Exact controllability to zero for general linear parabolic equations

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Why do we care about finding Carleman's inequalities?

An observability inequality

$$\int_{\Omega} |\varphi(0)|^2 dx \leq C \int \int_{\omega \times (0,\ T)} |\varphi|^2 dxdt$$

The problem of exact control to zero

$$\begin{cases} y_t - \Delta y = u 1_{\omega} \text{ in } Q = \Omega \times (0,\ T) \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,\ T) \\ y(0) = y_0 \text{ in } \Omega \end{cases}$$

$$y(T) = 0$$

Disadvantage: Carleman's inequality is true for functions in $C^2(\bar{Q})$.

What about the totally distributed control?

- It has not practical interest
- The existence of a control is easily guaranteed

Our motivation: the glioblastoma model

$$y_t - \nabla \cdot (D(x)\nabla y) = \rho y$$

Our approach:

1. Obtaining a totally distributed control from "any positive function" and for $y_0 \geq 0$
2. For any y_0, apply the results to y_0^+ and to y_0^-
3. Passing to a partially distributed control

Let be $u \in L^2(\Omega \times (0, T), u \geq c > 0, y_0 \in L^2(\Omega), y_0 \geq 0$, Is there a function $v^* \geq 0$ such that

$$\|y(T)\|\Psi_{v^*}(T) - \|\Psi_{v^*}(T)\|y(T)$$

has a constant sign in Ω?

$$\begin{aligned} &y_t - \Delta y = u & \qquad &(\Psi_{v^*})_t - \Delta\Psi_{v^*} = v^* \\ &y|_\Sigma = 0 & &\Psi_{v^*}|_\Sigma = 0 \\ &y(0) = y_0 & &\Psi_{v^*}(0) = y_0 \end{aligned}$$

We write $-\Delta$ for simplicity

Observe that this scalar product

$$(\|y(T)\|\Psi_{v^*}(T) - \|\Psi_{v^*}(T)\|y(T)\,,\, \|y(T)\|\Psi_{v^*}(T) + \|\Psi_{v^*}(T)\|y(T))$$

is zero. Then,

$$\|y(T)\|\Psi_{v^*}(T) - \|\Psi_{v^*}(T)\|y(T) = 0$$

and a control is

$$\hat{u} = \frac{\|y(T)\|v^* - \|\Psi_{v^*}(T)\|u}{\|y(T)\| - \|\Psi_{v^*}(T)\|}$$

Idea: building a sequence $\{v_k\}_k$ such that

$$\|\Psi_{v_{k-1}}(T)\|y(T) - \|y(T)\|\Psi_{v_k}(T) \leq 0$$

and passing to the limit
Difficulties:

- It is necessary a maximum principle weaker than the classical one, a maximum principle at the final time T
- How to avoid that the sequence converges to u?

The result when u is totally distributed

Theorem

Let be $u \in L^2(\Omega \times (0, T))$, $u \geq c > 0$ in $\Omega \times (0,\ T)$, $y_0 \in L^2(\Omega)$, $y_0 \geq 0$. Then, there exists $v^* \in L^2(\Omega \times (0, T))$, $0 \leq v^* \leq u$, $\|\Psi_{v^*}(T)\| < \|y(T)\|$, such that

$$\hat{u} = \frac{\|y(T)\| v^* - \|\Psi_{v^*}(T)\| u}{\|y(T)\| - \|\Psi_{v^*}(T)\|}$$

is an exact control to zero in $\Omega \times (0,\ T)$.

The proof: a maximum principle at the final time

Theorem

Let be $\beta \in C^1([0,T])$ *verifying*

$$\beta > 0, \quad \max_{[0,T]} \beta = \beta(T)$$

and let be $w \in W(0,T)$ *such that*

$$w_t - \Delta w \geq 0 \quad w|_\Sigma = 0 \quad w(0) \geq 0.$$

Then, the solution z *of the problem*

$$\begin{cases} z_t - \Delta z = -\beta' w \\ z|_\Sigma = 0 \\ z(0) = z_0 \leq 0. \end{cases}$$

verifies

$$z(T) \leq 0.$$

2. Building a sequence $\{v_k\}_k$ by recurrence, such that

$$0 \leq v_k \leq v_{k+1} \leq u$$

$$v_k = \frac{1}{2}u \text{ in } B \times I, \quad B \text{ a ball in } \Omega, \quad I \text{ an interval in } (0,\, T)$$

$$\|\Psi_{v_{k-1}}(T)\|y(T) - \|y(T)\|\Psi_{v_k}(T) \leq 0$$

How to build this sequence? An increasing sequence, built by recurrence

$$0 \leq v_1 \leq u, \quad v_1 = \frac{1}{2} u \text{ in } B \times I$$

$$\beta_1' \leq \frac{(\|y(T)\| - \|\Psi_{v_1}(T)\|) c}{\sup_{\Omega \times (0,\, T) \setminus \tilde{I}} w_1} \tag{1}$$

