Estimate of the Stokes system at a boundary

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$$\begin{cases} \partial_t U - \Delta U + \nabla q = F \\ \operatorname{div} U = h \end{cases}$$ on $\mathcal{M}$ smooth open set of $\mathbb{R}^d$

+ initial conditions

+ Boundary conditions. general precise conditions given below (Lopatinskiĭ-Šapiro type)

typical exemples: . Dirichlet $U|_\partial = G$

. Navier: $\nu \cdot U = G_\nu \quad \left((\nabla U + {}^t\nabla U)\cdot \nu \right)_{\tan} = G_{\tan}$

. Neumann $\left({}^t\nabla U + n \nabla U \right)\nu - q\nu = G$

Observability

$T > 0 \qquad \omega \subset M$ open subset

$$\int_0^T \left(\|u\|^2 + \|q\|^2 \right) dt \lesssim \int_0^T \left(\|u\|^2_{L^2(\omega)} + \|q\|^2_{L^2(\omega)} \right) dt$$

$$+ \int_0^T \left(\|F\|^2 + \|h\|^2 + |G|^2 \right) dt$$

G: boundary value.

Ref : homogeneous Dirichlet + flat Laplace

- Fernandez-Cara, Guerrero, Imanuvilov, Puel 04
- Buffe, Takahashi 25

Application observability / null controllability of Oseen system

NS (local result)

- Navier : Guerrero 2006

Fernandez-Cara, Gonzalez-Burgos, Imanuvilov, Puel (2006–2008)

<u>1st strategy</u> $\partial_t U - \Delta U = F - \nabla q$ + homog. Dirichlet

heat-type Carleman estimate for U.

Apply divergence and use $\operatorname{div} U = 0$

$\Delta q = \operatorname{div} F$ No boundary condition.

Multiplier + microlocal argument $\longrightarrow$ estimation of $q|_{\partial_y}$

Elliptic Carleman estimate for q + time integration

$\longrightarrow$ combine estimations.

Carleman estimates

Estimations of the form

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \leq C\,\|e^{\tau\varphi}Pu\|$$

- P order 2, elliptic
- $\varphi = \varphi(x)$ weight function.
- $u \in C_c^\infty$, $\tau > 0$ large

- if u defined at a boundary

boundary condition (order k)

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \leq C\left(\|e^{\tau\varphi}Pu\|_{L^2} + |e^{\tau\varphi}Bu|_{\frac{3}{2}-k}\right)$$

$$+\ \tau^{3/2}|e^{\tau\varphi}u|_\partial|_{L^2} + \tau^{1/2}|e^{\tau\varphi}\nabla u|_\partial|_{L^2}$$

+ Observation term

Carleman estimates   Parabolic version

$$\| \tilde{\tau}^{3/2} e^{\tau\eta\phi} u \|_{L^2} + \| \tilde{\tau}^{1/2} e^{\tau\eta\phi} \nabla u \|_{L^2} \leq C \, \| e^{\tau\eta\phi} P u \|_{L^2}$$

- $P = \partial_t - \Delta$. $u \in C_c^\infty$, $\tau > 0$ large

- $\varphi = \varphi(x)$ weight function. $\eta = \dfrac{1}{t\,(T-t)}$

- $\phi = \varphi - K < 0$

- $\tilde{\tau} = \tau \eta \varphi$

Carleman estimates

Elliptic estimate

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \lesssim C\,\|e^{\tau\varphi}Pu\|_{L^2}$$

$$P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi}.$$

$$\|P_\varphi v\| \gtrsim \tau^{3/2}\|v\| + \tau^{\frac12}\|\nabla v\|$$

Lemma if $|d\psi| \geq c > 0$ and $\varphi = e^{\gamma\psi}$ then subelliptic property

Dependence on γ (Carleman second large parameter)

$$\frac{\gamma\mu}{\tilde{\tau}}(q_2^2+q_1^2) + \{q_2, q_1\} \gtrsim \frac{\gamma}{\tilde{\tau}}\left(\tilde{\tau}^4 + |\xi|^4\right)$$

$$\|P_\varphi v\| \gtrsim \gamma^{\frac12}\left(\|\tilde{\tau}^{3/2}v\| + \|\tilde{\tau}^{\frac12}\nabla v\| + \|\tilde{\tau}^{-1/2}D^2 v\|\right)$$

