Optimal Control Problems Related to Chemo-repulsion Systems

Exequiel Mallea-Zepeda
Universidad Católica del Norte, Antofagasta, Chile

Collaborators:
F. Guillén-González, Universidad de Sevilla, Spain
M.A. Rodríguez-Bellido, Universidad de Sevilla, Spain
E.J. Villamizar-Roa, Universidad Industrial de Santander, Colombia

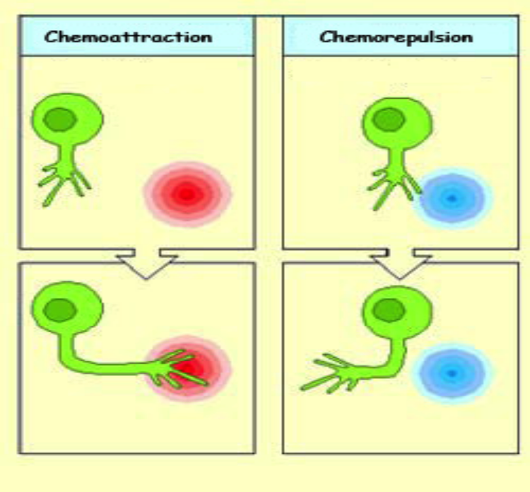
Workshop on PDEs and Control 2025 (PKM-60)
September 3-5, 2025

Universidad de Sevilla, Sevilla, Spain

Outline

1. Chemotaxis Systems:
    - Chemo-attraction-production
    - Chemo-repulsion-production
2. Optimal Control Problems
    - 2D case: linear and nonlinear production
    - 3D case: linear production
3. Other Cases

Chemotaxis



Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\begin{cases} \partial_t u - \Delta u = \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ \partial_t v - \Delta v + v = h(u) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ u(0, x) = u_0(x) \geq 0, \ v(0, x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, \ \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{cases} \tag{1}$$

- $u := u(t, x) \geq 0$ is cell density and $v := v(t, x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    Chemo − attraction
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    Chemo − repulsion

Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u &=& \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0,T) \times \Omega, \\ \partial_t v - \Delta v + v &=& h(u) \quad \text{in} \quad Q := (0,T) \times \Omega, \\ u(0,x) &=& u_0(x) \geq 0, \; v(0,x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &=& 0, \; \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega, \end{array} \right. \tag{1}$$

- $u := u(t,x) \geq 0$ is cell density and $v := v(t,x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    Chemo – attraction
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    Chemo – repulsion

Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\begin{cases} \partial_t u - \Delta u = \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ \partial_t v - \Delta v + v = h(u) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ u(0, x) = u_0(x) \geq 0, \ v(0, x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, \ \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{cases} \tag{1}$$

- $u := u(t, x) \geq 0$ is cell density and $v := v(t, x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    Chemo − attraction
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    Chemo − repulsion

Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u & = & \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ \partial_t v - \Delta v + v & = & h(u) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ u(0, x) & = & u_0(x) \geq 0, \;\; v(0, x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} & = & 0, \;\; \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{array} \right. \tag{1}$$

- $u := u(t, x) \geq 0$ is cell density and $v := v(t, x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    Chemo − attraction
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    Chemo − repulsion

Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\begin{cases} \partial_t u - \Delta u = \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ \partial_t v - \Delta v + v = h(u) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ u(0, x) = u_0(x) \geq 0, \; v(0, x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, \; \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{cases} \tag{1}$$

- $u := u(t, x) \geq 0$ is cell density and $v := v(t, x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    $\boxed{Chemo-attraction}$
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    $\boxed{Chemo-repulsion}$

Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\left\{\begin{array}{rcl} \partial_t u - \Delta u &=& \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ \partial_t v - \Delta v + v &=& h(u) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ u(0, x) &=& u_0(x) \geq 0, \;\; v(0, x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &=& 0, \;\; \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{array}\right. \tag{1}$$

- $u := u(t, x) \geq 0$ is cell density and $v := v(t, x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    $\boxed{Chemo - attraction}$
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    $\boxed{Chemo - repulsion}$

Chemotaxis model [E.F. Keller, L.A. Segel, 1971]

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain, with smooth boundary $\partial\Omega$ and $(0, T)$ a time interval.

$$\begin{cases} \partial_t u - \Delta u = \alpha \nabla \cdot (u \nabla v) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ \partial_t v - \Delta v + v = h(u) \quad \text{in} \quad Q := (0, T) \times \Omega, \\ u(0, x) = u_0(x) \geq 0, \; v(0, x) = v_0(x) \geq 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0, \; \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \end{cases} \tag{1}$$

- $u := u(t, x) \geq 0$ is cell density and $v := v(t, x) \geq 0$ is a chemical concentration.
- $h(u) \geq 0$ represents the production term.
- The term $\alpha u \nabla v$ models the transport of cells:
  - If $\alpha < 0$, towards the higher concentrations of chemical signal

    Chemo – attraction
  - If $\alpha > 0$, towards the lower concentrations of chemical signal

    Chemo – repulsion

Chemo-repulsion with potential production ($h(u) = u^p$)

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v), \\ \partial_t v - \Delta v + v = u^p, \quad p \in [1,2]. \end{cases}$$

- $p = 1$: Linear production [Cieślak, Laurençot, Morales-Rodrigo, 2008]
  - 2D: Existence and uniqueness of classical solutions
  - 3D: Existence of weak solutions
- $1 < p < 2$: Superlinear production [Guillén-González, Rodríguez-Bellido, Rueda-Gómez, 2020]
  - Existence of weak solutions in 2D and 3D domains
- $p = 2$: Quadratic production [Guillén-González, Rodríguez-Bellido, Rueda-Gómez, 2020]
  - 1D and 2D: Existence and uniqueness of weak solutions
  - 3D: Existence of weak solutions

Chemo-repulsion with potential production ($h(u) = u^p$)

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u & = & \nabla \cdot (u \nabla v), \\ \partial_t v - \Delta v + v & = & u^p, \quad p \in [1, 2]. \end{array} \right.$$

- $p = 1$: Linear production [Cieślak, Laurençot, Morales-Rodrigo, 2008]
  - 2D: Existence and uniqueness of classical solutions
  - 3D: Existence of weak solutions
- $1 < p < 2$: Superlinear production [Guillén-González, Rodríguez-Bellido, Rueda-Gómez, 2020]
  - Existence of weak solutions in 2D and 3D domains
- $p = 2$: Quadratic production [Guillén-González, Rodríguez-Bellido, Rueda-Gómez, 2020]
  - 1D and 2D: Existence and uniqueness of weak solutions
  - 3D: Existence of weak solutions

Chemo-repulsion with potential production ($h(u) = u^p$)

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u & = & \nabla \cdot (u \nabla v), \\ \partial_t v - \Delta v + v & = & u^p, \quad p \in [1,2]. \end{array} \right.$$

