Decay rates to solutions of some dissipative systems in Sobolev critical spaces

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Navier-Stokes equations

Navier-Stokes equations

- The Navier-Stokes equations describe the time evolution of the velocity of a homogeneous incompressible viscous fluid

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u + \nabla p &= \Delta u, \\ div\ u &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{NS}$$

where

- $u = u(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^n$ is the fluid velocity and
- $p = p(x,t) : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is the pressure.
- $u \cdot \nabla = \sum_{i=1}^{n} u_i \partial_{x_i}$.

Navier-Stokes equations

- At least formally, we have the energy equality

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\|\nabla u(t)\|_{L^2}^2.$$

- It is natural to ask whether $\|u(t)\|_{L^2}$ goes to zero or not when time goes to infinity.
- In the last paragraph of his article, Leray (1934) stated that he did not know an answer to this question.

> *N. B.* J'ignore si $W(t)$ tend nécessairement vers o quand t augmente indéfiniment.

N.B. I do not know if $W(t)$ goes necessarily to 0 when t grows indefinitely.

- Here $W(t)$ refers to the $L^2(\mathbb{R}^3)$-norm of the weak solution to (NS) that Leray constructed.

Navier-Stokes equations

- Using different techniques, Kato (1984) and Masuda (1985) showed

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0,$$

but they did not provide a rate of decay.

- Decay results for weak solutions using Fourier Splitting Method are due to M.E. Schonbek (1985,1986): for initial data $u_0 \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $n \geq 3$, with $1 \leq p < 2$, then

$$\|u(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{n}{2}\left(\frac{2}{p}-1\right)}, \qquad t > 0.$$

- Note that the decay rate is determined by the L^p part of the initial datum u_0.
- M.E. Schonbek (1986) also proved that if u_0 is just in L^2, then there are solutions that do not have a uniform algebraic decay rate.

Navier-Stokes equations

- Principle: "decay rate is determined by small frequencies of initial datum".
- Given $u_0 \in L^2(\mathbb{R}^n)$ we associate a decay character $r^* = r^*(u_0)$, $-\frac{n}{2} < r^* < \infty$ which measures its "algebraic order" near the origin. Roughly speaking says that

$$|\widehat{u_0}(\xi)| \approx |\xi|^{r^*} \text{ when } |\xi| \approx 0.$$

- Studied by Bjorland and M.E. Schonbek (2009), Niche and M.E. Schonbek (2015) and Brandolese (2016).
- For wide family of dissipative linear operators $\mathcal{L}$ like Laplacian, solutions to linear system obey

$$C_1(1+t)^{-\left(\frac{n}{2}+r^*\right)} \leq \|e^{t\mathcal{L}} u_0\|_{L^2}^2 \leq C_2(1+t)^{-\left(\frac{n}{2}+r^*\right)}.$$

Navier-Stokes: decay results

Navier-Stokes: decay results

- For solutions to Navier-Stokes equations, for $u_0 \in L^2(\mathbb{R}^3)$ with $r^* = r^*(u_0)$, it holds

$$\|u(t)\|_{L^2}^2 \leq C(1+t)^{-\min\left\{\frac{3}{2}+r^*, \frac{5}{2}\right\}}.$$

See Bjorland and M.E. Schonbek (2009), Niche and M.E. Schonbek (2015).

- Linear vs nonlinear part contribution: if $r^* \leq 1$, decay as linear part; if $r^* > 1$, decay as nonlinear part.
- For $u_0 \in H^s(\mathbb{R}^3)$, $s \geq 1$, it holds

$$\|u(t)\|_{\dot{H}^s}^2 \leq C(1+t)^{-\left(s+\min\left\{\frac{3}{2}+r^*, \frac{5}{2}\right\}\right)}.$$

See Niche and M.E. Schonbek (2015).

