On a model of flows in a deformable porous solid with small strain and density depending material modulus

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Common work with

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PKM-60

Outline

The model

MODEL

Introducing the model

- The model I shall present was proposed by Professor Rajagopal to Andrea Bonito, Diane Guignard and me.
- It is a model for the flow of slow incompressible fluids in a deformable porous solid, but it is different from the classical Biot model.
- The model for the solid's displacement is introduced and studied in "An implicit constitutive relation for describing the small strain response of porous elastic solids whose material moduli are dependent on the density", by Rajagopal, in Mathematics and Mechanics of Solids, 2021.
- This dependence has been linearized, but the equations of the model are nonlinear.
- Since the flow is slow, the motion of the fluid is governed by Stokes system.
- The flow and solid are coupled by a reaction term between the velocity of the solid and the velocity of the fluid.

Constitutive equation for the solid

- $\mathbf{T}_s$: symmetric Cauchy stress tensor of solid, $\mathbf{u}_s$: solid's displacement
  $\boldsymbol{\varepsilon}_s$: symmetric gradient tensor of $\mathbf{u}_s$, $\boldsymbol{\varepsilon}_s = \frac{1}{2}(\nabla\,\mathbf{u}_s + (\nabla\,\mathbf{u}_s)^T)$
  $\text{tr}(\mathbf{T}_s)$: trace of $\mathbf{T}_s$, $\text{tr}(\mathbf{T}_s) = \sum_{i=1}^{d}(T_s)_{i,i}$
  d : dimension, here we work in 3-D, i.e., $d = 3$
- Implicit constitutive relation for $\mathbf{T}_s$:

$$\boldsymbol{\varepsilon}_s = E_{1s}\big(1 + \lambda_2 \text{tr}(\boldsymbol{\varepsilon}_s)\big)\mathbf{T}_s + E_{2s}\big(1 + \lambda_3 \text{tr}(\boldsymbol{\varepsilon}_s)\big)\text{tr}(\mathbf{T}_s)\mathbf{I}. \tag{1}$$

The linearization is justified by the assumption :

$$\|\varepsilon_s\| \leq \delta \ll 1 \tag{2}$$

- $\|\cdot\|$: Frobenius norm
- Model applies to porous metals, bones, rocks, concrete, undergoing small deformations

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Constitutive equation for the solid–II

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- The linearized material moduli are :

$$E_{1s}\big(1+\lambda_2 \mathrm{tr}(\varepsilon_s)\big) \text{ and } E_{2s}\big(1+\lambda_3 \mathrm{tr}(\varepsilon_s)\big)$$

- They are related to a generalized Young's modulus and a generalized Poisson's ratio if we define

$$\frac{1+\nu_g}{E_g} = E_{1s}\big(1+\lambda_2 \mathrm{tr}(\varepsilon_s)\big), \quad \frac{-\nu_g}{E_g} = E_{2s}\big(1+\lambda_3 \mathrm{tr}(\varepsilon_s)\big)$$

- Where is the dependence on density ?
  The balance of mass and $\|\varepsilon_s\| \leq \delta \ll 1$ imply

$$\varrho_R = \varrho_s(1+\mathrm{tr}(\varepsilon_s))$$

- ϱ_R : reference density, ϱ_s : actual density of the solid.

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- From relations of E_{1s} and E_{2s} with E_g and ν_g, we expect

$$E_{1s} > 0 \text{ and } E_{2s} < 0 \text{ therefore } E_{2s} = -|E_{2s}|$$

- Assumption :

The parameters λ_2 and λ_3 are small in absolute value

- $\lambda_2 = 0$ and $\lambda_3 = 0$ imply standard linear elasticity

$$\boldsymbol{\varepsilon}_s = E_{1s}\mathbf{T}_s - |E_{2s}|\mathrm{tr}(\mathbf{T}_s)\mathbf{I}.$$

- Therefore the nonlinear model is a perturbation of a linear model.

Balance of linear momentum for the solid

- Assumption :
  Acceleration of the solid $\partial^2_{tt}\mathbf{u}_s$ is negligible
- Balance of linear momentum takes into account interaction with the fluid

$$\mathbf{div}(\mathbf{T}_s) + \alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s) = \mathbf{0}, \tag{3}$$

- $\mathbf{v}_f$: velocity of fluid, $\partial_t \mathbf{u}_s$: velocity of solid
- Interaction term :

$$\alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s)$$

Equation of motion of the fluid

- p_f : fluid's pressure, $\mu_f > 0$: constant viscosity of fluid, $\varrho_f > 0$: constant density of fluid
- Flow equation of the fluid taking into account interaction with the solid :

$$\alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s) - \mu_f \Delta \mathbf{v}_f + \nabla p_f = -\varrho_f \partial_t \mathbf{v}_f \tag{4}$$

- Conservation of mass :

$$\operatorname{div} \mathbf{v}_f = 0 \tag{5}$$

- (4)–(5) is a standard Stokes system + the interaction term

Preliminary summary of equations

$$\boldsymbol{\varepsilon}_s = E_{1s}\big(1+\lambda_2 \text{tr}(\boldsymbol{\varepsilon}_s)\big)\mathbf{T}_s - |E_{2s}|\big(1+\lambda_3 \text{tr}(\boldsymbol{\varepsilon}_s)\big)\text{tr}(\mathbf{T}_s)\mathbf{I}$$

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$$\text{div}\, \mathbf{v}_f = 0$$

