# Local null controllability of a cubic Ginzburg-Landau equation with dynamic boundary conditions

Roberto Morales Ponce (UD, Spain)[1]

Workshop on PDEs and Control - PKM60.

A conference to celebrate the 60th birthday of Kisko and Manolo.

September 3, 2025

[1]e-mail: roberto.morales@deusto.es

1 Introduction and main result

2 Existence and uniqueness results for solutions

3 A new Carleman estimate

4 Comments

# Introduction

# Introduction

In this talk, for simplicity, we restrict our attention to the case of **Cubic Ginzburg-Landau equation with dynamic boundary conditions** [2]:

$$\begin{cases} \partial_t u - a(1+\alpha i)\Delta u + \mathcal{F}(u) = \mathbb{1}_\omega h & \text{in } \Omega \times (0,T), \\ \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma + \mathcal{F}(u_\Gamma) = 0 & \text{on } \Gamma \times (0,T), \\ u\big|_\Gamma = u_\Gamma & \text{on } \Gamma \times (0,T), \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma, \end{cases}$$

where

- $i := \sqrt{-1}$ is the imaginary unit.
- $\partial_\nu$ denotes the normal derivative operator,
- $\Delta_\Gamma$ is the Laplace-Beltrami operator acting on $\Gamma$,
- $\mathcal{F}(w) = c(1+\gamma i)|w|^2 w$,
- $a, b > 0$, $c \neq 0$, $\alpha, \gamma \in \mathbb{R}$.

[2] Carreño, N., Mercado, A., & Morales, R. (2025). Local null controllability of a cubic Ginzburg–Landau equation with dynamic boundary conditions. Journal of Evolution Equations, 25(3), 62.

- We notice that this equation can be seen as a coupled system in the variables $(u, u_\Gamma)$, which is controlled by a single control $h$ in small subset.
- This means that the first equation is controlled directly by the action of the control, while the second equation is controlled through the side condition $u\big|_\Gamma = u_\Gamma$ on $\Gamma \times (0, T)$.
- These models have been considered by Corrêa et al [3] and studied well-posedness of linear/nonlinear of such models. In addition, long time behavior of solutions is characterized when Lipschitz nonlinearities are considered.

[3] Corrêa, W. J., & Özsarı, T. (2018). Complex Ginzburg–Landau equations with dynamic boundary conditions. Nonlinear Analysis: Real World Applications, 41, 607-641.

The objective is to establish the **local null controllability** in $X$ of this system: $\exists\,\delta > 0$ such that, for every initial state $(u_0, u_{\Gamma,0}) \in X$ which fulfills

$$\|(u_0, u_{\Gamma,0})\|_X \leq \delta,$$

we can find a control $h \in L^2(\omega \times (0,T))$ such that the solution $(u, u_\Gamma)$ fulfills

$$u(\cdot, T) = 0 \text{ in } \Omega, \quad u_\Gamma(\cdot, T) = 0 \text{ on } \Gamma.$$

For $1 \leq p \leq +\infty$, we consider the Banach space $\mathbb{L}^p := L^p(\Omega) \times L^p(\Gamma)$, endowed by

$$\|(u, u_\Gamma)\|_{\mathbb{L}^p}^2 := \|u\|_{L^p(\Omega)}^2 + \|u_\Gamma\|_{L^p(\Gamma)}^2.$$

In particular, for $p = 2$, the space $\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma)$ is a Hilbert space equipped with the scalar product

$$\langle (u, u_\Gamma), (v, v_\Gamma) \rangle_{\mathbb{L}^2} := \Re \int_\Omega u\overline{v}\, dx + \Re \int_\Gamma u_\Gamma \overline{u_\Gamma}\, dS.$$

In addition, for $k \geq 1$, we introduce the space

$$\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \,:\, y\big|_\Gamma = y_\Gamma\}.$$

