Numerical analysis of a diffusive SIS epidemic model with repulsive infected-taxis

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Plan

1. Introduction

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2. The model
   - Some properties
   - Difficulties in the discrete framework

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3. FE discretization
   - Regularized functions and Numerical scheme
   - Discrete properties and Simulations

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Depending on the stimulus:

1. Phototaxis (light)
2. Haptotaxis (Cancer cells and Extracellular matrix).
3. Chemotaxis (Chemical substance).
4. ...
5. **Infectedtaxis** (Susceptible individuals and Infected individuals).

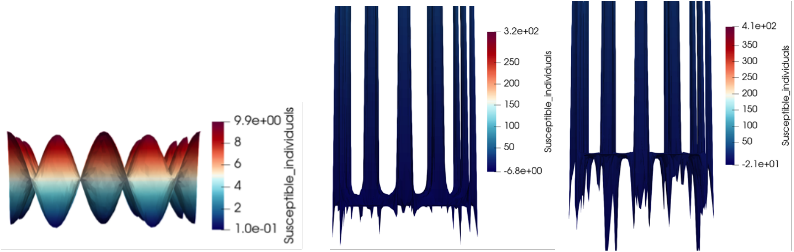
SIS epidemic model with repulsive infected-taxis

S. Wu, J. Wang, J. Shi, *Dynamics and pattern formation of a diffusive predator-prey model with predator-taxis.* Math. Mod. and Meth. in Appl. Sciences, Vol. 28, No. 11 (2018), 2275-2312.

$$\begin{cases} \partial_t u - d\Delta u = -g(u,v) \ \text{ in } \Omega,\ t>0, \\ \partial_t v - \Delta v - \xi\nabla\cdot(v\nabla u) = g(u,v) \text{ in } \ \Omega,\ t>0, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 \ \text{ on } \partial\Omega,\ t>0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0 \ \text{ in } \Omega, \end{cases}$$

- $v \geq 0$: Susceptible individuals.
- $u \geq 0$: Infected individuals.
- $g(u,v) := \gamma u - \beta \frac{uv}{u+v}$.

Approach 1: Backward Euler



(a) A (b) B (c) C

Figura: Spurious oscillations BE

Approach 2: Mass-lumping

$$(a,b)_h = \int_\Omega \Pi^h(ab)\,dx$$

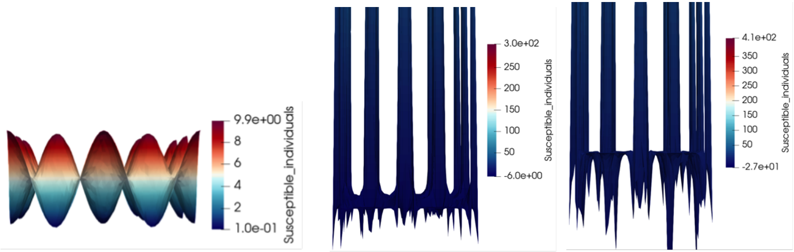


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Figura: Spurious oscillations ML

Some properties

$$\begin{cases} \partial_t u - d\Delta u = -\gamma u + \beta \frac{uv}{u+v} \\ \partial_t v - \Delta v - \xi \nabla \cdot (v \nabla u) = \gamma u - \beta \frac{uv}{u+v}. \end{cases}$$

- **Conservation**: $\int_\Omega [u(\cdot,t) + v(\cdot,t)]\, dx = \int_\Omega [u_0 + v_0]\, dx.$

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- **Nonnegativity of** u, v: $u, v \geq 0.$
- **Weak regularity for** u Testing u-eq by u:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + d\|\nabla u\|_{L^2}^2 + \gamma\|u\|_{L^2}^2 = \int_\Omega \beta u^2 \frac{v}{u+v}\, dx \leq \beta \|u\|_{L^2}^2, \tag{1}$$

which implies $u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$.