Back to Stokes — Away from the boundary

$$\partial_t U - \Delta U + \nabla q = F \qquad \operatorname{div}_g U = h$$

$$\gamma^{\frac{1}{2}} \left(\| \tilde{\tau}^{\frac{1}{2}} e^{\tau\eta\phi} q \|_{t,x} + \| \tilde{\tau}^{-\frac{1}{2}} e^{\tau\eta\phi} \nabla q \|_{t,x} \right)$$
$$+ \gamma \left(\| \tilde{\tau} e^{\tau\eta\phi} U \|_{t,x} + \| e^{\tau\eta\phi} \nabla U \|_{t,x} \right)$$
$$\lesssim \| e^{\tau\eta\phi} F \|_{t,x} + \| \tilde{\tau}^{-1} e^{\tau\eta\phi} (\partial_t - \Delta) h \|_{t,x}$$

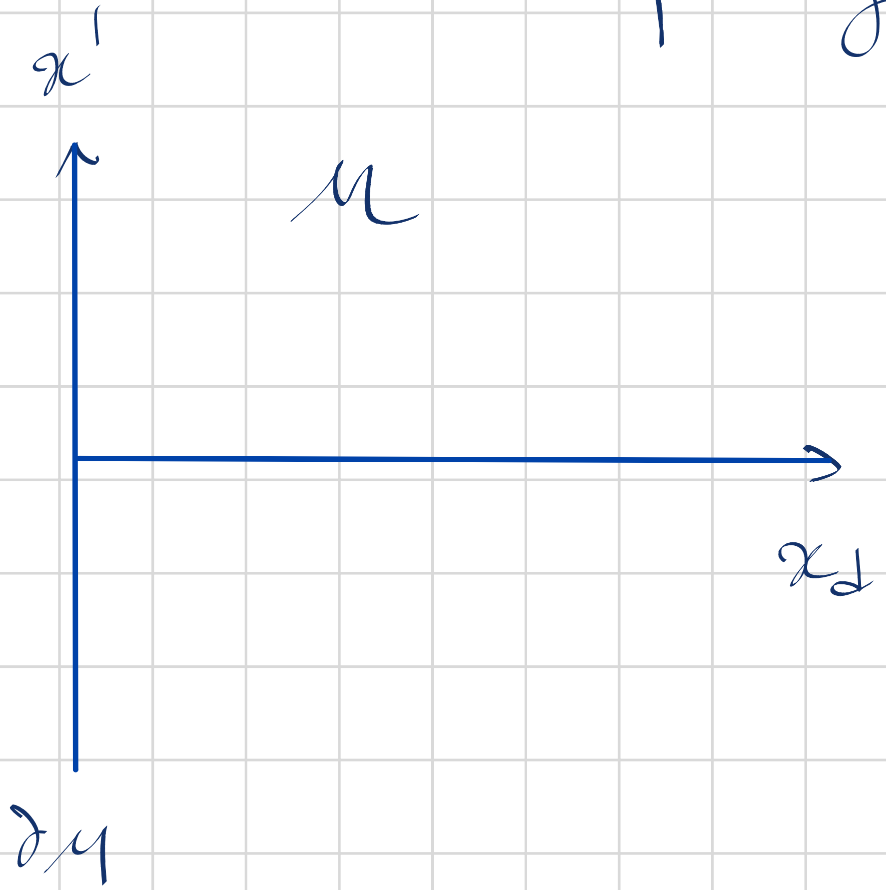
- pressure q : $\frac{1}{2}$ derivative loss
- velocity U full derivative loss

Near a boundary

Recall

$$P = -\Delta_g \qquad P_\varphi = e^{\tau\eta\phi} P e^{-\tau\eta\phi} = Q_2 + iQ_1$$

sub-ellipticity $\quad q_2 = q_1 = 0 \Longrightarrow \{q_2, q_1\}$.



$$P = D_d^2 + R(x, D') \qquad D = \frac{1}{i}\partial$$

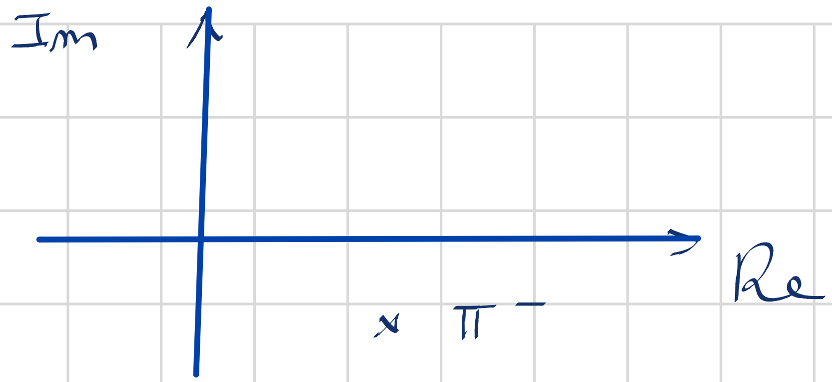
$$P_\varphi = \left(D_d + i\tilde{\tau}\partial_d\varphi\right)^2 + R_\varphi(x, \tau, D')$$