### Theorem (Carreño, Mercado, M., 2025)

Suppose that $d = 2$ or $d = 3$. Then, for every $T > 0$ and $\omega \Subset \Omega$, there exists $\delta > 0$ such that, for every $(u_0, u_{\Gamma,0}) \in \mathbb{H}^1$ satisfying

$$\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} \leq \delta,$$

there exists a control $h \in L^2(\omega \times (0,T))$ such that the unique corresponding solution of the GL equation satisfies

$$u(\cdot, T) = 0 \text{ in } \Omega, \quad u_\Gamma(\cdot, T) = 0 \text{ on } \Gamma.$$

# Strategy

To prove the main theorem, we follow the next steps:

- We first deduce a null controllability result for a linear system of the form:

$$\begin{cases} \partial_t y - a(1+\alpha i)\Delta y = f + \mathbb{1}_\omega h & \text{in } \Omega \times (0,T), \\ \partial_t y_\Gamma + a(1+\alpha i)\partial_\nu y - b(1+\alpha i)\Delta_\Gamma y_\Gamma = f_\Gamma & \text{on } \Gamma \times (0,T), \\ y\big|_\Gamma = y_\Gamma & \text{on } \Gamma \times (0,T), \\ (y, y_\Gamma)(0) = (y_0, y_{\Gamma,0}) & \text{in } \Omega \times \Gamma, \end{cases}$$

  where $(f, f_\Gamma)$ will be taken to decrease exponentially to zero in $t = T$.

- Then, we prove a new Carleman estimate for the adjoint system associated to the linearized equation. This will provide existence of a unique solution to a suitable variational problem, from which we define a solution $(y, y_\Gamma, h)$ such that

$$y(T) = 0 \text{ in } \overline{\Omega}.$$

- Finally, by an inverse mapping theorem, we deduce the null controllability of the nonlinear system.

# Existence and uniqueness results for solutions of GL equation with dynamic boundary conditions

For $a, b > 0$ and $\alpha \in \mathbb{R}$, let $\mathcal{A}_{GL} : D(\mathcal{A}_{GL}) \subset \mathbb{L}^2 \to \mathbb{L}^2$ is the operator defined by

$$\mathcal{A}_{GL}(U) := \begin{bmatrix} a(1+\alpha i)\Delta u \\ -a(1+\alpha i)\partial_\nu u + b(1+\alpha i)\Delta_\Gamma u_\Gamma \end{bmatrix},$$

for each $U := \begin{bmatrix} u \\ u_\Gamma \end{bmatrix} \in D(\mathcal{A}_{GL})$ with domain

$$D(\mathcal{A}_{GL}) := \left\{ U = \begin{bmatrix} u \\ u_\Gamma \end{bmatrix} \in \mathbb{H}^1 \, : \, (\Delta u, \Delta_\Gamma u_\Gamma) \in \mathbb{L}^2 \right\},$$

which is equivalent to $\mathbb{H}^2$ since $\Gamma$ is smooth [4].

[4] Gal, C. G. (2015). The role of surface diffusion in dynamic boundary conditions: Where do we stand?. Milan Journal of Mathematics, 83(2), 237-278.

We consider the linear problem in the abstract form

$$\begin{cases} U'(t) = \mathcal{A}_{GL} U(t) + F(t), \quad t \in (0, T), \\ U(0) = U_0. \end{cases}$$

We have the following result:

## Proposition (Existence and uniqueness)

- The operator $\mathcal{A}_{GL}$ is densely defined and generates an analytic semigroup $(e^{t\mathcal{A}_{GL}})_{t \geq 0}$ in $\mathbb{L}^2$.
- Suppose that $U_0 \in \mathbb{L}^2$ and $F \in L^2(0, T; (\mathbb{H}^1)')$. Then, the

$$U \in C^0([0, T]; \mathbb{L}^2) \cap L^2(0, T; \mathbb{H}^1).$$

- If $U_0 \in \mathbb{H}^1$ and $F \in L^2(0, T; \mathbb{L}^2)$. Then,

$$U \in H^1(0, T; \mathbb{L}^2) \cap C^0([0, T]; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^2).$$

Let us consider the nonlinear problem

$$\begin{cases} Lu + c(1+\gamma i)|u|^2 u = f & \text{in } \Omega \times (0,T), \\ L_\Gamma(u, u_\Gamma) + c(1+\gamma i)|u_\Gamma|^2 u_\Gamma = f_\Gamma & \text{in } \Gamma \times (0,T), \\ u\big|_\Gamma =_\Gamma & \text{on } \Gamma \times (0,T), \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma, \end{cases}$$

## Proposition (Existence of solutions in the nonlinear case)

Let $d = 2$ or $d = 3$. There exist $\epsilon > 0$ and $C > 0$ such that, for every $(f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$, $(u_0, u_{\Gamma,0}) \in \mathbb{H}^1$ such that

$$\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + \|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} \leq \epsilon,$$

there exists a unique solution $(u, u_\Gamma)$.