- 13 equations in 13 unknowns
- plus suitable initial and boundary conditions
- We must include hypothesis (2) that permitted the linearization :

$$\|\varepsilon_s\| \le \delta \ll 1$$

- It means that $\mathbf{u}_s$ is small in $W^{1,\infty}$ in space and time

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Truncation of the factors

- As it is, this system is unmanageable because the factors

$$1 + \lambda_2 \mathrm{tr}(\boldsymbol{\varepsilon}_s) \text{ and } 1 + \lambda_3 \mathrm{tr}(\boldsymbol{\varepsilon}_s)$$

  can only be controlled by hypothesis (2) : $\|\boldsymbol{\varepsilon}_s\| \le \delta \ll 1$
  but how can it be used ?
- One possibility is to truncate $\mathrm{tr}(\boldsymbol{\varepsilon}_s)$
- Recall the truncation operator T_k for real $k > 0$

$$T_k f = f \quad \text{if } |f| \le k, \quad T_k f = k \,\mathrm{sign}(f) \quad \text{if } |f| > k.$$

- Then $\|\boldsymbol{\varepsilon}_s\| \le \delta$ implies that $|\mathrm{tr}(\boldsymbol{\varepsilon}_s)| \le \sqrt{3}\delta$ and $\mathrm{tr}(\boldsymbol{\varepsilon}_s) = T_{\sqrt{3}\delta}\mathrm{tr}(\boldsymbol{\varepsilon}_s)$
- Therefore we can use

$$\boldsymbol{\varepsilon}_s = E_{1s}\big(1 + \lambda_2 T_{\sqrt{3}\delta}\mathrm{tr}(\boldsymbol{\varepsilon}_s)\big)\mathbf{T}_s - |E_{2s}|\big(1 + \lambda_3 T_{\sqrt{3}\delta}\mathrm{tr}(\boldsymbol{\varepsilon}_s)\big)\mathrm{tr}(\mathbf{T}_s)\mathbf{I}$$

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Inverting the displacement's constitutive relation

- To simplify denote $\sqrt{3}\delta$ by δ

$$\boldsymbol{\varepsilon}_s = E_{1s}\big(1 + \lambda_2 T_\delta \mathrm{tr}(\boldsymbol{\varepsilon}_s)\big)\mathbf{T}_s - |E_{2s}|\big(1 + \lambda_3 T_\delta \mathrm{tr}(\boldsymbol{\varepsilon}_s)\big)\mathrm{tr}(\mathbf{T}_s)\mathbf{I}.$$

- Linearization has been done directly on the implicit relation.
  But, mathematically the constitutive relation can be easily inverted and the inverted form is useful for mathematical analysis and discretization
- Take the trace of (1), recall $\mathrm{tr}(\boldsymbol{\varepsilon}_s) = \mathrm{div}\,\mathbf{u}_s$, $\mathrm{tr}(\mathbf{I}) = d = 3$

$$\mathbf{T}_s = \frac{1}{E_{1s}(1 + \lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_s)}\left(\boldsymbol{\varepsilon}_s + |E_{2s}|(1 + \lambda_3 T_\delta \mathrm{div}\,\mathbf{u}_s)\frac{\mathrm{div}\,\mathbf{u}_s}{F(\mathbf{u}_s)}\mathbf{I}\right)$$

where

$$F(\mathbf{u}_s) = E_{1s}(1 + \lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_s) - 3|E_{2s}|(1 + \lambda_3 T_\delta \mathrm{div}\,\mathbf{u}_s).$$

THE LITTLE ANALYSIS WE COULD DO

Summary of equations

- Domain Ω, boundary $\partial\Omega$, time interval $(0,T)$
- Unknowns $\mathbf{u}_s, \mathbf{v}_f, p_f$ satisfying a.e. $t \in (0,T)$

$$\mathbf{T}_s = \frac{1}{E_{1s}(1+\lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_s)}(\boldsymbol{\varepsilon}_s + |E_{2s}|(1+\lambda_3 T_\delta \mathrm{div}\,\mathbf{u}_s)\frac{\mathrm{div}\,\mathbf{u}_s}{F(\mathbf{u}_s)}\mathbf{I})$$

$$F(\mathbf{u}_s) = E_{1s}(1+\lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_s) - 3|E_{2s}|(1+\lambda_3 T_\delta \mathrm{div}\,\mathbf{u}_s)$$

$$\mathbf{div}(\mathbf{T}_s) + \alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s) = \mathbf{0}$$

$$\alpha(\mathbf{v}_f - \partial_t \mathbf{u}_s) - \mu_f \Delta\, \mathbf{v}_f + \nabla\, p_f = -\varrho_f \partial_t \mathbf{v}_f$$

$$\mathrm{div}\,\mathbf{v}_f = 0$$

- Boundary conditions (outrageously simplified in view of the analysis)

$$\mathbf{u}_s = \mathbf{0} \text{ on } \partial\Omega, \mathbf{v}_f = \mathbf{0} \text{ on } \partial\Omega$$

- Initial conditions

$$\mathbf{u}_s(0) = \mathbf{u}_{s,0},\ \mathbf{v}_f(0) = \mathbf{v}_{f,0},\ \mathrm{div}\,\mathbf{v}_{f,0} = 0,\ \mathbf{u}_{s,0} = \mathbf{v}_{f,0} = \mathbf{0} \text{ on } \partial\Omega$$