Key estimates coming from the continuous model

An estimate of a singular functional for v

$$\begin{cases} \partial_t u - d\Delta u = -\gamma u + \beta \frac{uv}{u+v} \\ \partial_t v - \Delta v - \xi \nabla \cdot (v \nabla u) = \gamma u - \beta \frac{uv}{u+v}. \end{cases} \tag{2}$$

Considering

$$G(v) := -\ln(v) \;\Rightarrow\; G'(v) = -\frac{1}{v} \;\Rightarrow\; G''(v) = \frac{1}{v^2} \quad \forall v > 0,$$

then testing $(2)_2$ by $G'(v)$:

$$\frac{d}{dt}\int_\Omega G(v)\,dx + \left\| \frac{\nabla v}{v} \right\|_{L^2}^2 = \int_\Omega \Big(\beta \frac{u}{u+v} - \gamma \frac{u}{v} - \xi \frac{1}{v} \nabla u \nabla v \Big) dx$$

$$\leq \beta |\Omega| + \frac{1}{2} \left\| \frac{\nabla v}{v} \right\|_{L^2}^2 + \frac{\xi^2}{2} \|\nabla u\|_{L^2}^2,$$

and then,

$$\frac{d}{dt}\int_\Omega G(v)\,dx + \frac{1}{2} \left\| \frac{\nabla v}{v} \right\|_{L^2}^2 \leq \beta |\Omega| + \frac{\xi^2}{2} \|\nabla u\|_{L^2}^2. \tag{3}$$

Problems when passing discrete level

Positivity?? test function for v??

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Solution: Regularized functions. Let $\varepsilon \in (0,1)$ and consider the truncated function

$$G''_\varepsilon(s) := \begin{cases} \dfrac{1}{\varepsilon^2} & \text{if} \quad s < \varepsilon, \\ \dfrac{1}{s^2} & \text{if} \quad s \geq \varepsilon. \end{cases} \tag{4}$$

We can integrate twice in (4) and we obtain a functional $G_\varepsilon(s)$ being a $\mathcal{C}^2$-approximation of $G(s)$ and such that $G_\varepsilon(s) = -\ln(s)$ and $G'_\varepsilon(s) = -1/s$ when $s \geq \varepsilon$.

Some results

Lemma 1: It is straightforward to see that $G_\varepsilon \in \mathcal{C}^2(\mathbb{R})$ verifies, for all $s \in \mathbb{R}$,

$$G'_\varepsilon(s) \leq 0 \text{ and } G''_\varepsilon(s) \geq 0$$

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Lemma 2: Let G_ε be the function defined from (4) and $v : \overline{\Omega} \to \mathbb{R}$ any continuous function. Then, for all $x \in \overline{\Omega}$, it holds

$$\frac{1}{2\varepsilon^2}(v_-(x))^2 \leq G_\varepsilon(v(x)) \cdot \chi_{\{x\in\overline{\Omega}\,:\,v(x)<0\}}(x). \tag{5}$$

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Lemma 3: Let $G_\varepsilon(s)$ and $G'_\varepsilon(s)$ be the functions defined from (4); then, it holds

$$|[s]_+ G'_\varepsilon(s)| \leq 1, \quad \forall s \in \mathbb{R}. \tag{6}$$

Numerical scheme

- **[Step 1]** Given $[u_h^{n-1}, v_h^{n-1}] \in Z_h \times Z_h$, find $v_h^n \in Z_h$ solving:

$$(\delta_t v_h^n, \bar{v}_h)^h + ((v_h^n)^2 \nabla \Pi^h (G'_\varepsilon(v_h^n)), \nabla \bar{v}_h) + \xi(v_h^n \nabla u_h^{n-1}, \nabla \bar{v}_h) = (g(u_h^{n-1}, [v_h^n]_+), \bar{v}_h)^h, \quad (7)$$

  for any $\bar{v}_h \in Z_h$.

- **[Step 2]** Given $[v_h^n, u_h^{n-1}] \in Z_h \times Z_h$, find $u_h^n \in Z_h$ solving:

$$(\delta_t u_h^n, \bar{u}_h)^h + d(\nabla u_h^n, \nabla \bar{u}_h) = -(g(u_h^{n-1}, [v_h^n]_+), \bar{u}_h)^h, \quad \forall \bar{u}_h \in Z_h, \quad (8)$$

where

- $(\cdot, \cdot)^h$: Mass lumping
- $g(u_h^{n-1}, [v_h^n]_+) := \gamma u_h^{n-1} - \beta \left(u_h^{n-1} \frac{[v_h^n]_+}{u_h^{n-1} + [v_h^n]_+} \right)$.

Some results

Lemma (Conservation)

Any solution $[u_h^n, v_h^n]$ satisfies

$$\int_\Omega [u_h^n + v_h^n]\,dx = \int_\Omega [u_h^0 + v_h^0]\,dx, \quad \forall n \geq 1.$$

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Assume acute triangulations $\mathcal{T}_h$. If $\Delta t \leq 1/\gamma$, then $u_h^n > 0$.