- $p = 1$: Linear production [Cieślak, Laurençot, Morales-Rodrigo, 2008]
  - 2D: Existence and uniqueness of classical solutions
  - 3D: Existence of weak solutions
- $1 < p < 2$: Superlinear production [Guillén-González, Rodríguez-Bellido, Rueda-Gómez, 2020]
  - Existence of weak solutions in 2D and 3D domains
- $p = 2$: Quadratic production [Guillén-González, Rodríguez-Bellido, Rueda-Gómez, 2020]
  - 1D and 2D: Existence and uniqueness of weak solutions
  - 3D: Existence of weak solutions

Injection/extraction

$$\left\{\begin{array}{rcl} \partial_t u - \Delta u & = & \nabla \cdot (u \nabla v) \quad \text{in} \quad Q, \\ \partial_t v - \Delta v + v & = & u^p + fv\, 1_{\Omega_c} \quad \text{in} \quad Q, \\ u(0,x) & = & u_0(x), \quad v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} & = & \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \end{array}\right. \tag{2}$$

$\Omega_c \subset \Omega$ is a *subdomain of control*

Some properties of system (2):

- $\dfrac{d}{dt}\left(\int_\Omega u\right) = 0 \Rightarrow \int_\Omega u(t) = \int_\Omega u_0 := m_0, \ \forall t > 0.$
- $\dfrac{d}{dt}\left(\int_\Omega v\right) + \int_\Omega v = \int_\Omega u^p + \int_{\Omega_c} fv.$ $\qquad \boxed{\int_\Omega u^p = m_0, \quad \text{for } p = 1}$

Injection/extraction

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v) \quad \text{in} \quad Q, \\ \partial_t v - \Delta v + v = u^p + f v \, 1_{\Omega_c} \quad \text{in} \quad Q, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \end{cases} \tag{2}$$

$\Omega_c \subset \Omega$ is a *subdomain of control*

Some properties of system (2):

- $\dfrac{d}{dt}\left(\int_\Omega u\right) = 0 \Rightarrow \int_\Omega u(t) = \int_\Omega u_0 := m_0, \ \forall t > 0.$
- $\dfrac{d}{dt}\left(\int_\Omega v\right) + \int_\Omega v = \int_\Omega u^p + \int_{\Omega_c} fv.$ $\quad \boxed{\int_\Omega u^p = m_0, \quad \text{for } p = 1}$

Injection/extraction

$$\begin{cases} \partial_t u - \Delta u &= \nabla\cdot(u\nabla v) \quad \text{in} \quad Q, \\ \partial_t v - \Delta v + v &= u^p + fv\, 1_{\Omega_c} \quad \text{in} \quad Q, \\ u(0,x) &= u_0(x), \quad v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega. \end{cases} \tag{2}$$

$\Omega_c \subset \Omega$ is a *subdomain of control*

Some properties of system (2):

- $\dfrac{d}{dt}\left(\int_\Omega u\right) = 0 \Rightarrow \int_\Omega u(t) = \int_\Omega u_0 := m_0, \ \forall t > 0.$
- $\dfrac{d}{dt}\left(\int_\Omega v\right) + \int_\Omega v = \int_\Omega u^p + \int_{\Omega_c} fv.$ $\quad \int_\Omega u^p = m_0, \quad \text{for } p = 1$

Injection/extraction

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v) \quad \text{in} \quad Q, \\ \partial_t v - \Delta v + v &= u^p + fv\, 1_{\Omega_c} \quad \text{in} \quad Q, \\ u(0,x) &= u_0(x), \quad v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \end{cases} \tag{2}$$

$\Omega_c \subset \Omega$ is a *subdomain of control*

Some properties of system (2):

- $\dfrac{d}{dt}\left(\int_\Omega u\right) = 0 \ \Rightarrow \ \int_\Omega u(t) = \int_\Omega u_0 := m_0, \ \forall t > 0.$
- $\dfrac{d}{dt}\left(\int_\Omega v\right) + \int_\Omega v = \int_\Omega u^p + \int_{\Omega_c} fv.$ $\quad \boxed{\int_\Omega u^p = m_0, \quad \text{for } p = 1}$

Existence of Strong Solutions in 2D Domains

[Guillén-González, M-Z, Rodríguez-Bellido, ESAIM COCV 2020 and SICON 2020]
[Guillén-González, M-Z, Villamizar-Roa, Acta Appl. Math. 2020]

$$X_q := \{u \in C([0,T]; W^{2-2/q,q}) \cap L^q(W^{2,q}) \, : \, \partial_t u \in L^q(Q) := L^q(L^q)\}$$

Hypothesis

- $f \in L^s(Q_c) := L^s(L^s(\Omega_c)), \quad s > 2$
- $(u_0, v_0) \in H^1(\Omega) \times W^{2-2/s,s}(\Omega)$, with $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a unique strong solution $(u, v) \in X_2 \times X_s$ such that

$$\|(u,v)\|_{X_2 \times X_s} \leq C(T, \|u_0\|_{H^1}, \|v_0\|_{W^{2-2/s,s}}, \|f\|_{L^s(Q_c)})$$

Existence of Strong Solutions in 2D Domains

[Guillén-González, M-Z, Rodríguez-Bellido, ESAIM COCV 2020 and SICON 2020]
[Guillén-González, M-Z, Villamizar-Roa, Acta Appl. Math. 2020]

$$X_q := \{u \in C([0,T]; W^{2-2/q,q}) \cap L^q(W^{2,q}) \,:\, \partial_t u \in L^q(Q) := L^q(L^q)\}$$

Hypothesis

- $f \in L^s(Q_c) := L^s(L^s(\Omega_c)), \quad s > 2$
- $(u_0, v_0) \in H^1(\Omega) \times W^{2-2/s,s}(\Omega)$, with $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a unique strong solution $(u,v) \in X_2 \times X_s$ such that

$$\|(u,v)\|_{X_2 \times X_s} \leq C(T, \|u_0\|_{H^1}, \|v_0\|_{W^{2-2/s,s}}, \|f\|_{L^s(Q_c)})$$

Existence of Strong Solutions in 2D Domains

[Guillén-González, M-Z, Rodríguez-Bellido, ESAIM COCV 2020 and SICON 2020]
[Guillén-González, M-Z, Villamizar-Roa, Acta Appl. Math. 2020]

$$X_q := \{u \in C([0,T]; W^{2-2/q,q}) \cap L^q(W^{2,q}) \, : \, \partial_t u \in L^q(Q) := L^q(L^q)\}$$

Hypothesis

- $f \in L^s(Q_c) := L^s(L^s(\Omega_c))$, $s > 2$
- $(u_0, v_0) \in H^1(\Omega) \times W^{2-2/s,s}(\Omega)$, with $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a unique strong solution $(u, v) \in X_2 \times X_s$ such that

$$\|(u,v)\|_{X_2 \times X_s} \leq C(T, \|u_0\|_{H^1}, \|v_0\|_{W^{2-2/s,s}}, \|f\|_{L^s(Q_c)})$$

Existence of Strong Solutions in 2D Domains

[Guillén-González, M-Z, Rodríguez-Bellido, ESAIM COCV 2020 and SICON 2020]
[Guillén-González, M-Z, Villamizar-Roa, Acta Appl. Math. 2020]

$$X_q := \{u \in C([0,T]; W^{2-2/q,q}) \cap L^q(W^{2,q}) \,:\, \partial_t u \in L^q(Q) := L^q(L^q)\}$$