Rough variational formulation

$$\frac{1}{E_{1s}}\int_0^t\int_\Omega \Big[\frac{1}{1+\lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_s}\boldsymbol{\varepsilon}(\mathbf{u}_s):\boldsymbol{\varepsilon}(\mathbf{z})$$
$$+\frac{|E_{2s}|(1+\lambda_3 T_\delta \mathrm{div}\,\mathbf{u}_s)}{(1+\lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_s)}\frac{\mathrm{div}\,\mathbf{u}_s}{F(\mathbf{u}_s)}\mathrm{div}\,\mathbf{z}\Big]+\alpha\int_0^t\int_\Omega \partial_t\mathbf{u}_s\cdot\mathbf{z}$$
$$=\alpha\int_0^t\int_\Omega \mathbf{v}_f\cdot\mathbf{z},$$

$$\int_0^t\int_\Omega \varrho_f\partial_t\mathbf{v}_f\cdot\mathbf{w}+\mu_f\int_0^t\int_\Omega \nabla\,\mathbf{v}_f:\nabla\,\mathbf{w}+\alpha\int_0^t\int_\Omega \mathbf{v}_f\cdot\mathbf{w}$$
$$-\int_0^t\int_\Omega p_f\mathrm{div}\,\mathbf{w}=\alpha\int_0^t\int_\Omega \partial_t\mathbf{u}_s\cdot\mathbf{w},$$
$$\mathrm{div}\,\mathbf{v}_f=0$$

- $\mathbf{z}$: test displacement in $L^2(0,T;H_0^1(\Omega)^3)$,
  $\mathbf{w}$: test velocity in $L^2(0,T;H_0^1(\Omega)^3)$

A little good news

Assumption on the parameters

$$E_{1s} - 3|E_{2s}| > 0$$

- and suppose that δ (and/or λ_2, λ_3) sufficiently small so that

$$0 < 1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_s,\ 0 < 1 + \lambda_3 T_\delta \mathrm{div}\, \mathbf{u}_s,\ 0 < F(\mathbf{u}_s)$$

are bounded above and away from zero
recall $F(\mathbf{u}_s) = E_{1s}(1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_s) - 3|E_{2s}|(1 + \lambda_3 T_\delta \mathrm{div}\, \mathbf{u}_s)$

- Then the system is unconditionally stable in basic norms :

$$\|\mathbf{v}_f\|_{L^\infty(0,t;L^2(\Omega))},\ \|\mathbf{v}_f\|_{L^2(Q_t)} \text{ and } \|\nabla \mathbf{v}_f\|_{L^2(Q_t)} \le C_1 \exp(C_2 t),$$

Q_t : time-space cylinder $\Omega\times]0,t[$
C_1 is small if $\|\mathbf{v}_{f,0}\|_{L^2(\Omega)}$ and $\|\mathbf{u}_{s,0}\|_{L^2(\Omega)}$ are small
C_2 depends on the parameters $\alpha, \mu_f, \varrho_f, E_{1s}, E_{2s}, \lambda_2, \lambda_3, \delta$

Stability estimates–II

- The displacement satisfies

$$\|\mathbf{u}_s\|_{L^\infty(0,t;L^2(\Omega))},\ \|\boldsymbol{\varepsilon}(\mathbf{u}_s)\|_{L^2(Q_t)},\ \|\mathrm{div}\,\mathbf{u}_s\|_{L^2(Q_t)}$$
$$\text{and } \|\partial_t\mathbf{u}_s\|_{L^2(0,t;H^{-1}(\Omega))} \le C_3\|\mathbf{v}_f\|_{L^2(Q_t)} + C_4$$

  C_3 depends on the parameters
  C_4 is small if $\|\mathbf{u}_{s,0}\|_{L^2(\Omega)}$ is small
- Similar bounds hold for the velocity plus pressure and for the pressure itself

$$\|\partial_t\mathbf{v}_f + \frac{1}{\varrho_f}\nabla p_f\|_{L^2(0,t;H^{-1}(\Omega))},\ \|p_f\|_{W^{-1,\infty}(0,t;L^2_0(\Omega))}$$

Very bas news

The system has neither compactness nor monotony

- No compactness : because of the term

$$\int_0^t \int_\Omega \frac{1}{1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_s} \boldsymbol{\varepsilon}(\mathbf{u}_s) : \boldsymbol{\varepsilon}(\mathbf{z})$$

weak convergence of $\boldsymbol{\varepsilon}(\mathbf{u}_s)$ in $L^2(Q_t)$ and weak convergence of $\frac{1}{1+\lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_s}$ in $L^\infty(Q_t)$ does not imply convergence of the product to the product of the limits

- No monotony : there is no reason why

$$\int_0^t \int_\Omega \Big[\frac{1}{1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_1} \boldsymbol{\varepsilon}(\mathbf{u}_1) - \frac{1}{1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_2} \boldsymbol{\varepsilon}(\mathbf{u}_2) \Big] : \boldsymbol{\varepsilon}(\mathbf{u}_1 - \mathbf{u}_2) > 0$$

An attempt with perturbation

- The fully linear problem is well-posed

$$\frac{1}{E_{1s}}\int_0^t\int_\Omega\Big[\boldsymbol{\varepsilon}(\mathbf{u}_s):\boldsymbol{\varepsilon}(\mathbf{z})+\frac{|E_{2s}|}{E_{1s}-3|E_{2s}|}\mathrm{div}\,\mathbf{u}_s\mathrm{div}\,\mathbf{z}\Big]+\alpha\int_0^t\int_\Omega\partial_t\mathbf{u}_s\cdot\mathbf{z}$$
$$=\alpha\int_0^t\int_\Omega\mathbf{v}_f\cdot\mathbf{z},$$

$$\int_0^t\int_\Omega\varrho_f\partial_t\mathbf{v}_f\cdot\mathbf{w}+\mu_f\int_0^t\int_\Omega\nabla\,\mathbf{v}_f:\nabla\,\mathbf{w}+\alpha\int_0^t\int_\Omega\mathbf{v}_f\cdot\mathbf{w}$$
$$-\int_0^t\int_\Omega p_f\mathrm{div}\,\mathbf{w}=\alpha\int_0^t\int_\Omega\partial_t\mathbf{u}_s\cdot\mathbf{w},$$
$$\mathrm{div}\,\mathbf{v}_f=0$$