To prove this result, we use the Banach fixed point Theorem.

# A new Carleman estimate

Let $\lambda, m > 1$ and given $\omega' \Subset \omega \Subset \Omega$, we introduce the **weight functions**

$$\varphi(x,t) := (t(T-t))^{-1}\left(e^{2\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty)+\eta^0(x)}\right),$$
$$\xi(x,t) := (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}$$

for $(x,t) \in \overline{\Omega} \times (0,T)$, where $\eta^0 \in C^2(\overline{\Omega})$ satisfies

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \Gamma, \quad |\nabla \eta^0| > 0 \text{ in } \overline{\Omega \setminus \omega'}.$$

# A new Carleman estimate

## Theorem (Carreño, Mercado, M., 2025)

Let $\omega \Subset \Omega$. Set $\omega' \Subset \omega$ and $\eta^0$ as before. Then, there exist constants $C, \lambda_0, s_0 > 0$ such that for all $\lambda \geq \lambda_0$ and $s \geq s_0$:

$$\begin{aligned}
&\int_0^T \int_\Omega e^{-2s\varphi}(s^3\lambda^4\xi^3|v|^2 + s\lambda^2\xi|\nabla v|^2 + s^{-1}\xi^{-1}(|\partial_t v|^2 + |\Delta v|^2)) \\
&+ \int_0^T \int_\Gamma e^{-2s\varphi}(s^3\lambda^3\xi^3|v_\Gamma|^2 + s\lambda\xi\left(|\nabla_\Gamma v_\Gamma|^2 + |\partial_\nu v|^2\right)) \\
&+ \int_0^T \int_\Gamma e^{-2s\varphi} s^{-1}\xi^{-1}(|\partial_t v_\Gamma|^2 + |\Delta_\Gamma v_\Gamma|^2) \\
\leq & C s^3\lambda^4 \int_0^T \int_\omega e^{-2s\varphi}\xi^3|v|^2 + C \int_0^T \int_\Omega |\partial_t v + a(1-\alpha i)\Delta v|^2 \\
&+ C \int_0^T \int_\Gamma e^{-2s\varphi}|\partial_t v_\Gamma - a(1-\alpha i)\partial_\nu v + b(1-\alpha i)\Delta_\Gamma v_\Gamma|^2
\end{aligned}$$

for all $(v, v_\Gamma) \in H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^2)$.

- To prove this result, we follow the ideas developed by L. Rosier and B. Y. Zhang [5]. The main difficulty in our case is to deal with the new boundary terms appeared in the conjugation process.
- The assumption $b > 0$ is crucial to keep the term $|\nabla_\Gamma v_\Gamma|^2$ on the left-hand side.
- With this estimate at hand, we can establish the observability inequality needed to our purposes. Therefore, the null controllability for the linear case is given.
- Combining these results with a local inversion argument in Banach spaces, the main result is obtained.

[5] Rosier, L., & Zhang, B. Y. (2009, February). Null controllability of the complex Ginzburg–Landau equation. In Annales de l'Institut Henri Poincaré C, Analyse non linéaire (Vol. 26, No. 2, pp. 649-673).

# Comments

# Comments

- Similar results can be obtained using different nonlinearities $\mathcal{F}$, but the delicate point is choosing the functional spaces appropriately.
- For the 1D problem, the treatment is easier, since the tangential derivatives are not included in the equations.
- Other control problems can be considered for the Ginzburg-Landau equation with dynamic boundary conditions.

Figure: Happy birthday Kisko and Manolo! ;)