Hypothesis

- $f \in L^s(Q_c) := L^s(L^s(\Omega_c))$, $\quad s > 2$
- $(u_0, v_0) \in H^1(\Omega) \times W^{2-2/s,s}(\Omega)$, with $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a unique strong solution $(u, v) \in X_2 \times X_s$ such that

$$\|(u,v)\|_{X_2 \times X_s} \leq C(T, \|u_0\|_{H^1}, \|v_0\|_{W^{2-2/s,s}}, \|f\|_{L^s(Q_c)})$$

Keys of the proof

1. Leray-Schauder fixed-point Theorem:

$$(\overline{u}, \overline{v}) \mapsto (u, v)$$

solving

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (\overline{u}_+ \nabla v), \\ \partial_t v - \Delta v + v &= \overline{u}^p + f \overline{v}_+ 1_{\Omega_c} \end{cases}$$

2. Energy estimates
   1. Linear case: $(\ln(u), -\Delta v)$

$$\frac{d}{dt}\left(\int_\Omega u \ln u + \frac{1}{2}\|\nabla v\|^2\right) + 4\|\nabla\sqrt{u}\|^2 + \|\Delta v\|^2 + \|\nabla v\|^2 = -\int_{\Omega_c} f v \Delta v,$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v\right)^2 + \left(\int_\Omega v\right)^2 = m_0\left(\int_\Omega v\right) + \left(\int_{\Omega_c} f v\right)\left(\int_\Omega v\right)$$

Keys of the proof

1. Leray-Schauder fixed-point Theorem:

$$(\overline{u}, \overline{v}) \mapsto (u, v)$$

solving

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (\overline{u}_+ \nabla v), \\ \partial_t v - \Delta v + v &= \overline{u}^p + f \overline{v}_+ 1_{\Omega_c} \end{cases}$$

2. Energy estimates
   1. Linear case: $(\ln(u), -\Delta v)$

$$\frac{d}{dt}\left(\int_\Omega u \ln u + \frac{1}{2}\|\nabla v\|^2\right) + 4\|\nabla\sqrt{u}\|^2 + \|\Delta v\|^2 + \|\nabla v\|^2 = -\int_{\Omega_c} f v \Delta v,$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v\right)^2 + \left(\int_\Omega v\right)^2 = m_0\left(\int_\Omega v\right) + \left(\int_{\Omega_c} f v\right)\left(\int_\Omega v\right)$$

Keys of the proof

1. Leray-Schauder fixed-point Theorem:

$$(\overline{u}, \overline{v}) \mapsto (u, v)$$

solving

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u & = & \nabla \cdot (\overline{u}_+ \nabla v), \\ \partial_t v - \Delta v + v & = & \overline{u}^p + f \overline{v}_+ 1_{\Omega_c} \end{array} \right.$$

2. Energy estimates
   1. Linear case: $(\ln(u), -\Delta v)$

$$\frac{d}{dt}\left(\int_\Omega u \ln u + \frac{1}{2}\|\nabla v\|^2\right) + 4\|\nabla \sqrt{u}\|^2 + \|\Delta v\|^2 + \|\nabla v\|^2 = -\int_{\Omega_c} f v \Delta v,$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v\right)^2 + \left(\int_\Omega v\right)^2 = m_0\left(\int_\Omega v\right) + \left(\int_{\Omega_c} f v\right)\left(\int_\Omega v\right)$$

2. Energy estimates
   2. Superlinear case $1 < p \leq 2$: $(\frac{1}{p-1}u^{p-1}, -\frac{1}{p}\Delta v)$

$$\frac{d}{dt}\left(\frac{1}{p(p-1)}\|u^{p/2}\|^2 + \frac{1}{2p}\|\nabla v\|^2\right) + \frac{4}{p^2}\|\nabla(u^{p/2})\|^2$$
$$+\frac{1}{2p}\|\Delta v\|^2 + \frac{1}{p}\|\nabla v\|^2 = -\frac{1}{p}\int_{\Omega_c} fv\Delta v,$$
$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v\right)^2 + \left(\int_\Omega v\right)^2 = \left(\int_\Omega u^p\right)\left(\int_\Omega v\right) + \left(\int_{\Omega_c} fv\right)\left(\int_\Omega v\right)$$

Problem: To estimate $\left(\int_\Omega u^p\right)^2$
Key: Gagliardo-Nirenberg inequality implies

$$\left(\int_\Omega u^p\right)^2 = \|u\|_{L^p}^{2p} \leq \frac{2}{p^2}\|\nabla(u^{p/2})\|^2 + C(\|u\|_{L^1}^{2p})$$

3. Using a bootstrapping argument, via a L^p-regularity result for the heat-Neumann equation

2. Energy estimates
   2. Superlinear case $1 < p \leq 2$: $\left(\frac{1}{p-1}u^{p-1}, -\frac{1}{p}\Delta v\right)$

$$\frac{d}{dt}\left(\frac{1}{p(p-1)}\|u^{p/2}\|^2 + \frac{1}{2p}\|\nabla v\|^2\right) + \frac{4}{p^2}\|\nabla(u^{p/2})\|^2$$
$$+\frac{1}{2p}\|\Delta v\|^2 + \frac{1}{p}\|\nabla v\|^2 = -\frac{1}{p}\int_{\Omega_c} fv\Delta v,$$
$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v\right)^2 + \left(\int_\Omega v\right)^2 = \left(\int_\Omega u^p\right)\left(\int_\Omega v\right) + \left(\int_{\Omega_c} fv\right)\left(\int_\Omega v\right)$$

Problem: To estimate $\left(\int_\Omega u^p\right)^2$
Key: Gagliardo-Nirenberg inequality implies

$$\left(\int_\Omega u^p\right)^2 = \|u\|_{L^p}^{2p} \leq \frac{2}{p^2}\|\nabla(u^{p/2})\|^2 + C(\|u\|_{L^1}^{2p})$$

3. Using a bootstrapping argument, via a L^p-regularity result for the heat-Neumann equation

Weak Solutions in 3D Domains: Linear case ($p = 1$)

[Guillén-González, M-Z, Rodríguez-Bellido, SICON 2020]

Hypothesis

- $f \in L^4(Q_c)$
- $(u_0, v_0) \in L^{p_0}(\Omega) \times H^1(\Omega)$ ($p_0 > 1$), $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a *weak-strong* solution (u, v) such that

- $u \in L^{5/3}(Q) \cap L^{5/4}(W^{1,5/4})$, $\partial_t u \in L^{10/9}((W^{1,10})')$
- $v \in L^\infty(H^1) \cap L^2(H^2)$, $\partial_t v \in L^{5/3}(Q)$
- The weak formulation holds for the u-equation

$$\int_0^T \langle \partial_t u, \overline{u} \rangle + \int_0^T (\nabla u, \nabla \overline{u}) + \int_0^T (u \nabla v, \nabla \overline{u}) = 0 \quad \forall \overline{u} \in L^{10}(W^{1,10})$$

- The v-equation holds pointwisely a.e. $(t, x) \in Q$

Weak Solutions in 3D Domains: Linear case ($p = 1$)