- We hope that a small perturbation has also a solution

General idea of perturbation

- Let $G^s(\lambda_2, \lambda_3, \mathbf{u}_s, \mathbf{v}_f)$ be the operator for the solid's displacement
  let $G^f(\mathbf{u}_s, \mathbf{v}_f, p_f)$ be the operator for the flow
  let $\mathcal{F}(\lambda_2, \lambda_3, \mathbf{u}_s, \mathbf{v}_f, p_f)$ be the operator for the nonlinear system including the initial conditions :

$$\mathcal{F}(\lambda_2, \lambda_3, \mathbf{u}_s, \mathbf{v}_f, p_f) := \Big(G^s(\lambda_2, \lambda_3, \mathbf{u}_s, \mathbf{v}_f), G^f(\mathbf{u}_s, \mathbf{v}_f, p_f), \mathbf{u}_s(0) - \mathbf{u}_{s,0}, \mathbf{v}_f(0) - \mathbf{v}_{f,0}\Big)$$

- The nonlinear system reads

$$\mathcal{F}(\lambda_2, \lambda_3, \mathbf{u}_s, \mathbf{v}_f, p_f) = \big(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\big).$$

- The fully linear system corresponds to $\lambda_2 = \lambda_3 = 0$
  It has a unique solution, say $\mathbf{u}_{\rm lin}, \mathbf{v}_{\rm lin}, p_{\rm lin}$:

$$\mathcal{F}(0, 0, \mathbf{u}_{\rm lin}, \mathbf{v}_{\rm lin}, p_{\rm lin}) = \big(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\big).$$

General idea of perturbation–II

- Set $\boldsymbol{\omega} = (\mathbf{u}, \mathbf{v}, p)$ and $\boldsymbol{\omega}_{\rm lin} = (\mathbf{u}_{\rm lin}, \mathbf{v}_{\rm lin}, p_{\rm lin})$
  Suppose $\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\rm lin})$ is differentiable with respect to $\boldsymbol{\omega}$
  Let $D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\rm lin})$ be the derivative
- If $D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\rm lin})$ is invertible , then we can define the auxiliary function

$$\mathcal{H}(\lambda_2, \lambda_3, \boldsymbol{\omega}) = \boldsymbol{\omega} - [D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\rm lin})]^{-1}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega})$$

  and the fixed points of $\mathcal{H}$ are the zeros of $\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega})$

Computing the inverse of the derivative

- Three steps
  1. Compute $D_{\boldsymbol{\omega}}\mathcal{F}(0,0,\boldsymbol{\omega}_{\rm lin})$
  2. Compare with $D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_1,\lambda_2,\boldsymbol{\omega}_{\rm lin})$
  3. Deduce the inverse if the difference is less that one
- Start with the linear operator $\mathcal{F}(0,0,\omega)$
  Spaces : argument ω in $\mathcal{X}$ and values in $\mathcal{Y}$
- Pick an increment $\boldsymbol{\theta} = (\boldsymbol{\theta}_u, \boldsymbol{\theta}_v, \theta_p)$ in $\mathcal{X}$
  Since $\mathcal{F}(0,0,\omega)$ is linear, $D_{\boldsymbol{\omega}}\mathcal{F}(0,0,\omega)\boldsymbol{\theta}$ has a simple expression
- And computing the inverse $[D_{\boldsymbol{\omega}}\mathcal{F}(0,0,\omega)]^{-1}$ amounts to solve for any set of values $(\mathbf{g}_u, \mathbf{g}_v, \mathbf{a}, \mathbf{b})$ in $\mathcal{Y}$ the system : Find $\boldsymbol{\theta} \in \mathcal{X}$

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$$D_{\boldsymbol{\omega}}\mathcal{F}(0,0,\boldsymbol{\omega})\boldsymbol{\theta} = (\mathbf{g}_u, \mathbf{g}_v, \mathbf{a}, \mathbf{b})$$

Computing the inverse of the derivative-II

- Compute $D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_1, \lambda_2, \boldsymbol{\omega}_{\rm lin})\boldsymbol{\theta}$
- All terms are good except

$$-\frac{\lambda_2}{E_{1s}}\frac{\boldsymbol{\varepsilon}(\mathbf{u}_{\rm lin}) : \boldsymbol{\varepsilon}(\mathbf{z})}{(1+\lambda_2 T_\delta \mathrm{div}\,\mathbf{u}_{\rm lin})^2} s(\mathrm{div}\,\mathbf{u}_{\rm lin})\mathrm{div}\,\boldsymbol{\theta}_u,$$

here $s(\cdot)$ comes from differentiating the truncation :
$s(f) = 1$ if $|f| \leq \delta$ and $s(f) = 0$ if $|f| > \delta$

- It can be controlled if we assume

$$\|\boldsymbol{\varepsilon}(\mathbf{u}_{\rm lin})\|_{L^\infty(Q_T)} \leq \delta$$

- Then

$$\|D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\rm lin}) - D_{\boldsymbol{\omega}}\mathcal{F}(0, 0, \boldsymbol{\omega}_{\rm lin})\| \leq \frac{\delta}{E_{1s}} A_1$$

where A_1 is small if λ_2 and λ_3 are small.