Navier-Stokes: decay results

- We could ask whether similar decay results can be obtained in function spaces X where the Navier-Stokes equations can be solved in $\mathbb{R}^3$ but where u does not necessarily have finite L^2-norm.
- A particularly important such family is that of critical spaces, namely those for which the natural scaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

leads to a new solution u_λ whose norm is invariant, i.e.

$$\|u_\lambda\|_X = \|u\|_X, \text{ for all } \lambda > 0.$$

- As examples of these spaces X in $\mathbb{R}^n$ we mention

$$\dot{H}^{\frac{n}{2}-1}; \quad L^n; \quad \dot{B}^{-1+\frac{n}{p}}_{p,\infty}, p \geq n; \quad \dot{B}^{-1}_{\infty,\infty}.$$

Navier-Stokes: decay results

- $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is critical for the Navier-Stokes equations.
- For small $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, mild solutions obey

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = 0.$$

  See Gallagher, Iftimie and Planchon (2002).
- No decay results in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, there are "technical issues".

Our decay result: big picture

- We prove algebraic decay for solutions to Navier-Stokes equations in the critical space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for mild solutions.
- Main ideas for the proof:
  - $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$-norm is a Lyapunov function,
  - rigorous a priori estimates,
  - Fourier Splitting Method and Decay Character.

Fourier Splitting Method

- The Fourier Splitting Method was developed by M.E. Schonbek to study decay of energy for solutions to parabolic conservations laws (1980) and to Navier-Stokes equations (1985, 1986).
- It rests on the observation that for these equations for large enough times, remaining energy is concentrated at the low frequencies.
- The method amounts to estimating a differential inequality for the L^2-norm, which is bounded from above by the average of the solution in a small, time dependent shrinking ball around the origin in frequency space.

Decay Character

Let

$$P_r(v_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} \left| \widehat{v_0}(\xi) \right|^2 d\xi,$$

where $B(\rho)$ denotes the ball at the origin with radius ρ.

Definition

The *decay character of* $v_0 \in L^2(\mathbb{R}^n)$, denoted by $r^* = r^*(v_0)$ is the unique $r \in \left(-\frac{n}{2}, \infty\right)$ such that $0 < P_r(v_0) < \infty$, provided that this number exists. We set

$$r^* = \begin{cases} -\frac{n}{2}, & \text{if } P_r(v_0) = \infty \text{ for all } r \in \left(-\frac{n}{2}, \infty\right) \\ \infty, & \text{if } P_r(v_0) = 0 \text{ for all } r \in \left(-\frac{n}{2}, \infty\right). \end{cases}$$

- It is possible to explicitly compute the decay character for many important examples.
- When $v_0 \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $1 < p < 2$ and $v_0 \notin L^{\bar{p}}(\mathbb{R}^n)$ for $\bar{p} < p$, we have that $r^*(v_0) = -n\left(1 - \frac{1}{p}\right)$.

Navier-Stokes: decay results

Theorem (Ikeda, Kosloff, Niche, P. (2024))

Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, *with div* $u_0 = 0$ *and* $\|u_0\|_{\dot{H}^{\frac{1}{2}}} < \epsilon$, *for small enough* $\epsilon > 0$ *and* $-\frac{3}{2} < q^* = r^*(\Lambda^{\frac{1}{2}} u_0) < \infty$. *Then, for any mild solution to (NS), we have that*

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq C(1+t)^{-\min\{\frac{3}{2}+q^*, 1\}}.$$

- Notation: $\Lambda = (-\Delta)^{\frac{1}{2}}$.
- Improves result by Gallagher et al (2002).
- As in the L^2 case the linear and nonlinear parts drive the decay for different sets of initial data.

Sketch of the proof of decay in critical space

Sketch of proof in critical space

- $\dot{H}^{\frac{1}{2}}$-norm is a Lyapunov function,

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq \|u(s)\|^2_{\dot{H}^{\frac{1}{2}}}, \quad t \geq s.$$

See Bahouri et al. (2011).