[Guillén-González, M-Z, Rodríguez-Bellido, SICON 2020]

Hypothesis

- $f \in L^4(Q_c)$
- $(u_0, v_0) \in L^{p_0}(\Omega) \times H^1(\Omega)$ $(p_0 > 1)$, $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a *weak-strong* solution (u, v) such that

- $u \in L^{5/3}(Q) \cap L^{5/4}(W^{1,5/4})$, $\partial_t u \in L^{10/9}((W^{1,10})')$
- $v \in L^\infty(H^1) \cap L^2(H^2)$, $\partial_t v \in L^{5/3}(Q)$
- The weak formulation holds for the u-equation

$$\int_0^T \langle \partial_t u, \overline{u} \rangle + \int_0^T (\nabla u, \nabla \overline{u}) + \int_0^T (u \nabla v, \nabla \overline{u}) = 0 \quad \forall \overline{u} \in L^{10}(W^{1,10})$$

- The v-equation holds pointwisely a.e. $(t, x) \in Q$

Weak Solutions in 3D Domains: Linear case ($p = 1$)

[Guillén-González, M-Z, Rodríguez-Bellido, SICON 2020]

Hypothesis

- $f \in L^4(Q_c)$
- $(u_0, v_0) \in L^{p_0}(\Omega) \times H^1(\Omega)$ $(p_0 > 1)$, $u_0 \geq 0$ and $v_0 \geq 0$ in Ω

Then, system (2) has a *weak-strong* solution (u, v) such that

- $u \in L^{5/3}(Q) \cap L^{5/4}(W^{1,5/4})$, $\partial_t u \in L^{10/9}((W^{1,10})')$
- $v \in L^\infty(H^1) \cap L^2(H^2)$, $\partial_t v \in L^{5/3}(Q)$
- The weak formulation holds for the u-equation

$$\int_0^T \langle \partial_t u, \overline{u} \rangle + \int_0^T (\nabla u, \nabla \overline{u}) + \int_0^T (u \nabla v, \nabla \overline{u}) = 0 \quad \forall \overline{u} \in L^{10}(W^{1,10})$$

- The v-equation holds pointwisely a.e. $(t, x) \in Q$

Keys of the proof

- Regularization: $\forall \varepsilon \in (0,1)$ solution of

$$\left\{ \begin{array}{rcl} \partial_t u^\varepsilon - \Delta u^\varepsilon & = & \nabla \cdot (u^\varepsilon \nabla v(z^\varepsilon)) \\ \partial_t z^\varepsilon - \Delta z^\varepsilon + z^\varepsilon & = & u^\varepsilon + f v(z^\varepsilon)_+ 1_{\Omega_c}, \end{array} \right. \quad (3)$$

with $v^\varepsilon := v(z^\varepsilon)$ the solution of

$$\left\{ \begin{array}{rcl} v^\varepsilon - \varepsilon \Delta v^\varepsilon & = & z^\varepsilon \\ \dfrac{\partial v^\varepsilon}{\partial \mathbf{n}} & = & 0 \end{array} \right.$$

- System (3) has a unique strong solution $(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}$, via Leray-Schauder fixed-point theorem
- Energy estimates, independent of ε
- Passing to the limit as $\varepsilon \to 0$, one has existence of weak solutions

Keys of the proof

- Regularization: $\forall \varepsilon \in (0,1)$ solution of

$$\left\{ \begin{array}{rcl} \partial_t u^\varepsilon - \Delta u^\varepsilon & = & \nabla \cdot (u^\varepsilon \nabla v(z^\varepsilon)) \\ \partial_t z^\varepsilon - \Delta z^\varepsilon + z^\varepsilon & = & u^\varepsilon + f v(z^\varepsilon)_+ 1_{\Omega_c}, \end{array} \right. \tag{3}$$

with $v^\varepsilon := v(z^\varepsilon)$ the solution of

$$\left\{ \begin{array}{rcl} v^\varepsilon - \varepsilon \Delta v^\varepsilon & = & z^\varepsilon \\ \dfrac{\partial v^\varepsilon}{\partial \mathbf{n}} & = & 0 \end{array} \right.$$

- System (3) has a unique strong solution $(u^\varepsilon, z^\varepsilon) \in X_{5/3} \times X_{10/3}$, via Leray-Schauder fixed-point theorem
- Energy estimates, independent of ε
- Passing to the limit as $\varepsilon \to 0$, one has existence of weak solutions

Problem: Lack of the regularity of the weak solutions

Regularity Criterion

- $(u_0, v_0) \in W^{3/2,4}(\Omega) \times W^{3/2,4}(\Omega)$
- (u, v) weak solution of (2) such that

$$\boxed{u \in L^{20/7}(Q)}$$

Then $(u, v) \in X_4 \times X_4$ is also a strong solution of (2). Moreover, there exists a positive constant $K := K(T, \|u_0\|_{W^{3/2,4}}, \|v_0\|_{W^{3/2,4}}, \|f\|_{L^4})$ such that

$$\|(u, v)\|_{X_4 \times X_4} \leq K$$

Key of the proof:
Bootstrapping argument

Problem: Lack of the regularity of the weak solutions

Regularity Criterion

- $(u_0, v_0) \in W^{3/2,4}(\Omega) \times W^{3/2,4}(\Omega)$
- (u, v) weak solution of (2) such that

$$u \in L^{20/7}(Q)$$

Then $(u, v) \in X_4 \times X_4$ is also a strong solution of (2). Moreover, there exists a positive constant $K := K(T, \|u_0\|_{W^{3/2,4}}, \|v_0\|_{W^{3/2,4}}, \|f\|_{L^4})$ such that

$$\|(u, v)\|_{X_4 \times X_4} \leq K$$

Key of the proof:
Bootstrapping argument

Problem: Lack of the regularity of the weak solutions

Regularity Criterion

- $(u_0, v_0) \in W^{3/2,4}(\Omega) \times W^{3/2,4}(\Omega)$
- (u, v) weak solution of (2) such that

$$\boxed{u \in L^{20/7}(Q)}$$

Then $(u, v) \in X_4 \times X_4$ is also a strong solution of (2). Moreover, there exists a positive constant $K := K(T, \|u_0\|_{W^{3/2,4}}, \|v_0\|_{W^{3/2,4}}, \|f\|_{L^4})$ such that

$$\|(u, v)\|_{X_4 \times X_4} \leq K$$

Key of the proof:
Bootstrapping argument

Optimal Control Problems

- $\mathcal{F} \subset L^s(Q_c)$ is nonempty, convex and closed set
- $f \in \mathcal{F}$ is a bilinear control acting on the v-equation

$$\begin{aligned} \min J(u,v,f) &= \frac{\gamma_u}{r}\int_0^T \|u(t)-u_d(t)\|_{L^r}^r + \frac{\gamma_v}{2}\int_0^T \|v(t)-v_d(t)\|^2 \\ &+ \frac{\gamma_f}{s}\int_0^T \|f(t)\|_{L^s}^s \end{aligned} \tag{4}$$

subject to (u,v) being a strong solution of

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v) \\ \partial_t v - \Delta v + v &= u^p + fv\, 1_{\Omega_c} \end{cases} \tag{5}$$

$$r := \begin{cases} 2 & \text{in} \quad 2D \\ 20/7 & \text{in} \quad 3D \end{cases} \qquad s := \begin{cases} > 2 & \text{in} \quad 2D \\ 4 & \text{in} \quad 3D \end{cases}$$