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- Then

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  where A_1 is small if λ_2 and λ_3 are small.

Finding the fixed points of $\mathcal{H}(\lambda_2, \lambda_3, \boldsymbol{\omega})$

- Recall

$$\mathcal{H}(\lambda_2, \lambda_3, \boldsymbol{\omega}) = \boldsymbol{\omega} - [D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\text{lin}})]^{-1}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega})$$

- Standard technique : construct a sequence $\boldsymbol{\omega}_k$ with

$$\boldsymbol{\omega}_{k+1} = \boldsymbol{\omega}_k - [D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\text{lin}})]^{-1}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_k)$$

with obvious starting function $\boldsymbol{\omega}_0 = \boldsymbol{\omega}_{\text{lin}}$

- If it converges, the limit solves the problem

There is no free lunch

- To show convergence, consider two consecutive iterates $\boldsymbol{\omega}_k, \boldsymbol{\omega}_{k+1}$ set $\boldsymbol{\theta} = \boldsymbol{\omega}_{k+1} - \boldsymbol{\omega}_k$
- As part of the calculation, we have to estimate

$$\int_0^1 \big(D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_{\text{lin}}) - D_{\boldsymbol{\omega}}\mathcal{F}(\lambda_2, \lambda_3, \boldsymbol{\omega}_k + \tau\boldsymbol{\theta})\big)\boldsymbol{\theta}\, d\tau$$

  for this we assume that

$$\|\boldsymbol{\varepsilon}(\mathbf{u}_k)\|_{L^\infty(Q_T)} \leq \delta$$

  this is the weak point of the analysis because it is very restrictive
- If it holds then it gives existence.

FINITE-ELEMENT DISCRETIZATION

General Numerical Scheme

Discretization in space

- Assume domain is Lispchitz polyhedron, h : meshsize
  $\mathcal{T}_h$ shape-regular mesh of $\bar{\Omega}$
- Discrete space for $\mathbf{u}_s$: $\boldsymbol{U}_h \subset H_0^1(\Omega)^3$, eg. piecewise $(\mathbb{P}_2)^3$
- Pair of discrete spaces for $\mathbf{v}_f, p^f$: $\boldsymbol{X}_h \subset H_0^1(\Omega)^3$, $Q_h \subset L_0^2(\Omega)$
  stable for the divergence , eg. Hood–Taylor $(\mathbb{P}_2)^3 - \mathbb{P}_1$
- Approximation of the divergence-free functions

$$\boldsymbol{V}_h = \{\mathbf{v}_h \in \boldsymbol{X}_h \,; \int_\Omega q_h \mathrm{div}\,\mathbf{v}_h = 0, \forall q_h \in Q_h\}$$

- Π_h : interpolation operator from $H_0^1(\Omega)^3$ into $\boldsymbol{X}_h$ or $\boldsymbol{U}_h$

Discretization in time

- Choose $N > 1$, set time step : $\Delta t = \frac{T}{N}$, discrete times : $t_n = n\Delta t$
- Use backward Euler for the time derivative, implicit but time-lagging for the nonlinear factors

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The numerical scheme

- Starting from

$$\mathbf{u}_h^0 = \Pi_h \mathbf{u}_{0,s}, \quad \mathbf{v}_h^0 = \Pi_h \mathbf{v}_{0,f},$$

- then given $\mathbf{u}_h^{n-1} \in \boldsymbol{U}_h$ and $\mathbf{v}_h^{n-1} \in \boldsymbol{V}_h$ find $\mathbf{u}_h^n \in \boldsymbol{U}_h$, $\mathbf{v}_h^n \in \boldsymbol{V}_h$ and $p_h^n \in Q_h$ satisfying for all $\mathbf{z}_h \in \boldsymbol{U}_h$,

$$\frac{\alpha}{\Delta t} \int_\Omega (\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) \cdot \mathbf{z}_h + \frac{1}{E_{1s}} \int_\Omega \frac{1}{1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_h^{n-1}} \boldsymbol{\varepsilon}(\mathbf{u}_h^n) : \boldsymbol{\varepsilon}(\mathbf{z}_h)$$
$$+ \frac{|E_{2s}|}{E_{1s}} \int_\Omega \frac{1 + \lambda_3 T_\delta \mathrm{div}\, \mathbf{u}_h^{n-1}}{(1 + \lambda_2 T_\delta \mathrm{div}\, \mathbf{u}_h^{n-1})} \frac{1}{F(\mathbf{u}_h^{n-1})} (\mathrm{div}\, \mathbf{u}_h^n)(\mathrm{div}\, \mathbf{z}_h)$$
$$= \alpha \int_\Omega \mathbf{v}_h^n \cdot \mathbf{z}_h,$$

where $F(\mathbf{u}_h^{n-1}) = E_{1s} - 3|E_{2s}| + \big(E_{1s}\lambda_2 - 3|E_{2s}|\lambda_3\big) T_\delta \mathrm{div}\, \mathbf{u}_h^{n-1}$

The numerical scheme–II

- and for all $\mathbf{w}_h \in \boldsymbol{X}_h$,

$$\frac{\varrho_f}{\Delta t}\int_\Omega (\mathbf{v}_h^n - \mathbf{v}_h^{n-1})\cdot \mathbf{w}_h - \int_\Omega p_h^n \operatorname{div} \mathbf{w}_h + \mu_f \int_\Omega \nabla \mathbf{v}_h^n : \nabla \mathbf{w}_h$$
$$+\,\alpha \int_\Omega \mathbf{v}_h^n \cdot \mathbf{w}_h = \frac{\alpha}{\Delta t}\int_\Omega (\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\cdot \mathbf{w}_h$$