$$I \subsetneq \tilde{I} \subset (0,\ T)$$

$$\beta_1' < -\frac{\|\Psi_{v_1}(T)\| \sup_{B \times I} u}{\inf_{B \times I} w_1} \text{ in } I \tag{2}$$

$$\beta_1' \leq 0 \text{ in } \tilde{I} \tag{3}$$

We define $\tilde{v_1}$:

$$\|\Psi_{v_1}(T)\|u - \|y(T)\|\tilde{v_1} = -\beta_1' w_1$$

It verifies

$$\tilde{v_1} \leq u$$

$$\tilde{v_1} < 0 \text{ in } B \times I$$

Then,

$$v_2 = \max(v_1, \tilde{v_1})$$

And it satisfies

$$v_1 \leq v_2$$

$$0 \leq v_2 \leq u$$

$$v_2 = v_1 = \frac{1}{2}u$$

$$\|\Psi_{v_1}(T)\|y(T) - \|y(T)\|\Psi_{v_2}(T) \leq \|\Psi_{v_1}(T)\|y(T) - \|y(T)\|\Psi_{\tilde{v_1}}(T) \leq 0$$

3. Passing to the limit

$$\exists \lim v_k = v^*$$

4. Obtaining the control

Since

$$\Psi_{v^*}(T)\,\|y(T)\| - \|\Psi_{v^*}(T)\|\,y(T) = 0$$

$$\hat{u} = \frac{\|y(T)\|v^* - \|\Psi_{v^*}(T)\|u}{\|y(T)\| - \|\Psi_{v^*}(T)\|}$$

The partially distributed control

Theorem

Let be $u \in L^2(\Omega \times (0, T))$, $u \geq c > 0$ *in* $\Omega \times (0, T)$, $y_0 \in L^2(\Omega)$, $y_0 \geq 0$, $\omega \subset \Omega$ *an open set. Then, there exists* $v^* \in L^2(\Omega \times (0, T))$, $v^* = 0$ *in* $\Omega \setminus \omega \times (0, T)$, $v^* \geq 0$ *such that*

$$\hat{u} = \frac{\|y(T)\| v^* - \|\Psi_{v^*}(T)\| u 1_\omega}{\|y(T)\| - \|\Psi_{v^*}(T)\|}$$

is an exact control to zero in $\omega \times (0, T)$.

The proof

1. Defining

$$u_n = \begin{cases} u & \text{if} \quad x \in \omega \\ \frac{1}{n} u & \text{if} \quad x \in \Omega \setminus \omega \end{cases}$$

and applying the previous theorem to each u_n,

$$\exists v_n^*$$

2. Passing to the limit in n:

$$u_n \to u 1_\omega$$

$$v_n^* \text{ weakly converges to } v^*$$

3. The states $\Psi_{v_n^*}(T)$ converge strongly in $L^2(\Omega)$ to $\Psi_{v^*}(T)$

4. Since $0 \leq v_n^* \leq u_n$,

$$v^* = 0 \text{ in } \Omega \setminus \omega \times (0,\ T)$$

Conclussions

- Obtaining a control from any function u
- The solution of a linear parabolic problem can be written by a fixed point equation

$$y(T) = \|y(T)\| \frac{\Psi_{v^*}(T)}{\|\Psi_{v^*}(T)\|}$$

- It is possible to control with discontinuous diffusion coefficients and boundary $C^{0,1}$

- Benabdallah, A.; Dermenjian, Y.; Thevenet, L. Carleman estimates for some non-smooth anisotropic media. *Comm. Partial. Differ. Equations* **2013**, *38*, 1763–1790.
- Fernández-Cara, E.; Guerrero, S. Global Carleman inequalities for parabolic systems and applications to controllability. *SIAM J. Control Optim.* **2006**, *45*, 1399–1446.
- Fernández-Cara, E.; Zuazua, E. On the null controllability of the one-dimensional heat equation with BV coefficients. *Comput. Appl. Math.* **2002**, *21*, 167–190.
- Gayte Delgado, I.; Marín-Gayte, I. A new method for the exact controllability of linear parabolic equations. *Mathematics* **2025**, *13*, 344, https://doi.org/10.3390/math13030344
- K.R. Swanson, E.C. Alvord, J.D. Murray, *Dynamics of a model for brain tumors reveals a small window for therapeutic intervention*, Mathematical models in cancer (Nashville, TN, 2002). Discrete Contin. Dyn. Syst. Ser. B **4** (2004), no. 1, 289-295.