$$p_\varphi = \left(\xi_d + i\tilde{\tau}\partial_d\varphi\right)^2 + r_\varphi(x, \tau, \xi')$$
$$= (\xi_d - \pi^+)(\xi_d - \pi^-)$$

$$r_\varphi(x, \tau, \xi') = r(x, \xi' + i\tilde{\tau} d\varphi') = \beta^2 \quad \text{with } \mathrm{Re}\,\beta \geq 0$$

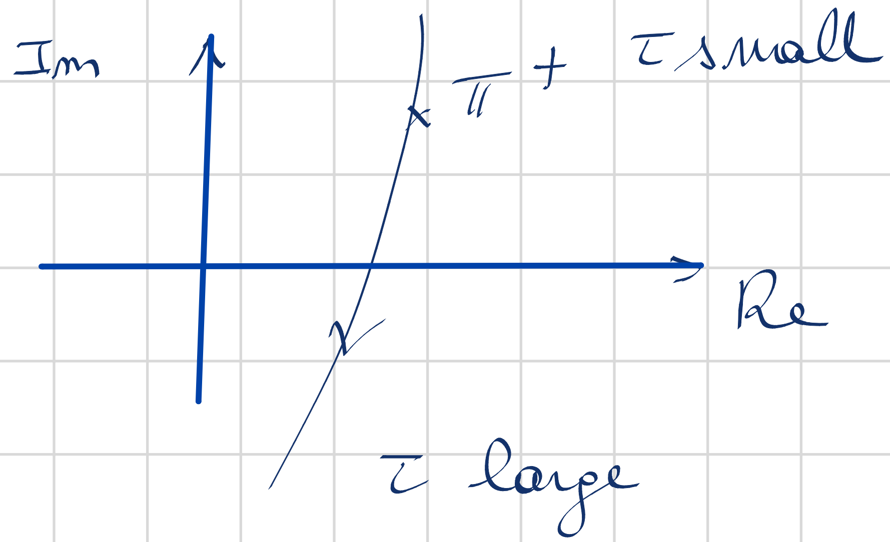
$$\pi^+ = i\beta - i\tilde{\tau}\partial_d\varphi \qquad \pi^- = -i\beta - i\tilde{\tau}\partial_d\varphi$$

- $D_d - Op(\pi^-) \longrightarrow$ elliptic estimate



$$\|\tilde{\tau} v\|_{L^2} + \|\nabla v\|_{L^2} + |Op(\lambda_T^{\frac{1}{2}}) v_{|\partial}|_{L^2} \lesssim \|(D_d - Op(\pi^-)) v\|_{L^2}$$

$$\lambda_T^2 = \tilde{\tau}^2 + |\xi'|^2 .$$



- $D_d - Op(\pi^+) \longrightarrow$ sub-elliptic estimate

$$\gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{\frac{1}{2}} v\|_{L^2} + \|\tilde{\tau}^{-\frac{1}{2}} \nabla v\|_{L^2}\right) \lesssim \|(D_d - Op(\pi^+)) v\|_{L^2} + |Op(\lambda_T^{\frac{1}{2}}) v_{|\partial}|_{L^2}$$

Parabolic operator

Similar analysis $P = \partial_t - \Delta_g$. Change of time variable

$$s = \tan\left(\frac{\pi t}{T} - \frac{\pi}{2}\right) \qquad \partial_t = \frac{\pi \langle s\rangle^2}{T}\partial_s$$

$$\eta = \frac{T^2}{t(T-t)} = \pi^2\left(\frac{\pi}{2} + \arctan s\right)^{-1}\left(\frac{\pi}{2} - \arctan s\right)^{-1} \approx \langle s\rangle$$

$$p = \frac{\pi\langle s\rangle^2}{T} i\sigma - r(x,\xi)$$

Conjugation $\quad P_\varphi = e^{\tau\eta\phi} P e^{-\tau\eta\phi} = (D_d + i\tilde{\tau}\partial_d\psi)^2 + M_\varphi$

$$p_\varphi = (\xi_d + i\tilde{\tau}\partial_d\psi)^2 + m_\varphi \;; \qquad m_\varphi = \frac{\pi\langle s\rangle^2}{T}(i\sigma - \tau\eta'\phi) + r_\varphi(x,\tau,\xi)$$

$$= (\xi_d - \pi^+)(\xi_d - \pi^-) \qquad = \alpha^2, \text{ with } \operatorname{Re}\alpha \geq 0$$