- At each step n, this coupled linear system in finite dimension has a unique solution

Stability independent of the exact solution

- Assumptions

1. Stability and approximation property of Π_h, $\forall \mathbf{u} \in H_0^1(\Omega)^3$

   $\|\nabla \Pi_h \mathbf{u}\|_{L^2(\Omega)} \le C\|\nabla \mathbf{u}\|_{L^2(\Omega)}$ and $\|\mathbf{u} - \Pi_h \mathbf{u}\|_{L^2(\Omega)} \le Ch\|\nabla \mathbf{u}\|_{L^2(\Omega)}$

   used for the initial data and for estimating in H^{-1} the discrete time derivative of $\mathbf{u}_h$
2. Relation between the steps

   $$h^2 \le \Delta t$$

   also used for estimating in H^{-1} the discrete time derivative of $\mathbf{u}_h$
3. Restriction on the time step : Δt sufficiently small

   $$\Delta t \le \tau, \quad \tau \text{ fixed, depending on the parameters}$$

   used in applying discrete Gronwall's lemma

Stability independent of the exact solution–II

- Then

$$\|\mathbf{v}_h^n\|_{L^2(\Omega)}^2,\ \sum_{i=1}^{n-1}\|\mathbf{v}_h^i-\boldsymbol{v}_h^{i-1}\|_{L^2(\Omega)}^2,\sum_{i=1}^{n}\Delta\, t\|\nabla\,\mathbf{v}_h^i\|_{L^2(\Omega)}^2,\ \sum_{i=1}^{n}\Delta\, t\|\mathbf{v}_h^i\|_{L^2(\Omega)}^2 \leq C_1\mathsf{exp}(C_2 t_n)$$

$$\|\mathbf{u}_h^n\|_{L^2(\Omega)}^2,\ \sum_{i=1}^{n}\|\mathbf{u}_h^i-\mathbf{u}_h^{i-1}\|_{L^2(\Omega)}^2,\ \sum_{i=1}^{n}\Delta\, t\|\boldsymbol{\varepsilon}(\mathbf{u}_h^i)\|_{L^2(\Omega)}^2,\ \sum_{i=1}^{n}\Delta\, t\|\mathrm{div}\,\mathbf{u}_h^i\|_{L^2(\Omega)}^2 \leq C_3\mathsf{exp}(C_2 t_n)$$

- Similar estimate for $\|p_h\|_{W^{-1,\infty}(0,T;L_0^2(\Omega))}$

A priori error estimates – degree one in space

- Errors

$$\mathbf{e}_u^n := \Pi_h \mathbf{u}(t_n) - \mathbf{u}_h^n, \quad \mathbf{e}_v^n := \Pi_h \mathbf{v}(t_n) - \mathbf{v}_h^n$$

- If the exact solution is sufficiently smooth but not necessarily small

$$\|\mathbf{e}_v^n\|_{L^2(\Omega)}^2, \ \sum_{i=1}^n \Delta t \|\nabla \mathbf{e}_v^i\|_{L^2(\Omega)}^2, \ \sum_{i=1}^n \Delta t \|\mathbf{e}_v^i\|_{L^2(\Omega)}^2 \le \big(h^2 + (\Delta t)^2\big) C_4 \exp(C_5 t_n)$$

$$\|\mathbf{e}_u^n\|_{L^2(\Omega)}^2, \ \sum_{i=1}^n \|\mathbf{e}_u^i - \mathbf{e}_u^{i-1}\|_{L^2(\Omega)}^2, \ \sum_{i=1}^n \Delta t \|\boldsymbol{\varepsilon}(\mathbf{e}_u^i)\|_{L^2(\Omega)}^2,$$
$$\sum_{i=1}^n \Delta t \|\operatorname{div} \mathbf{e}_u^i\|_{L^2(\Omega)}^2 \le \big(h^2 + (\Delta t)^2\big) C_6 \exp(C_7 t_n)$$

- Similar estimate for $\|e_p\|_{W^{-1,\infty}(0,T;L_0^2(\Omega))}$

Numerical experiment in $2 - D$

- Results using *deal.ii* and Paraview for visualization
- Domain : $\Omega = (0,1)^2$, unit cube, final time : $T = 1$

$$\alpha = 1,\ \mu_f = 1,\ \varrho_f = 1,\ \lambda_2 = \lambda_3 = 0.1,\ E_{1s} = 1,\ E_{2s} = -0.2,\ \delta = 0.4$$

- Spaces $\boldsymbol{U}_h : \mathbb{Q}_2,\ \boldsymbol{X}_h : \mathbb{Q}_2,\ \ Q_h : \mathbb{Q}_1$
- Manufactured solution

  time factor

$$\theta(t) = \theta_1(t) := \exp(-t) \qquad \text{or} \qquad \theta(t) = \theta_2(t) := \sin(3\pi t).$$

$$\mathbf{u}(t,\mathbf{x}) = \frac{\theta(t)}{100}\begin{pmatrix} \cos(2\pi x_1)\sin(2\pi x_2) \\ \sin(2\pi x_1)\cos(2\pi x_2) \end{pmatrix}$$

$$\mathbf{v}(t,\mathbf{x}) = \theta(t)\begin{pmatrix} \sin(2\pi x_1)\cos(2\pi x_2) \\ -\cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix}$$

$$p(\mathbf{x},t) = \theta(t)(60x_1^2 x_2 - 20x_2^3 - 5)$$

force terms and non-homogeneous Dirichlet boundary conditions are added to the system