Remark

In 3D we follows [E.Casas, K. Chrysafinos, 2016]

Optimal Control Problems

- $\mathcal{F} \subset L^s(Q_c)$ is nonempty, convex and closed set
- $f \in \mathcal{F}$ is a bilinear control acting on the v-equation

$$\min J(u,v,f) = \frac{\gamma_u}{r}\int_0^T \|u(t)-u_d(t)\|_{L^r}^r + \frac{\gamma_v}{2}\int_0^T \|v(t)-v_d(t)\|^2 + \frac{\gamma_f}{s}\int_0^T \|f(t)\|_{L^s}^s \quad (4)$$

subject to (u,v) being a strong solution of

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v) \\ \partial_t v - \Delta v + v = u^p + fv\,1_{\Omega_c} \end{cases} \quad (5)$$

$$r := \begin{cases} 2 & \text{in } 2D \\ 20/7 & \text{in } 3D \end{cases} \qquad s := \begin{cases} >2 & \text{in } 2D \\ 4 & \text{in } 3D \end{cases}$$

Remark

In 3D we follows [E.Casas, K. Chrysafinos, 2016]

Optimal Control Problems

- $\mathcal{F} \subset L^s(Q_c)$ is nonempty, convex and closed set
- $f \in \mathcal{F}$ is a bilinear control acting on the v-equation

$$\begin{aligned} \min J(u,v,f) &= \frac{\gamma_u}{r}\int_0^T \|u(t)-u_d(t)\|_{L^r}^r + \frac{\gamma_v}{2}\int_0^T \|v(t)-v_d(t)\|^2 \\ &+ \frac{\gamma_f}{s}\int_0^T \|f(t)\|_{L^s}^s \end{aligned} \tag{4}$$

subject to (u,v) being a strong solution of

$$\left\{ \begin{array}{rcl} \partial_t u - \Delta u &=& \nabla\cdot(u\nabla v) \\ \partial_t v - \Delta v + v &=& u^p + fv\,1_{\Omega_c} \end{array} \right. \tag{5}$$

$$r := \left\{ \begin{array}{ccc} 2 & \text{in} & 2D \\ 20/7 & \text{in} & 3D \end{array} \right. \qquad s := \left\{ \begin{array}{ccc} >2 & \text{in} & 2D \\ 4 & \text{in} & 3D \end{array} \right.$$

Remark

In 3D we follows [E.Casas, K. Chrysafinos, 2016]

Optimal Control Problems

- $\mathcal{F} \subset L^s(Q_c)$ is nonempty, convex and closed set
- $f \in \mathcal{F}$ is a bilinear control acting on the v-equation

$$\begin{aligned} \min J(u,v,f) &= \frac{\gamma_u}{r}\int_0^T \|u(t)-u_d(t)\|_{L^r}^r + \frac{\gamma_v}{2}\int_0^T \|v(t)-v_d(t)\|^2 \\ &+ \frac{\gamma_f}{s}\int_0^T \|f(t)\|_{L^s}^s \end{aligned} \tag{4}$$

subject to (u,v) being a strong solution of

$$\begin{cases} \partial_t u - \Delta u &= \nabla\cdot(u\nabla v) \\ \partial_t v - \Delta v + v &= u^p + fv\,1_{\Omega_c} \end{cases} \tag{5}$$

$$r := \begin{cases} 2 & \text{in} & 2D \\ 20/7 & \text{in} & 3D \end{cases} \qquad s := \begin{cases} >2 & \text{in} & 2D \\ 4 & \text{in} & 3D \end{cases}$$

Remark

In 3D we follows [E.Casas, K. Chrysafinos, 2016]

Optimal Control Problems

- $\mathcal{F} \subset L^s(Q_c)$ is nonempty, convex and closed set
- $f \in \mathcal{F}$ is a bilinear control acting on the v-equation

$$\min J(u,v,f) = \frac{\gamma_u}{r}\int_0^T \|u(t)-u_d(t)\|_{L^r}^r + \frac{\gamma_v}{2}\int_0^T \|v(t)-v_d(t)\|^2 + \frac{\gamma_f}{s}\int_0^T \|f(t)\|_{L^s}^s \tag{4}$$

subject to (u,v) being a strong solution of

$$\begin{cases} \partial_t u - \Delta u = \nabla\cdot(u\nabla v) \\ \partial_t v - \Delta v + v = u^p + fv\,1_{\Omega_c} \end{cases} \tag{5}$$

$$r := \begin{cases} 2 & \text{in } 2D \\ 20/7 & \text{in } 3D \end{cases} \qquad s := \begin{cases} > 2 & \text{in } 2D \\ 4 & \text{in } 3D \end{cases}$$

Remark

In 3D we follows [E.Casas, K. Chrysafinos, 2016]

Existence of Global Optimal Solution

Admissible set

$$\mathcal{S}_{ad} = \{(u, v, f) \in X_2 \times X_s \times \mathcal{F} \,:\, (u, v, f) \text{ is strong solution}\}$$

Hypothesis

- Either $\gamma_f > 0$ or $\gamma_f = 0$ and $\mathcal{F}$ is bounded in $L^s(Q_c)$
- In 3D, $\mathcal{S}_{ad} \neq \emptyset$ ($u \in L^{20/7}(Q_c)$) and $\gamma_u > 0$

Then, there exists a global optimal solution $(\widehat{u}, \widehat{v}, \widehat{f})$ via minimizing sequences and the fact that J is weakly lower semicontinuous on $\mathcal{S}_{ad}$

Remark

In 3D, $\mathcal{S}_{ad} \neq \emptyset$ if the control acts on the whole domain ($\Omega_c = \Omega$) and $v_0 \geq \alpha > 0$ in Ω.

Existence of Global Optimal Solution

Admissible set

$$\mathcal{S}_{ad} = \{(u, v, f) \in X_2 \times X_s \times \mathcal{F} \, : \, (u, v, f) \text{ is strong solution}\}$$

Hypothesis

- Either $\gamma_f > 0$ or $\gamma_f = 0$ and $\mathcal{F}$ is bounded in $L^s(Q_c)$
- In 3D, $\mathcal{S}_{ad} \neq \emptyset$ $(u \in L^{20/7}(Q_c))$ and $\gamma_u > 0$

Then, there exists a global optimal solution $(\widehat{u}, \widehat{v}, \widehat{f})$ via minimizing sequences and the fact that J is weakly lower semicontinuous on $\mathcal{S}_{ad}$

Remark

In 3D, $\mathcal{S}_{ad} \neq \emptyset$ if the control acts on the whole domain ($\Omega_c = \Omega$) and $v_0 \geq \alpha > 0$ in Ω.