- Then

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} = -\|\nabla u(t)\|^2_{\dot{H}^{\frac{1}{2}}} - \langle \Lambda^{\frac{1}{2}} u(t), \Lambda^{\frac{1}{2}} \mathbb{P}\,\nabla \cdot (u \otimes u)(t)\rangle.$$

- As

$$\langle u, \mathbb{P}\,\nabla \cdot (u \otimes u)\rangle_{\dot{H}^{\frac{1}{2}}} \leq C\|u\|_{\dot{H}^{\frac{1}{2}}} \|\nabla u\|^2_{\dot{H}^{\frac{1}{2}}}$$

then

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq -\left(1 - C\|u(t)\|_{\dot{H}^{\frac{1}{2}}}\right) \|\nabla u(t)\|^2_{\dot{H}^{\frac{1}{2}}}.$$

Sketch of proof in critical space

- As solution is mild, small initial data in $\dot{H}^{\frac{1}{2}}$ leads to
$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq -C\|\nabla u(t)\|^2_{\dot{H}^{\frac{1}{2}}}.$$
which is analogous to the energy inequality but for the $\dot{H}^{\frac{1}{2}}$-norm.
- We consider a ball $B(t)$ around the origin in frequency space with continuous, time-dependent radius $r(t)$ such that
$$B(t) = \left\{\xi \in \mathbb{R}^3 : |\xi| \leq r(t) = \left(\frac{g'(t)}{Cg(t)}\right)^{\frac{1}{2}}\right\},$$
with g an increasing continuous function such that $g(0) > 0$.
- We then adapt the Fourier Splitting Method to this context to obtain
$$\frac{d}{dt}\left(g(t)\|\Lambda^{\frac{1}{2}}u(t)\|^2_{L^2}\right) \leq g'(t)\int_{B(t)}\left||\xi|^{\frac{1}{2}}|\widehat{u}(\xi,t)\right|^2 d\xi.$$
- A pointwise estimate for $\left||\xi|^{\frac{1}{2}}\widehat{u}(\xi,t)\right|^2$ in $B(t)$ with a carefully chosen decreasing radius $g(t)$, leads to the result.

□

Navier-Stokes-Coriolis equations

Navier-Stokes-Coriolis

- Similar results are obtained for mild solutions to the Navier-Stokes-Colioris equations

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u + \nabla p + \Omega\, e_3 \times u &= \Delta u, \\ div\ u &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{NSC}$$

  for constant rotation velocity Ω around $e_3 = (0,0,1)$.
- These equations are a prototype for geophysical models with strong rotation around a fixed axis, this being modelled by the Coriolis term.
- Rotation of Earth is important: Earth's rotation speed at Equator ≈ 1700 km/h, typical velocity on oceans/atmosphere $O(10\, km/h)$.
- Rotation of Earth, then Coriolis Force.

Navier-Stokes-Coriolis

- $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is critical for the Navier-Stokes-Coriolis equations.
- For $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, Iwabuchi and Takada (2013) proved existence of mild solutions for fast enough rotation (but bound depends on datum profile instead of its norm) such that

$$u \in C([0,\infty); \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^4([0,\infty); \dot{H}_3^{\frac{1}{2}}(\mathbb{R}^3)).$$

Navier-Stokes-Coriolis

Theorem (Ikeda, Kosloff, Niche, P. (2024))

Let $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, *with div* $u_0 = 0$ *and* $\|u_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} < \epsilon$, *for small enough* $\epsilon > 0$. *Then, there exists* $\omega(u_0) > 0$ *such that for any* Ω *with* $|\Omega| > \omega(u_0)$, *if* $-\frac{3}{2} < q^* = r^*(\Lambda^{\frac{1}{2}} u_0) < \infty$, *then, for any mild solution to (NSC), we have that*

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \leq C(1+t)^{-\min\{\frac{3}{2}+q^*, 1\}}.$$

- Improves result by Iwabuchi and Takada (2013).
- Same decay as for the Navier-Stokes equations.
- Proof is similar to that the decay for solutions to (NS).
- The main difference for the Navier-Stokes-Coriolis equations lies on the initial set up given that the Coriolis term does not contribute to the energy estimates.

