Some Questions About Second-Grade Fluid Equations

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In honour of Catedráticos Manolo y Kisko

In collaboration with:

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Contents

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$$\left\{\begin{array}{rcl} \dfrac{\partial}{\partial t}\,(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \operatorname{curl}(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) \times \boldsymbol{u} + \nabla q & = & \boldsymbol{f},\ \text{in } Q, \\ \operatorname{div}\boldsymbol{u} & = & 0,\ \text{in } Q, \\ \boldsymbol{u} & = & 0,\ \text{on } \Sigma, \\ \boldsymbol{u}(0) & = & \boldsymbol{u}_0,\ \text{in } \Omega, \end{array}\right. \tag{1}$$

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where $\boldsymbol{u}$ is the velocity of the fluid and $q := p - \alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{4}|\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t|^2) - \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}$, being p the pressure, here $Q := \Omega \times [0,T]$ and $\Sigma := \partial\Omega\times]0,T[$.

The first articles

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- The first mathematical analysis of (1) was done by Cioranescu and Ouazar (1984), in this paper the authors prove the first general existence and uniqueness in two and three dimensional domains Ω, and supposing small data.
- Ciorarescu y Girault (1997) completed these results by showing the global existence in time of weak and classical solutions and uniqueness in the three-dimensional domain Ω under the assumption that the data are also small enough (Excellent work and the most important).

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$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}_2} = (\boldsymbol{u}, \boldsymbol{v}) + \alpha(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + (\text{curl}(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}), \text{curl}(\boldsymbol{v} - \alpha\Delta\boldsymbol{v})) \quad (2)$$

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are equivalent in $\boldsymbol{V}_2$, where $(\cdot,\cdot)$ is the usual $\boldsymbol{L}^2$ inner product.

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$$(\boldsymbol{u}', \boldsymbol{v}) + \alpha\, (\nabla \boldsymbol{u}', \nabla \boldsymbol{v}) + \mu\, (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \alpha\, b(\boldsymbol{u}, \Delta \boldsymbol{u}, \boldsymbol{v}) + \alpha\, b(\boldsymbol{v}, \Delta \boldsymbol{u}, \boldsymbol{u}) = 0.$$

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Reproductive solution (Friz, Kisko, Marko 2009, 2010)

For any $\boldsymbol{f} \in L^2(0,T;H(\text{curl};\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, there exists a reproductive solution.

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For any $\boldsymbol{f} \in L^2(0,T;H(\text{curl};\Omega)) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, there exists a reproductive solution.

The proof is made following the ideas of Kaniel and Shinbrot (1967) (see also the classical book of J.-L. Lions (1969) and Cionarescu and Girault (1997) + adequate estimates.

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  holds.
- The behavior of the solution of problem (1) changes continuously with the data, that is, the solution is uniformly stable.

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$$E_{\boldsymbol{u}}(t) = \frac{1}{2}\left\{\|\boldsymbol{u}(t)\|^2 + \alpha\|\nabla\boldsymbol{u}(t)\|^2\right\} = \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^1_\alpha}$$

  of problem (1) has an exponential decay rate, under appropriated assumptions.

These results are in Clark, Friz and Rojas-Medar (2025)

Comparison solution and Galerkin approximations

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Let $\boldsymbol{u}_n^\alpha$ be the Galerkin aprroximations of a second grade flow and $\boldsymbol{u}$ the solutions of Navier-Stokes equations, we like to study the convergences rates when $\alpha \to 0$ and $n \to \infty$ in terms of eigenfunctions of Stokes operator.

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J.V. Gutiérrez-Santacreu M.A. Rojas-Medar 2023, On the approximation of turbulent fluid flow by the Navier-Stokes - α equations on bounded domains, Physica D: Nonlinear Phenomena 448 (2023) 133724 local and uniform estimates in t.

Asymptotic Equivalence and Decay

Cruz, Perusato, Niche, Rojas-Medar 2024 proven the following results:

Theorem

Let $\mathbf{u}_0 \in H^1_\alpha(\mathbb{R}^3)$, *with* $\operatorname{div} \mathbf{u}_0 = 0$. *If* $\mathbf{u}_0$ *satisfies*

$$P_r(\mathbf{u}_0)_+ = \limsup_{\rho \to 0^+} \rho^{-2r^*-3} \int_{|\xi|<\rho} |\widehat{\mathbf{u}}_0(\xi)|^2 \, d\xi < \infty,$$

for some $r^* = r^*(\mathbf{u}_0) \in (-3/2, \infty)$, *then any weak solution* $\mathbf{u}(t)$ *to* (1) *satisfies,*

$$\left\|\mathbf{u}(t)\right\|^2_{H^1_\alpha(\mathbb{R}^3)} \leq C\,(t+1)^{-\min\left\{\frac{3}{2}+r^*, \frac{5}{2}\right\}}, \qquad \forall\, t \geq 0.$$

Where the constant $C > 0$ *depends only on* $\|\mathbf{u}_0\|_{H^1_\alpha(\mathbb{R}^3)}, \alpha, r^*, P_r(\mathbf{u}_0)_+, \mu.$

We also compare the evolution of solutions $\mathbf{u}(\cdot,t)$ to (1) with the solutions $\overline{\mathbf{u}}(\cdot,t)$ of the linear system associated, which is the following pseudo-parabolic equation in $\mathbb{R}^3\times(0,\infty)$:

$$\begin{cases} \partial_t(\overline{\mathbf{u}}-\alpha\Delta\overline{\mathbf{u}})-\mu\Delta\overline{\mathbf{u}}=\mathbf{0}, \\ \operatorname{div}\overline{\mathbf{u}}=0, \\ \overline{\mathbf{u}}(0)=\mathbf{u}_0\in H^1_\alpha(\mathbb{R}^3). \end{cases} \tag{3}$$

Theorem

Let $\mathbf{u}$ *be a weak solution of problem* (1), *and* $\overline{\mathbf{u}}$ *the solution to the linear part* (3) *with the same initial data* $\mathbf{u}_0 \in H^4(\mathbb{R}^3)$, *with* $\operatorname{div} \mathbf{u}_0 = 0$. *If* $\mathbf{u}_0$ *satisfies*

$$P_r(\mathbf{u}_0)_+ = \limsup_{\rho \to 0^+} \rho^{-2r^*-3} \int_{|\xi|<\rho} |\widehat{\mathbf{u}}_0(\xi)|^2 \, d\xi < \infty,$$

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$$\left\| \mathbf{u}(t) - \overline{\mathbf{u}}(t) \right\|^2_{H^1_\alpha(\mathbb{R}^3)} \le C\,(t+1)^{-\min\left\{\frac{5}{2}+\frac{3}{2}r^*, \frac{5}{2}\right\}}, \qquad \forall\, t \ge 0,$$

Where the constant $C > 0$ *depends only on* $\|\mathbf{u}_0\|_{H^1_\alpha(\mathbb{R}^3)}, \alpha, r^*, P_r(\mathbf{u}_0), \mu$. *In other words, the solution* $\mathbf{u}$ *of* (1) *is asymptotically equivalent to the solution* $\overline{\mathbf{u}}$ *of the pseudo-parabolic equation* (3) *with the same data.*

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- Question 2:

  Consider a known weak solution. What additional velocity
  and/or pressure conditions allow us to prove it is a single solution?

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- Question 5:

  Will it be possible to obtain point error estimates (in t)
  for the $H^2(\Omega)$ norm?

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