Existence of Global Optimal Solution

Admissible set

$$\mathcal{S}_{ad} = \{(u, v, f) \in X_2 \times X_s \times \mathcal{F} \, : \, (u, v, f) \text{ is strong solution}\}$$

Hypothesis

- Either $\gamma_f > 0$ or $\gamma_f = 0$ and $\mathcal{F}$ is bounded in $L^s(Q_c)$
- In $3D$, $\mathcal{S}_{ad} \neq \emptyset$ $(u \in L^{20/7}(Q_c))$ and $\gamma_u > 0$

Then, there exists a global optimal solution $(\widehat{u}, \widehat{v}, \widehat{f})$ via minimizing sequences and the fact that J is weakly lower semicontinuous on $\mathcal{S}_{ad}$

Remark

In 3D, $\mathcal{S}_{ad} \neq \emptyset$ if the control acts on the whole domain $(\Omega_c = \Omega)$ and $v_0 \geq \alpha > 0$ in Ω.

Optimality System

We consider the spaces

$$\mathbb{X} := \widehat{X}_2 \times \widehat{X}_s \times L^s(Q_c), \qquad \mathbb{Y} := L^2(Q) \times L^s(Q)$$

with $\widehat{X}_2 = \{u \in X_2 \,:\, u(0) = 0\}$, $\widehat{X}_s = \{u \in X_s \,:\, u(0) = 0\}$ and the operator $G = (G_1, G_2) : \mathbb{X} \to \mathbb{Y}$ given by

$$\begin{aligned} G_1(u, v, f) &= \partial_t u - \Delta u - \nabla \cdot (u \nabla v) \\ G_2(u, v, f) &= \partial_t v - \Delta v - u^p - fv\, 1_{\Omega_c} \end{aligned}$$

Then, the optimal control problem (4)-(5) is reformulated as

$$\min_{(u,v,f) \in \mathbb{M}} J(u, v, f) \quad \text{subject to} \quad G(u, v, f) = (0, 0)$$

$\mathbb{M} := (\widehat{u}, \widehat{v}, \widehat{f}) + \widehat{X}_2 \times \widehat{X}_s \times (\mathcal{F} - \widehat{f})$

Optimality System

We consider the spaces

$$\mathbb{X} := \widehat{X}_2 \times \widehat{X}_s \times L^s(Q_c), \qquad \mathbb{Y} := L^2(Q) \times L^s(Q)$$

with $\widehat{X}_2 = \{u \in X_2 \,:\, u(0) = 0\}$, $\widehat{X}_s = \{u \in X_s \,:\, u(0) = 0\}$ and the operator $G = (G_1, G_2) : \mathbb{X} \to \mathbb{Y}$ given by

$$\begin{aligned} G_1(u, v, f) &= \partial_t u - \Delta u - \nabla \cdot (u \nabla v) \\ G_2(u, v, f) &= \partial_t v - \Delta v - u^p - fv\, 1_{\Omega_c} \end{aligned}$$

Then, the optimal control problem (4)-(5) is reformulated as

$$\min_{(u,v,f)\in\mathbb{M}} J(u, v, f) \quad \text{subject to} \quad G(u, v, f) = (0, 0)$$

$$\mathbb{M} := (\widehat{u}, \widehat{v}, \widehat{f}) + \widehat{X}_2 \times \widehat{X}_s \times (\mathcal{F} - \widehat{f})$$

Step 1: Regular point

Let $(\widehat{u},\widehat{v},\widehat{f}) \in \widehat{X}_2 \times \widehat{X}_s \times L^s(Q_c)$ be a local optimal solution and $(g_u, g_v) \in \mathbb{Y}$. Then $(U, V, F) \in \widehat{X}_2 \times \widehat{X}_s \times \mathcal{C}(\widehat{f})$ is the unique solution of system

$$\left\{ \begin{array}{rcl} \partial_t U - \Delta U - \nabla\cdot(U\nabla\widehat{v}) - \nabla\cdot(\widehat{u}\nabla V) & = & g_u, \\ \partial_t V - \Delta V + V - p\widehat{u}^{p-1}U - \widehat{f}\,V\,1_{\Omega_c} - F\widehat{v}\,1_{\Omega_c} & = & g_v, \\ U(0) = V(0) & = & 0, \\ \dfrac{\partial U}{\partial \mathbf{n}} = \dfrac{\partial V}{\partial \mathbf{n}} & = & 0, \end{array} \right.$$

where

$$\mathcal{C}(\widehat{f}) := \{\theta(f - \widehat{f}) \,:\, \theta \geq 0,\ f \in \mathcal{F}\}$$

Step 2: Lagrange multipliers [Zowe, Kurcyusz, 1979]

There exists Lagrange multipliers $(\widehat{\lambda}, \widehat{\eta}) \in L^2(Q) \times [L^s(Q)]'$ such that

$$\gamma_u \int_0^T \int_\Omega (\widehat{u} - u_d) U + \gamma_v \int_0^T \int_\Omega (\widehat{v} - v_d) V + \gamma_f \int_0^T \int_{\Omega_c} \operatorname{sgn}(\widehat{f}) |\widehat{f}|^{s-1} F$$
$$+ \int_0^T \int_\Omega (\partial_t U - \Delta U - \nabla \cdot (U \nabla \widehat{v}) - \nabla \cdot (\widehat{u} \nabla V)) \widehat{\lambda}$$
$$+ \int_0^T \int_\Omega (\partial_t V - \Delta V + V - p \widehat{u}^{p-1} U - \widehat{f} V 1_{\Omega_c}) \widehat{\eta} + \int_0^T \int_{\Omega_c} F \widehat{v} \widehat{\eta}$$
$$\geq 0 \quad \forall (U, V, F) \in \widehat{X}_2 \times \widehat{X}_s \times \mathcal{C}(\widehat{f})$$

Step 3: Existence of Strong Solutions of the Adjoint Problem

$$\left\{\begin{array}{rcl} -\partial_t\lambda - \Delta\lambda + \nabla\lambda\cdot\nabla\widehat{v} - p\widehat{u}^{p-1}\eta & = & J'_{\widehat{u}} \\ -\partial_t\eta - \Delta\eta - \nabla\cdot(\widehat{u}\nabla\lambda) + \eta - \widehat{f}\eta\, 1_{\Omega_c} & = & J'_{\widehat{v}} \\ \lambda(T) & = & \eta(T)=0 \\ \dfrac{\partial\lambda}{\partial\mathbf{n}} & = & \dfrac{\partial\eta}{\partial\mathbf{n}} = 0 \end{array}\right. \tag{6}$$

$$\left\{\begin{array}{l} (\lambda,\eta)\in X_2\times X_{s'} \text{ in 2D, } \frac{1}{s}+\frac{1}{s'}=1 \\ (\lambda,\eta)\in X_{20/13}\times X_{20/13} \text{ in 3D} \end{array}\right. \tag{7}$$

Step 4: Regularity of the Lagrange Multipliers

$(\widehat{\lambda},\widehat{\eta})$ is the very-weak solution of the adjoint problem (6). Also, the Lagrange multiplier is unique and satisfies

$$\|\lambda-\widehat{\lambda}\|^2_{L^2} + \|\eta-\widehat{\eta}\|^{s'}_{L^{s'}} = 0.$$

Therefore, $(\widehat{\lambda},\widehat{\eta})$ has the strong regularity (7).