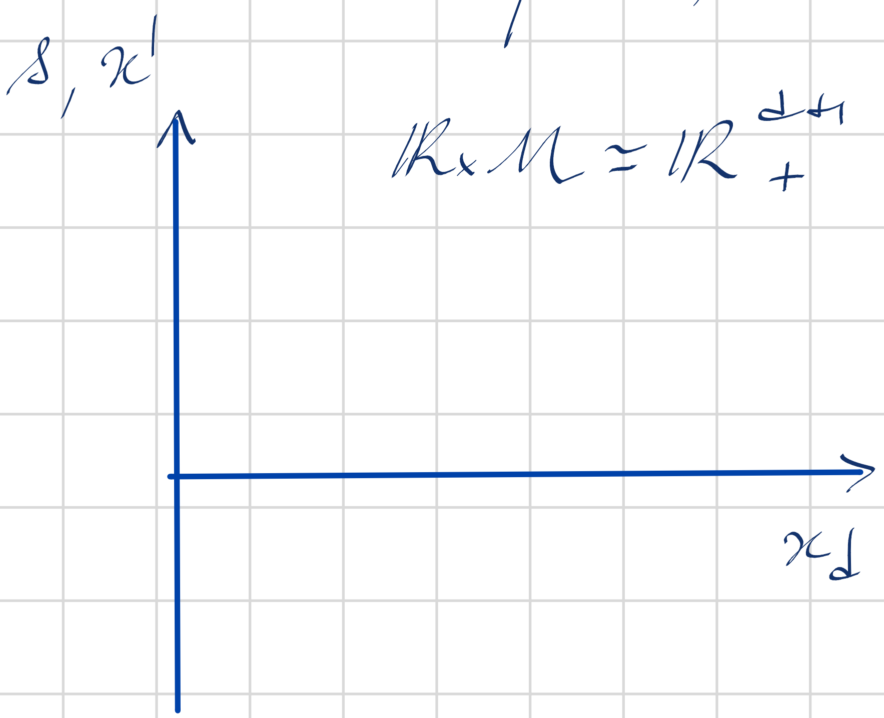
$$\pi^{\pm} = i\alpha - i\tilde{\tau}\partial_d\psi \qquad \pi^- = -i\alpha - i\tilde{\tau}\partial_d\psi$$

Back to Stokes (part 2)

$$(\partial_t - \Delta)U + \nabla q = F \longrightarrow \left(\frac{\pi\langle s\rangle^2}{T}\partial_s - \Delta\right)U + \nabla q = F$$

Conjugaison $e^{\tau\eta^\phi}$ as in the parabolic case

$$U^\phi = Op(\chi)\, e^{\tau\eta^\phi} U \qquad q^\phi = i\, Op(\chi)\, e^{\tau\eta^\phi} q$$



$\Theta = Op_T(\vartheta)$ order 1

$\tilde V = \Theta U^\phi \in \mathbb{C}^d$ $\qquad D_d^\phi = D_d + i\tilde\tau\,\partial_d\psi$

$\hat V' = D_d^\phi U^{\phi\prime}$ $\qquad \hat V \in \mathbb{C}^d$

$\hat V_d = q^\phi$

$$V = \begin{pmatrix}\tilde V\\ \hat V\end{pmatrix} \in \mathbb{C}^{2d}$$

$$\boxed{D_d^\phi V = AV + G}$$

$$D_d^\phi V = AV + G \qquad \text{with } A = \begin{pmatrix} 0 & 0 & \Theta & 0 \\ i\,\mathrm{div}^{\phi\prime} & 0 & 0 & 0 \\ -M_\phi \Theta^{-1} & 0 & 0 & i\nabla^{\phi\prime} \\ 0 & -M_\phi \Theta^{-1} & -i\,\mathrm{div}^{\phi\prime} & 0 \end{pmatrix}$$

$$M_\phi = e^{\tau\eta\phi}\left(\frac{\Pi}{T}\langle s\rangle^2 \partial_s\right)e^{-\tau\eta\phi} + R_\phi \quad \text{with} \quad R_\phi = e^{\tau\eta\phi} R(x, D')\, e^{-\tau\eta\phi}$$