Some comments

- We have to show that $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$-norm is a Lyapunov function.
- Semigroup associated to (NSC) has some subtle dispersive features.
- Spaces where Fixed Point Theorem is used look "strange" but then provide right setting for some estimates in the proof of decay (criticality).
- Form of symbol matrix associated to semigroup does not allow the direct use of results in Niche and M.E. Schonbek (2015) concerning decay of linear part, have to tweak things around.

Energy-critical nonlinear heat equation

Energy-critical nonlinear heat equation

The energy-critical nonlinear heat equation in $\mathbb{R}^n$,

$$\begin{aligned} \partial_t u &= \Delta u + |u|^{\frac{4}{n-2}} u, \\ u(x,0) &= u_0(x) \in \dot{H}^1(\mathbb{R}^n), \end{aligned} \tag{nNHE}$$

where $n \geq 3$, has recently been extensively studied due to its rich structure.

Energy-critical nonlinear heat equation

- Using the natural scaling

$$u_\lambda(x,t) = \lambda^{\frac{n-2}{2}} u(\lambda x, \lambda^2 t), \quad \lambda > 0,$$

from a solution u we obtain a new one u_λ such that in the critical space $\dot{H}^1(\mathbb{R}^n)$ we have

$$\|u_\lambda(t)\|_{\dot{H}^1} = \|u(t)\|_{\dot{H}^1}, \qquad t > 0.$$

- This scaling leaves the energy

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t)|^2 \, dx - \frac{n-2}{2n}\int_{\mathbb{R}^n} |u(t)|^{\frac{2n}{n-2}} \, dx,$$

invariant, i.e. $E(u_\lambda(t)) = E(u(t))$, for all $t, \lambda > 0$. Moreover, each of the terms is also invariant by this scaling.

Energy-critical nonlinear heat equation

- Equation (nNHE) has a stationary solution with remarkable properties, the Aubin-Talenti bubble (1976)

$$W(x) = \frac{1}{\left(1 + \frac{1}{n(n-2)}|x|^2\right)^{\frac{n-2}{2}}},$$

which is, up to the previous scaling and translations, the unique extremum of the Sobolev embedding $\dot{H}^1(\mathbb{R}^n) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^n)$, i.e. W realizes the equality in

$$\frac{\|u\|_{L^{\frac{2n}{n-2}}}}{\|u\|_{\dot{H}^1}} \leq C, \quad \text{for } C = \sqrt{\frac{1}{\pi n(n-2)}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}}.$$

Energy-critical nonlinear heat equation

- Recently, Gustafson and Roxanas (2018) proved global existence of mild solutions to (4NHE) provided the initial datum u_0 is small compared to the ground state and established its behaviour at infinity.
- Let $u_0 \in \dot{H}^1(\mathbb{R}^4)$ such that

$$E(u_0) \leq E(W), \quad \|\nabla u_0\|_{L^2} \leq \|\nabla W\|_{L^2}.$$

Then the mild solution to (4NHE) is global and dissipates, i.e.

$$\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0.$$

- We extend the result to (nNHE) and also we prove the decay of the $\dot{H}^1$ norm of solutions to (nNHE).

Decay for energy-critical nonlinear heat equation

Decay of the $\dot{H}^1$-norm of solutions to (nNHE)

Theorem (Kosloff, Niche, P. (2024), Ikeda, Niche, P. (2025))

Let $n \geq 3$ and u be a dissipative solution. Let $q^* = r^*(\Lambda u_0) > -\frac{n}{2}$. Then we have

$$\|u(t)\|_{\dot{H}^1}^2 \leq \begin{cases} C(1+t)^{-\min\left\{\frac{n}{2}+q^*,1\right\}}, & n \leq 10 \\ C[\ln(e+t)]^{-2}, & n > 10 \end{cases}$$

for large enough t.

- The decay character can be used to establish the role of the linear and nonlinear part of the solution in the decay.

Decay for energy-critical nonlinear heat equation

Some comments

- We have to show that $\dot{H}^1(\mathbb{R}^n)$-norm is a Lyapunov function,
- For $n > 10$ we can only prove an inverse logarithmic decay rate.
- This is the first decay estimate we obtain for any $n \geq 3$, which we then use to bootstrap in order to obtain faster decay rates.
- However, the nonlinearity has such a structure that for $n > 10$ we cannot obtain improvements in the pointwise estimates needed.