Step 3: Existence of Strong Solutions of the Adjoint Problem

$$\left\{\begin{array}{rcl} -\partial_t\lambda - \Delta\lambda + \nabla\lambda\cdot\nabla\widehat{v} - p\widehat{u}^{p-1}\eta & = & J'_{\widehat{u}} \\ -\partial_t\eta - \Delta\eta - \nabla\cdot(\widehat{u}\nabla\lambda) + \eta - \widehat{f}\eta\, 1_{\Omega_c} & = & J'_{\widehat{v}} \\ \lambda(T) & = & \eta(T)=0 \\ \dfrac{\partial\lambda}{\partial\mathbf{n}} & = & \dfrac{\partial\eta}{\partial\mathbf{n}} = 0 \end{array}\right. \tag{6}$$

$$\left\{\begin{array}{l} (\lambda,\eta)\in X_2\times X_{s'} \text{ in 2D, } \frac{1}{s}+\frac{1}{s'}=1 \\ (\lambda,\eta)\in X_{20/13}\times X_{20/13} \text{ in 3D} \end{array}\right. \tag{7}$$

Step 4: Regularity of the Lagrange Multipliers

$(\widehat{\lambda},\widehat{\eta})$ is the very-weak solution of the adjoint problem (6). Also, the Lagrange multiplier is unique and satisfies

$$\|\lambda-\widehat{\lambda}\|^2_{L^2}+\|\eta-\widehat{\eta}\|^{s'}_{L^{s'}}=0.$$

Therefore, $(\widehat{\lambda},\widehat{\eta})$ has the strong regularity (7).

Step 3: Existence of Strong Solutions of the Adjoint Problem

$$\left\{\begin{array}{rcl} -\partial_t\lambda - \Delta\lambda + \nabla\lambda\cdot\nabla\widehat{v} - p\widehat{u}^{p-1}\eta & = & J'_{\widehat{u}} \\ -\partial_t\eta - \Delta\eta - \nabla\cdot(\widehat{u}\nabla\lambda) + \eta - \widehat{f}\eta\, 1_{\Omega_c} & = & J'_{\widehat{v}} \\ \lambda(T) & = & \eta(T)=0 \\ \dfrac{\partial\lambda}{\partial\mathbf{n}} & = & \dfrac{\partial\eta}{\partial\mathbf{n}} = 0 \end{array}\right. \tag{6}$$

$$\left\{\begin{array}{l} (\lambda,\eta)\in X_2\times X_{s'} \text{ in 2D, } \frac{1}{s}+\frac{1}{s'}=1 \\ (\lambda,\eta)\in X_{20/13}\times X_{20/13} \text{ in 3D} \end{array}\right. \tag{7}$$

Step 4: Regularity of the Lagrange Multipliers

$(\widehat{\lambda},\widehat{\eta})$ is the very-weak solution of the adjoint problem (6). Also, the Lagrange multiplier is unique and satisfies

$$\|\lambda-\widehat{\lambda}\|_{L^2}^2 + \|\eta-\widehat{\eta}\|_{L^{s'}}^{s'} = 0.$$

Therefore, $(\widehat{\lambda},\widehat{\eta})$ has the strong regularity (7).

Optimality System

(u, v, f, λ, η) such that

- State equations

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v) \\ \partial_t v - \Delta v + v &= u^p + fv\, 1_{\Omega_c} \\ u(0) &= u_0, \quad v(0) = v_0 \\ \dfrac{\partial u}{\partial \mathbf{n}} &= \dfrac{\partial v}{\partial \mathbf{n}} = 0 \end{cases}$$

- Adjoint system

$$\begin{cases} -\partial_t \lambda - \Delta \lambda + \nabla \widehat{v} \cdot \nabla \lambda - p \widehat{u}^{p-1} \eta &= J'_u \\ -\partial_t \eta - \Delta \eta - \nabla \cdot (\widehat{u} \nabla \lambda) + \eta - f \eta\, 1_{\Omega_c} &= J'_v \\ \lambda(T) &= \eta(T) = 0 \\ \dfrac{\lambda}{\partial \mathbf{n}} &= \dfrac{\partial \eta}{\partial \mathbf{n}} = 0 \end{cases}$$

- Optimality condition

$$\int_0^T \int_{\Omega_c} (J'_f + \widehat{v} \eta)(F - \widehat{f}) \geq 0 \quad \forall F \in \mathcal{F}$$

Other Cases

- Stationary Case with Linear Production [S. Lorca, E. Mallea-Zepeda, E.J. Villamizar-Roa, Bull. Braz. Math. Soc. 2023]:

$$\Omega \subset \mathbb{R}^n, n = 1, 2, 3, \text{ and } f \in L^3(\Omega_c) \text{ with } \|f\|_{L^3} << 1$$

$$\begin{cases} -\Delta u &= \nabla \cdot (u \nabla v) \quad \text{in} \quad \Omega, \\ -\Delta v + v &= u + fv\, 1_{\Omega_c} \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- Parabolic-Elliptic Case with Superlinear Production [A. Ancoma-Huarachi, E. Mallea-Zepeda, Hacettepe J. Math. Stat. 2023]:

$$\Omega \subset \mathbb{R}^2, f \in L^q(\Omega_c), 2 < q < \infty, \text{ with } \|f\|_{L^q} << 1 \text{ and } p \in [1, 2]$$

$$\begin{cases} \partial_t u - \Delta u &= \nabla \cdot (u \nabla v) \quad \text{in} \quad \Omega, \\ -\Delta v + v &= u^p + fv\, 1_{\Omega_c} \quad \text{in} \quad \Omega, \\ v(0, x) &= v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &= \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Other Cases

- **Stationary Case with Linear Production [S. Lorca, E. Mallea-Zepeda, E.J. Villamizar-Roa, Bull. Braz. Math. Soc. 2023]:**

$$\Omega \subset \mathbb{R}^n,\ n = 1, 2, 3,\ \text{ and } f \in L^3(\Omega_c) \text{ with } \|f\|_{L^3} << 1$$

$$\begin{cases} -\Delta u = \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ -\Delta v + v = u + fv\, 1_{\Omega_c} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

- **Parabolic-Elliptic Case with Superlinear Production [A. Ancoma-Huarachi, E. Mallea-Zepeda, Hacettepe J. Math. Stat. 2023]:**

$$\Omega \subset \mathbb{R}^2,\ f \in L^q(\Omega_c), 2 < q < \infty,\ \text{ with } \|f\|_{L^q} << 1 \text{ and } p \in [1, 2]$$

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ -\Delta v + v = u^p + fv\, 1_{\Omega_c} & \text{in } \Omega, \\ v(0,x) = v_0(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Other Cases

- **Stationary Case with Linear Production [S. Lorca, E. Mallea-Zepeda, E.J. Villamizar-Roa, Bull. Braz. Math. Soc. 2023]:**

$$\Omega \subset \mathbb{R}^n, n = 1, 2, 3, \text{ and } f \in L^3(\Omega_c) \text{ with } \|f\|_{L^3} << 1$$

$$\begin{cases} -\Delta u = \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ -\Delta v + v = u + fv\, 1_{\Omega_c} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