Numerical results–I

- $N = 20, 40, 80, 160$, $m = 5, 6, 7$, $\mathcal{T}_h$: 2^{2m} square elements $h = \sqrt{2}(2^{-m})$

m	N	$\mathbf{e}_u^{H_0^1}$	$\mathbf{e}_u^{L^2}$	$\mathbf{e}_v^{H_0^1}$	$\mathbf{e}_v^{L^2}$
5	20	$6.0694{\cdot}10^{-5}$	$2.5565{\cdot}10^{-6}$	$5.8579{\cdot}10^{-3}$	$1.1499{\cdot}10^{-4}$
5	40	$5.9344{\cdot}10^{-5}$	$1.3452{\cdot}10^{-6}$	$5.8790{\cdot}10^{-3}$	$6.7737{\cdot}10^{-5}$
5	80	$5.9183{\cdot}10^{-5}$	$7.5595{\cdot}10^{-7}$	$5.9028{\cdot}10^{-3}$	$4.9307{\cdot}10^{-5}$
5	160	$5.9238{\cdot}10^{-5}$	$5.1058{\cdot}10^{-7}$	$5.9180{\cdot}10^{-3}$	$4.3989{\cdot}10^{-5}$
6	20	$2.3337{\cdot}10^{-5}$	$2.5496{\cdot}10^{-6}$	$1.7063{\cdot}10^{-3}$	$1.0966{\cdot}10^{-4}$
6	40	$1.7316{\cdot}10^{-5}$	$1.3086{\cdot}10^{-6}$	$1.5342{\cdot}10^{-3}$	$5.6185{\cdot}10^{-5}$
6	80	$1.5458{\cdot}10^{-5}$	$6.6689{\cdot}10^{-7}$	$1.4922{\cdot}10^{-3}$	$2.8774{\cdot}10^{-5}$
6	160	$1.4975{\cdot}10^{-5}$	$3.3824{\cdot}10^{-7}$	$1.4837{\cdot}10^{-3}$	$1.5162{\cdot}10^{-5}$
7	20	$1.8670{\cdot}10^{-5}$	$2.5505{\cdot}10^{-6}$	$9.7393{\cdot}10^{-4}$	$1.0965{\cdot}10^{-4}$
7	40	$9.9300{\cdot}10^{-6}$	$1.3091{\cdot}10^{-6}$	$5.8362{\cdot}10^{-4}$	$5.6058{\cdot}10^{-5}$
7	80	$5.9230{\cdot}10^{-6}$	$6.6644{\cdot}10^{-7}$	$4.3337{\cdot}10^{-4}$	$2.8421{\cdot}10^{-5}$
7	160	$4.3675{\cdot}10^{-6}$	$3.3601{\cdot}10^{-7}$	$3.8703{\cdot}10^{-4}$	$1.4363{\cdot}10^{-5}$

TABLE – Errors for the manufactured solution with $\theta = \theta_1$.

Numerical results–II

m	N	$\mathbf{e}_u^{H_0^1}$	$\mathbf{e}_u^{L^2}$	$\mathbf{e}_v^{H_0^1}$	$\mathbf{e}_v^{L^2}$
5	20	$1.5468\cdot 10^{-3}$	$2.3046\cdot 10^{-4}$	$8.6103\cdot 10^{-2}$	$1.0423\cdot 10^{-2}$
5	40	$7.9397\cdot 10^{-4}$	$1.1828\cdot 10^{-4}$	$4.3897\cdot 10^{-2}$	$5.2890\cdot 10^{-3}$
5	80	$4.0548\cdot 10^{-4}$	$5.9907\cdot 10^{-5}$	$2.2736\cdot 10^{-2}$	$2.6595\cdot 10^{-3}$
5	160	$2.1132\cdot 10^{-4}$	$3.0172\cdot 10^{-5}$	$1.2661\cdot 10^{-2}$	$1.3342\cdot 10^{-3}$
6	20	$1.5454\cdot 10^{-3}$	$2.3044\cdot 10^{-4}$	$8.5882\cdot 10^{-2}$	$1.0421\cdot 10^{-2}$
6	40	$7.9143\cdot 10^{-4}$	$1.1826\cdot 10^{-4}$	$4.3460\cdot 10^{-2}$	$5.2872\cdot 10^{-3}$
6	80	$4.0060\cdot 10^{-4}$	$5.9882\cdot 10^{-5}$	$2.1880\cdot 10^{-2}$	$2.6575\cdot 10^{-3}$
6	160	$2.0194\cdot 10^{-4}$	$3.0145\cdot 10^{-5}$	$1.1051\cdot 10^{-2}$	$1.3319\cdot 10^{-3}$
7	20	$1.5453\cdot 10^{-3}$	$2.3044\cdot 10^{-4}$	$8.5868\cdot 10^{-2}$	$1.0421\cdot 10^{-2}$
7	40	$7.9127\cdot 10^{-4}$	$1.1826\cdot 10^{-4}$	$4.3433\cdot 10^{-2}$	$5.2871\cdot 10^{-3}$
7	80	$4.0029\cdot 10^{-4}$	$5.9880\cdot 10^{-5}$	$2.1825\cdot 10^{-2}$	$2.6574\cdot 10^{-3}$
7	160	$2.0133\cdot 10^{-4}$	$3.0144\cdot 10^{-5}$	$1.0942\cdot 10^{-2}$	$1.3318\cdot 10^{-3}$

TABLE – Errors for the manufactured solution with $\theta = \theta_2$.