$$a = \begin{pmatrix} 0 & 0 & \Theta & 0 \\ -{}^t\zeta' & 0 & 0 & 0 \\ -\Theta^{-1} m_\phi & 0 & 0 & -R\zeta' \\ 0 & -\Theta^{-1} m_\phi & {}^t\zeta' & 0 \end{pmatrix} \quad \text{where } {}^t\zeta' R\, \zeta' = R(x, \zeta')$$

$$\text{and } \zeta' = \xi' + i\tilde{\tau}\, d\varphi'$$

One seeks the eigenvalues of ${}^t a$ and its eigenvectors

4 regions

1. $m_\phi \neq 0$, $r_\phi \neq 0$, and $m_\phi \neq r_\phi$
2. $m_\phi \neq 0$, $r_\phi \sim 0$
3. $m_\phi \sim 0$
4. $m_\phi \neq 0$, $r_\phi \neq 0$, and $m_\phi \sim r_\phi$

Region 1

$m_\phi \neq 0 \,,\; r_\phi \neq 0 \,,\; m_\phi \neq r_\phi$

$\alpha^2 = m_\phi \qquad \mathrm{Re}\,\alpha \geq 0$

$\beta^2 = r_\phi \qquad \mathrm{Re}\,\beta \geq 0$

- $$W_\vartheta^\pm = \begin{pmatrix} \pm\mu \;\; \theta^{-1}\left(({}^t\zeta' R\vartheta)\,\zeta' - m_\phi \vartheta\right) \\ -\theta^{-1}\, {}^t\zeta' R \vartheta \\ \vartheta \\ \mp \mu^{-1}\, {}^t\zeta' R \vartheta \end{pmatrix}$$

$\vartheta \in \mathbb{C}^{d-1} \,,\quad \mu = i\alpha$

$$\boxed{{}^t a\, W_\vartheta^\pm = \pm\mu\, W_\vartheta^\pm}$$

- $$W_\nu^\pm = \begin{pmatrix} 0 \\ -\theta^{-2} m_\phi \\ \theta^{-1}\zeta' \\ \pm\theta^{-1}\nu \end{pmatrix}$$

$\nu = i\beta \,,$

$$\boxed{{}^t a\, W_\nu^\pm = \pm\nu\, W_\nu^\pm}$$

$$Z = Op({}^t W_\vartheta^{\pm})\, V \qquad D_d^\phi V = AV + G$$

$$Op({}^t W_\vartheta^{\pm})\, D_d^\phi V = Op({}^t W_\vartheta^{\pm})\, AV + Op({}^t W_\vartheta^{\pm})\, G$$

commutators calculus

$$D_d^\phi Z = \pm Op(\mu)\, Z + RHS + \text{remainders}$$

$$\boxed{(D_d - Op(\pi^{\pm}))\, Z = RHS + \text{remainders}}$$

with $\pi^{\pm} = \pm \mu - i\tilde{\tau}\partial_d \varphi = \pm i\alpha - i\tilde{\tau}\partial_d\varphi$

- $\mathrm{Im}\,\pi^- < 0$ Elliptic estimate for $Z = Op(W_\vartheta^-)\, V$

$$\|Z\|_{\lambda,1} + |Z|_\partial|_{\lambda,\frac{1}{2}} \lesssim \|(D_d - Op(\pi^-))Z\|$$

- $\mathrm{Im}\,\pi^+$ change sign, subelliptic estimate for $Z = Op(W_\vartheta^+)V$

$$\gamma^{\frac{1}{2}} \|\tilde{\tau}^{-\frac{1}{2}} Z\|_{\lambda,1} \lesssim \|(D_d - Op(\pi^+))\, Z\| + |Z|_\partial|_{\lambda,\frac{1}{2}}$$

Similarly

- Elliptic estimate for $Z = Op(W_\nu^-)V$

$$\|Z\|_{\lambda,1} + \left|\,|Z|_\partial\,\right|_{\lambda,\frac{1}{2}} \lesssim \|(D_d - Op(\pi^-))Z\|$$

$$\pi^- = -i\nu - i\tilde{\tau}\partial_d\varphi$$

- Subelliptic estimate for $Z = Op(W_\nu^+)V$

$$\gamma^{\frac{1}{2}}\|\tilde{\sigma}^{-\frac{1}{2}}Z\|_{\lambda,1} \lesssim \|(D_d - Op(\pi^+))Z\| + \left|\,|Z|_\partial\,\right|_{\lambda,\frac{1}{2}}$$

$$\pi^+ = +i\nu - i\tilde{\tau}\partial_d\varphi$$

Lopatinskii ellipticity condition

$$\mathrm{Span}\,\{B_\alpha\} \cup \{W_\vartheta^-,\ \vartheta \in \mathbb{C}^{d-1}\} \cup \{W_\nu^-\} = \mathbb{C}^{2d}$$

