| > 1)$ assuming $K \star f$ in $L^\infty([0,T]; W^{1,\infty}(\Omega))$ and

$$\int_0^T \Big(\int_{\mathbb{D}} \big(|\nabla_v \log f|^2 + |\tfrac{1}{f}\nabla^2_{vv} f|^2 + |\tfrac{1}{f}\nabla^2_{xv} f|^2\big)\, f\Big)^{\frac12} < \infty.$$

The duality argument:

- Allows to rather consider the solution of a backward dual Liouville equation with final data having a linear structure
- the problem is then reduced to studying instead how this linear structure propagates.
- The advantage of this reformulation is as follows: the defect of linearity is naturally measured by means of linear correlation functions.

Our method

- Requires the interaction kernel K to be in L^2.
- Captures the structure of correlations in the dynamics.
- Works with or without diffusion
- Works for first and second order systems.
- Valuable for general domains.

A Duality approach (on first order for the talk)

The joint law solves the Liouville or Forward Kolmogorov equation

$$\partial_t F_N + \frac{1}{N-1}\sum_{i=1}^N \sum_{j\neq i} K(X_i - X_j)\cdot \nabla_{x_j} F_N = \alpha \sum_{i=1}^N \Delta_{x_i} F_N$$

with

$$F_N|_{t=0} = (f^0)^{\otimes N}.$$

Dual Backward Kolmogorov equation

$$\partial_t \Phi_N + \frac{1}{N-1}\sum_{i=1}^N \sum_{j\neq i} K(x_i - X_j)\cdot \nabla_{x_j} \Phi_N = -\alpha \sum_{i=1}^N \Delta_{x_i} \Phi_N$$

with

$$\Phi_N|_{t=T} = \overline{\Phi}_N.$$

We have

$$\int F_N(t=T)\overline{\Phi}_N dx_1\cdots dx_N = \int (f^0)^{\otimes N}\Phi_N(t=0)dx_1\cdots dx_N.$$

The Goal : Show that as $N\to\infty$ for the limiting solution f

$$\int (f(t=T))^{\otimes N}\overline{\Phi}_N dx_1\cdots dx_N - \int (f^0)^{\otimes N}\Phi_N(t=0)dx_1\cdots dx_N \to 0.$$

Choosing $\overline{\Phi}_N = \frac{1}{N}\sum_{i=1}^N \overline{\varphi}(X_i)$

$$\int F_N(t=T)\overline{\Phi}_N dx_1\cdots dx_N - \int (f(t=T))^{\otimes N}\overline{\Phi}_N dx_1\cdots dx_N$$

$$= \int F_{N,1}(t=T,x)\overline{\varphi}(x)dx - \int f(t=T,x)\overline{\varphi}(x)dx \to 0$$

That means weak convergence of the 1-marginal
(similar formula for other marginals)

We want to prove when $N \to +\infty$

$$\frac{d}{dt} \int (f(t))^{\otimes N} \Phi_N dx_1 \cdots dx_N \to 0$$

How to do?

Formal calculation seems to give a first order term.....

Sketch of the ideas:

- Once a priori estimates for linear dual correlations are established, we will be able to analyze the BBGKY-type hierarchy of equations satisfied by those quantities.
- At vanishing temperature $\alpha = 0$, it is well-known that BBGKY hierarchies cannot be of any rigorous use due to the loss of derivatives.
- Yet, in the dual reformulation, thanks to a priori estimates on linear dual correlations, we discover that the loss of derivatives in the hierarchy of equations for the latter only occurs in perturbative terms, which vanish in the macroscopic limit $N \to \infty$.

- $\Longrightarrow$ Establish a uniqueness principle for the limit hierarchy using the linearity and vanishing final data.
- $\Longrightarrow$ As linear dual correlations vanish at final time in the limit, we may then deduce that this cancellation propagates over time, thus allowing to conclude the desired mean-field limit result.

Remark: Convergence rates are further obtained by a more detailed stability analysis of the limit hierarchy, viewing the exact N-dependent hierarchy satisfied by linear dual correlations as a (singular) perturbation.

Sketch of proof:

Using the equations and data on Φ_N and on f, we can show

$$\int_{\Omega^k} \psi^{\otimes k}\big(F_{N,k}(T) - f(T)^{\otimes k}\big) \\ = -N \int_0^T \Big(\int_{\Omega^N} V_f(z_1, z_2)\, \Phi_N\, f^{\otimes N} \Big) dt,$$

where we have defined

$$V_f(z_i, z_j) := \big(K(x_i - x_j) - K * f(x_i)\big) \cdot (\nabla_v \log f)(z_i).$$

Thus the validity of the mean-field convergence

$$\int_{\Omega^k} \psi^{\otimes k} F_{N,k}(T) \xrightarrow{N\uparrow\infty} \Big(\int_\Omega \psi f(T)\Big)^k$$

is equivalent to the dual convergence property

$$N \int_0^T \Big(\int_{\Omega^N} V_f(z_1, z_2)\, \Phi_N\, f^{\otimes N}\Big)\, dt \xrightarrow{N\uparrow\infty} 0,$$

- If the dual solution Φ_N was known to approximatively keep the special additive structure of its final condition then we would conclude.
- We are thus led to examining how much the dual Φ_N approximatively keeps its additive structure over time.
- The defect of linearity of the dual solution Φ_N is naturally measured by means of the linear correlation function $C_{N,n}$ that can be expressed as linear combinations of the marginals

$$M_{N,n}(z_1, \dots, z_n) := \int_{\Omega^{N-n}} \Phi_N(z_1, \dots, z_N) \, f^{\otimes N-n}(z_{n+1}, \dots, z_N) \, dz_{n+1} \dots dz_N.$$

- Write the system on $M_{N,n}$ and therefore on $C_{N,n}$.
- Remark that

$$N\int_0^T\Big(\int_{\Omega^N} V_f(z_1,z_2)\,\Phi_N\, f^{\otimes N}\Big)\,dt$$
$$= N\int_0^T\Big(\int_{\Omega^2} V_f\, M_{N,2}\, f^{\otimes 2}\Big)\,dt = N\int_0^T\Big(\int_{\Omega^2} V_f\, C_{N,2}\, f^{\otimes 2}\Big)\,dt.$$

- Show that

$$N^{\frac{n}{2}}\,C_{N,n} \overset{*}{\rightharpoonup} n!^{\frac{1}{2}}\,\bar{C}_n, \quad \text{in } L^\infty(0,T;L^2_f(\Omega^n))$$

- Write the equation on $\bar{C}_n$ and appropriate final condition to perform a uniqueness result to show that $\bar{C}_2 = 0$ and its friends.

Thank you !! Bon anniversaire Kisco et Manolo !!