Numerical results–III

		$\theta = \theta_1$		$\theta = \theta_2$	
m	N	$\mathbf{e}_u^{H_0^1}$	rate	$\mathbf{e}_u^{H_0^1}$	rate
4	10	$2.2859 \cdot 10^{-4}$	–	$2.9561 \cdot 10^{-3}$	–
5	40	$5.9344 \cdot 10^{-5}$	1.946	$7.9397 \cdot 10^{-4}$	1.897
6	160	$1.4975 \cdot 10^{-5}$	1.987	$2.0194 \cdot 10^{-4}$	1.975

m	N	$\mathbf{e}_v^{H_0^1}$	rate	$\mathbf{e}_v^{H_0^1}$	rate
4	10	$2.2632 \cdot 10^{-2}$	–	$1.6862 \cdot 10^{-1}$	–
5	40	$5.8790 \cdot 10^{-3}$	1.945	$4.3897 \cdot 10^{-2}$	1.942
6	160	$1.4837 \cdot 10^{-3}$	1.986	$1.1051 \cdot 10^{-2}$	1.990

m	N	$e_p^{L^2}$	rate	$e_p^{L^2}$	rate
4	10	$9.0339 \cdot 10^{-3}$	–	$1.3004 \cdot 10^{-1}$	–
5	40	$2.3427 \cdot 10^{-3}$	1.946	$3.3711 \cdot 10^{-2}$	1.948
6	160	$5.9101 \cdot 10^{-4}$	1.987	$8.4744 \cdot 10^{-3}$	1.992

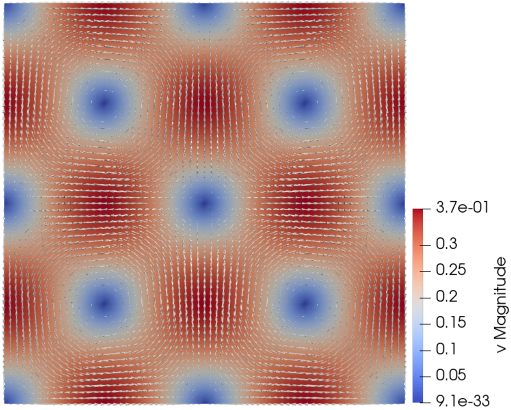
TABLE – Errors and convergence rate when $\Delta t = 12.8h^2$.

m	3	4	5	6
$\mathbf{e}_u^{H_0^1}$	$9.5525{\cdot}10^{-4}$	$2.3769{\cdot}10^{-4}$	$5.9353{\cdot}10^{-5}$	$1.4838{\cdot}10^{-5}$
rate	–	2.00681	2.00168	2.00002
$\mathbf{e}_u^{L^2}$	$2.8425{\cdot}10^{-5}$	$3.5025{\cdot}10^{-6}$	$4.3565{\cdot}10^{-7}$	$7.2611{\cdot}10^{-8}$
rate	–	3.02067	3.00718	2.58490
$\mathbf{e}_v^{H_0^1}$	$9.4809{\cdot}10^{-2}$	$2.3725{\cdot}10^{-2}$	$5.9325{\cdot}10^{-3}$	$1.4833{\cdot}10^{-3}$
rate	–	1.99859	1.99971	1.99983
$\mathbf{e}_v^{L^2}$	$2.7696{\cdot}10^{-3}$	$3.4737{\cdot}10^{-4}$	$4.3459{\cdot}10^{-5}$	$5.7305{\cdot}10^{-6}$
rate	–	2.99514	2.99873	2.92292
$e_p^{L^2}$	$3.7580{\cdot}10^{-2}$	$9.3790{\cdot}10^{-3}$	$2.3439{\cdot}10^{-3}$	$5.8608{\cdot}10^{-4}$
rate	–	2.00247	2.00054	1.99971

TABLE – Errors and convergence rate for $\theta = \theta_1$ when $\Delta t = 0.001$.

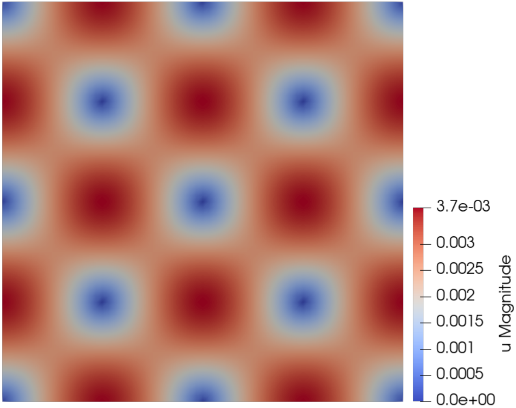
Fluid velocity, Plot

Fluid velocity



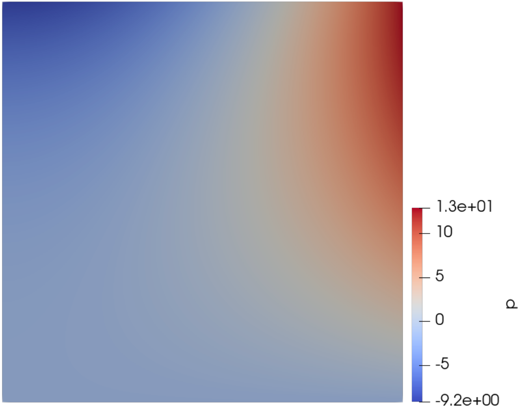
Displacement, Plot

Norm of solid displacement



Pressure, Plot

Pressure



Thanks

Happy Birthday Francisco and Manuel