Dirichlet ✓ Navier ✓ Neumann ✓

Estimate

$$|V''|_\partial|_{\lambda,\frac{1}{2}} + |\hat{V}_d|_\partial|_{\tilde{\lambda},\frac{1}{2}} \lesssim \sum_\vartheta |\mathrm{Op}({}^tW_\vartheta^-)\,V|_\partial|_{\lambda,\frac{1}{2}} + |\mathrm{Op}({}^tW_\nu^-)V|_\partial|_{\lambda,\frac{1}{2}}$$

$$+ \sum_\alpha |B_\alpha V|_\partial|_{\lambda,\frac{1}{2}}$$

Result

$$\gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{-1/2}V''\|_{\lambda,1} + \|\tilde{\tau}^{-1/2}\hat{V}_d\|_{\tilde{\lambda},1}\right) + |V''|_\partial|_{\lambda,\frac{1}{2}} + |\hat{V}_d|_\partial|_{\tilde{\lambda},\frac{1}{2}}$$

$$\lesssim \sum_\alpha |B_\alpha V|_\partial|_{\lambda,\frac{1}{2}} + \|RHS\|$$

Region 2 similar to region 1

Region 3 peculiar, $d-2$ Jordan blocks of size 2
1 Jordan block of size 4

All roots have negative imaginary parts

$\longrightarrow$ Elliptic estimation, no boundary condition needed.

Region 4 $m_\phi \neq 0$, $r_\phi \neq 0$, $m_\phi \sim r_\phi$ $\mu \sim \nu$

2 Jordan blocks of size 2

$W_\nu^\pm$ as in region 1 $\longrightarrow$ $2d-2$ eigenvectors

$W_\nu^\pm$ coincides with $W_\vartheta^\pm$ for $\nu = \Theta^{-1}\zeta$ if $\mu = \nu$

Set $W_\mu^\pm = \begin{pmatrix} 2\theta^{-1}\xi' \\ \mp\theta^{-1}\mu \\ \pm\mu^{-1}\xi' \\ 0 \end{pmatrix}$ $\quad t_a W_\mu^\pm = \pm\mu W_\mu^\pm + \theta W_\nu^\pm + \mu^{-1}\nu(\nu-\mu)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

Estimate: with loss of $\frac{1}{2}$ a derivative for W_θ^+, W_ν^+

- zero loss for W_θ^-, W_ν^-
- loss of a full derivative for $W_\mu^\pm$

$\longrightarrow$ loss of a full derivative for V'', that is, the velocity U.

important observation $W_\nu^+ - W_\nu^- = \begin{pmatrix}0\\0\\0\\2i\theta^{-1}\beta\end{pmatrix}$

$\longrightarrow$ loss of $\frac{1}{2}$ a derivative for $\hat{V}_4$, that is, the pressure.

Final estimate at the boundary

Theorem

$$\gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{\frac{1}{2}}e^{\tau\eta\phi}q\|_{t,x}+\|\tilde{\tau}^{-\frac{1}{2}}e^{\tau\eta\phi}\nabla q\|_{t,x}\right)$$

$$+\gamma\left(\|\tilde{\tau}e^{\tau\eta\phi}U\|_{t,x}+\|e^{\tau\eta\phi}\nabla U\|_{t,x}\right)$$

$$+\quad\left|\tilde{\tau}^{\frac{3}{2}}e^{\tau\eta\phi}U|_{\partial M}\right|_{t,x}+\left|\tilde{\tau}^{\frac{1}{2}}e^{\tau\eta\phi}\nabla U|_{\partial M}\right|_{t,x}+\left|\tilde{\tau}^{\frac{1}{2}}e^{\tau\eta\phi}q|_{\partial M}\right|_{t,x}$$

$$\lesssim\quad\|e^{\tau\eta\phi}F\|_{t,x}+\|\tilde{\tau}^{-1}e^{\tau\eta\phi}h\|_{\partial,t}\quad+\sum_j\left|B_j u|_{\partial M}\right|_{t,x}$$

Thank you for you attention