- **Parabolic-Elliptic Case with Superlinear Production [A. Ancoma-Huarachi, E. Mallea-Zepeda, Hacettepe J. Math. Stat. 2023]:**

$$\Omega \subset \mathbb{R}^2, f \in L^q(\Omega_c), 2 < q < \infty, \text{ with } \|f\|_{L^q} << 1 \text{ and } p \in [1, 2]$$

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v) & \text{in } \Omega, \\ -\Delta v + v = u^p + fv\, 1_{\Omega_c} & \text{in } \Omega, \\ v(0, x) = v_0(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

- **Parabolic-Elliptic Case with Linear Production [J. Cen, J. Huayta-Centeno, E. Mallea-Zepeda, S. Zeng, Appl. Math. Optim. 2024]:**

$$\Omega \subset \mathbb{R}^2 \text{ and } f \in L^q(\Omega_c), 2 < q < \infty, \text{ with } \|f\|_{L^q} << 1$$

$$\begin{cases} \partial_t u - \Delta u & = & \nabla \cdot (u \nabla v) \quad \text{in} \quad Q, \\ -\Delta v + v & = & u + fv\,1_{\Omega_c} \quad \text{in} \quad Q, \\ v(0,x) & = & v_0(x) \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} & = & \dfrac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T) \times \partial\Omega. \end{cases}$$

- **Attraction-Repulsion Systems [J. Huayta-Centeno, E. Mallea-Zepeda, E.J. Villamizar-Roa, Optim. Letters 2025]:**

$$\Omega \subset \mathbb{R}^2, g \in L^{2+}(Q_c), h \in L^{2+}(Q_d)$$

$$\begin{cases} \partial_t u - \Delta u & = & -\nabla \cdot (u \nabla v) + \nabla \cdot (u \nabla w) + u(a - bu) \quad \text{in } Q, \\ \partial_t v - \Delta v + v & = & u + gv\,1_{\Omega_c} \quad \text{in } Q, \\ \partial_t w - \Delta w + w & = & u^p + hw\,1_{\Omega_d} \quad \text{in } Q. \end{cases}$$

- **Parabolic-Elliptic Case with Linear Production [J. Cen, J. Huayta-Centeno, E. Mallea-Zepeda, S. Zeng, Appl. Math. Optim. 2024]:**

$$\Omega \subset \mathbb{R}^2 \text{ and } f \in L^q(\Omega_c), 2 < q < \infty, \text{ with } \|f\|_{L^q} << 1$$

$$\begin{cases} \partial_t u - \Delta u = \nabla \cdot (u \nabla v) & \text{in } Q, \\ -\Delta v + v = u + fv\, 1_{\Omega_c} & \text{in } Q, \\ v(0,x) = v_0(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

- **Attraction-Repulsion Systems [J. Huayta-Centeno, E. Mallea-Zepeda, E.J. Villamizar-Roa, Optim. Letters 2025]:**

$$\Omega \subset \mathbb{R}^2, g \in L^{2+}(Q_c), h \in L^{2+}(Q_d)$$

$$\begin{cases} \partial_t u - \Delta u = -\nabla \cdot (u \nabla v) + \nabla \cdot (u \nabla w) + u(a - bu) & \text{in } Q, \\ \partial_t v - \Delta v + v = u + gv\, 1_{\Omega_c} & \text{in } Q, \\ \partial_t w - \Delta w + w = u^p + hw\, 1_{\Omega_d} & \text{in } Q. \end{cases}$$

- **Stationary Case with Nonlinear Production [Y.R. Linares, E. Mallea-Zepeda, I. Villarreal-Tintaya, Appl. Math. Optim. 2025]**

$\Omega \subset \mathbb{R}^2$, $p \in (1,2]$ and $f \in L^2(\Omega_c)$ with $\|f\|_{L^2} << 1$

$$\left\{\begin{array}{rcl} -\Delta u &=& \nabla \cdot (u \nabla v) \ \text{ in } \ \Omega, \\ -\Delta v + v &=& u^p + fv\, 1_{\Omega_c} \ \text{ in } \ \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} &=& \dfrac{\partial v}{\partial \mathbf{n}} = 0 \ \text{ on } \ \partial\Omega. \end{array}\right.$$

Work in Progress

- **Parabolic-Neumann Problem in 3D domains and Superlinear Production [F. Guillén-González, E. Mallea-Zepeda, M.A. Rodríguez-Bellido, E.J. Villamizar-Roa, 2025]:**
  - $\Omega \subset \mathbb{R}^3$, $p = [1, 5/3]$ and $f \in L^{5/2}(L^{5/2+}(\Omega_c))$

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& \nabla \cdot (u \nabla v) & \text{in } Q, \\ \partial_t v - \Delta v + v &=& u^p + f v \, 1_{\Omega_c} & \text{in } Q. \end{array} \right.$$

  - **Logistic Problem:** $\Omega \subset \mathbb{R}^3$, $p > 1$ and $f \in L^{5/2}(L^{5/2+}(\Omega_c))$

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& \nabla \cdot (u \nabla v) + ru - \mu u^p & \text{in } Q, \\ \partial_t v - \Delta v + v &=& u^p + f v \, 1_{\Omega_c} & \text{in } Q. \end{array} \right.$$

Work in Progress

- **Parabolic-Neumann Problem in 3D domains and Superlinear Production [F. Guillén-González, E. Mallea-Zepeda, M.A. Rodríguez-Bellido, E.J. Villamizar-Roa, 2025]:**
  - $\Omega \subset \mathbb{R}^3$, $p = [1, 5/3]$ and $f \in L^{5/2}(L^{5/2+}(\Omega_c))$

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u & = & \nabla \cdot (u \nabla v) & \text{in} \quad Q, \\ \partial_t v - \Delta v + v & = & u^p + fv\, 1_{\Omega_c} & \text{in} \quad Q. \end{array} \right.$$

  - **Logistic Problem:** $\Omega \subset \mathbb{R}^3$, $p > 1$ and $f \in L^{5/2}(L^{5/2+}(\Omega_c))$

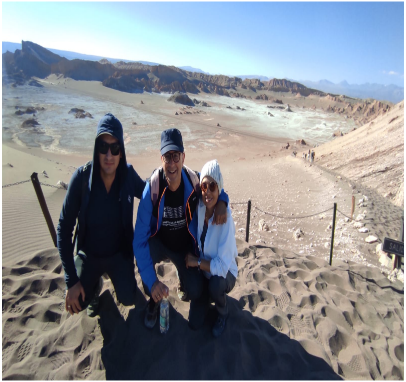
$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u & = & \nabla \cdot (u \nabla v) + ru - \mu u^p & \text{in} \quad Q, \\ \partial_t v - \Delta v + v & = & u^p + fv\, 1_{\Omega_c} & \text{in} \quad Q. \end{array} \right.$$

Empirical Research: Observing saddle points

Empirical Research: Observing saddle points

Empirical Research: Observing saddle points

Thank you very much for your friendship!



Thank you!