Hardy-Sobolev parabolic equation

Hardy-Sobolev parabolic equation

The Hardy-Sobolev parabolic equation on $\mathbb{R}^n$,

$$\begin{aligned} \partial_t u &= \Delta u + |x|^{-\gamma}|u|^{2^*(\gamma)-2}u \\ u(x,0) &= u_0(x) \in \dot{H}^1(\mathbb{R}^n), \end{aligned} \tag{HS}$$

where $n \geq 3$, $0 < \gamma < 2$ and

$$2^*(\gamma) = \frac{2(n-\gamma)}{n-2}$$

is the critical Hardy-Sobolev exponent.

Hardy-Sobolev parabolic equation

- Equation (HS) is energy-critical because the energy

$$E_\gamma(u(t)) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(t)|^2\,dx - \frac{1}{2^*(\gamma)}\int_{\mathbb{R}^n} \frac{|u(t)|^{2^*(\gamma)}}{|x|^\gamma}\,dx \qquad (1)$$

  is invariant under the natural scaling

$$u_\lambda(x,t) = \lambda^{\frac{2-\gamma}{2^*(\gamma)-2}} u(\lambda x, \lambda^2 t) = \lambda^{\frac{n-2}{2}} u(\lambda x, \lambda^2 t), \quad \lambda > 0. \qquad (2)$$

- Both terms in (1) are invariant under (2).
- $\dot{H}^1(\mathbb{R}^n)$ is a critical space for (HS).
- Chikami, Ikeda and Taniguchi (2021) gave conditions on the initial datum to the mild solution to (HS) be global and dissipate

$$\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0.$$

Hardy-Sobolev parabolic equation

Decay of the $\dot{H}^1$-norm of solutions to (HS)

Theorem (Ikeda, Niche, P. (2025))

Let $n \geq 5$, u be a dissipative solution of (HS) and $q^* = r^*(\Lambda u_0) > -\frac{n}{2}$, where $\Lambda = (-\Delta)^{1/2}$. Then we have

$$\|u(t)\|_{\dot{H}^1}^2 \leq \begin{cases} C(1+t)^{-\min\left\{\frac{n}{2}+q^*,1\right\}}, & n \leq 10-4\gamma, \\ C[\ln(e+t)]^{-2}, & n > 10-4\gamma, \end{cases}$$

for large enough t.

Some comments

- Due to the fact that we use the Rellich inequality to show that the critical norm is a Lyapunov function, we are restricted to $n \geq 5$.
- The singularity at $x = 0$ in the nonlinear term forces us to make some significant modifications in the proof, when compared to the case $\gamma = 0$.
- These are implemented through delicate estimates in Lorentz spaces.

Final remarks

Final remarks

- Method used for proving decay in critical space could be applied to some (many!) dissipative equations.
- Namely:
    - find mild solution in critical, scale-invariant space;
    - show norm is Lyapunov function;
    - use Fourier Splitting Method and Decay Character to prove decay.

References

- M. Ikeda, L. Kosloff, C. Niche, G. Planas, Algebraic decay rates for 3D Navier-Stokes and Navier-Stokes-Coriolis equations in $\dot{H}^{1/2}$, *Journal of Evolution Equations*, 24(58), 2024.
- L. Kosloff, C. Niche, G. Planas, Decay rates for the 4D energy-critical nonlinear heat equation, *Bulletin of the London Mathematical Society*, 56, 1468-1482, 2024.
- M. Ikeda, C.J. Niche, G. Planas, Global dynamics for the energy-critical nonlinear heat equation, *Discrete and Continuous Dynamical Systems*, online, 2025.
- M. Ikeda, C.J. Niche, G. Planas, Dynamics of dissipative solutions to the Hardy-Sobolev Parabolic Equation, *Dynamics of Partial Differential Equations*, to appear, 2025.

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