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Idea of the proof: Compare with

$$\text{(Minorant discrete ODE)} \begin{cases} \delta_t w^n + \gamma w^{n-1} = 0, \\ w^0 = \min\limits_{\Omega} u_h^0, \end{cases}$$

$$\Rightarrow u_h^n \geq w^n = (1 - \gamma \Delta t)^n \Big(\min_{\overline{\Omega}} u_h^0 \Big) \geq \Big(\min_{\overline{\Omega}} u_h^0 \Big) e^{-2\gamma t_n}.$$

Weak estimates for u_h^n

Proposition (Weak estimates for u_h^n)

It holds

$$\|u_h^n\|_{L^2}^2 + \Delta t \sum_{m=1}^{n} \|\nabla u_h^m\|_{L^2}^2 \le C, \quad \forall n \ge 1,$$

with the constant $C > 0$ independent of $[\Delta t, h]$.

Proof: Taking $\bar{u}_h = u_h^n \in Z_h$

$$\begin{aligned} \frac{1}{2}\delta_t \|u_h^n\|_{L^2}^2 \; + \frac{\Delta t}{2}\|\delta_t u_h^n\|_{L^2}^2 + d\|\nabla u_h^n\|_{L^2}^2 \\ = -\gamma \int_\Omega u_h^n u_h^{n-1} dx + \beta \int_\Omega \frac{[v_h^n]_+}{u_h^{n-1} + [v_h^n]_+} u_h^{n-1} u_h^n \, dx \\ \le \frac{\Delta t}{4}\|\delta_t u_h^n\|_{L^2}^2 + C(\Delta t + 1)\|u_h^{n-1}\|_{L^2}^2 . \end{aligned}$$

Discrete estimate for v_h^n

Taking $\bar{v}_h = \Pi^h(G'_\varepsilon(v_h^n))$ in (7), we obtain that

$$(\delta_t v_h^n, G'_\varepsilon(v_h^n))^h + \|v_h^n \nabla \Pi^h(G'_\varepsilon(v_h^n))\|_{L^2}^2 = \gamma(u_h^{n-1}, \Pi^h(G'_\varepsilon(v_h^n)))^h$$
$$-\xi(v_h^n \nabla u_h^{n-1}, \nabla \Pi^h(G'_\varepsilon(v_h^n))) - \beta\Big([v_h^n]_+ \frac{u_h^{n-1}}{u_h^{n-1} + [v_h^n]_+}, \Pi^h(G'_\varepsilon(v_h^n))\Big)^h.$$

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Notice that:

$$(\delta_t v_h^n, G'_\varepsilon(v_h^n))^h \geq \delta_t(G_\varepsilon(v_h^n), 1)^h.$$

$$|\xi(v_h^n \nabla u_h^{n-1}, \nabla \Pi^h(G'_\varepsilon(v_h^n)))| \leq \frac{1}{2}\|v_h^n \nabla \Pi^h(G'_\varepsilon(v_h^n))\|_{L^2}^2 + \frac{\xi^2}{2}\|\nabla u_h^{n-1}\|_{L^2}^2.$$

$$-\beta\left([v_h^n]_+ \frac{u_h^{n-1}}{u_h^{n-1} + [v_h^n]_+}, \Pi^h(G'_\varepsilon(v_h^n))\right)^h \leq \beta|\Omega|.$$

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Then,

$$\delta_t(G_\varepsilon(v_h^n), 1)^h + \frac{1}{2}\|v_h^n \nabla \Pi^h(G'_\varepsilon(v_h^n))\|_{L^2}^2 \leq \frac{\xi^2}{2}\|\nabla u_h^{n-1}\|_{L^2}^2 + \beta|\Omega|. \quad (9)$$

Implications

- **Approximated positivity**

$$\frac{1}{2\varepsilon^2}\int_\Omega (\Pi^h([v_h^n]_-))^2 dx \le \frac{1}{2\varepsilon^2}\int_\Omega \Pi^h(([v_h^n]_-)^2) dx \le \int_\Omega \Pi^h(G_\varepsilon(v_h^n)) dx \le C,$$

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- $\|v_h^n\|_{L^1} \le C$:

$$\int_\Omega |v_h^n|\, dx \le \int_\Omega \Pi^h |v_h^n|\, dx = \int_\Omega v_h^n\, dx - 2\int_\Omega \Pi^h([v_h^n]_-)\, dx \le C.$$

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- **Existence of solution for v-scheme:** Using a fixed point argument.

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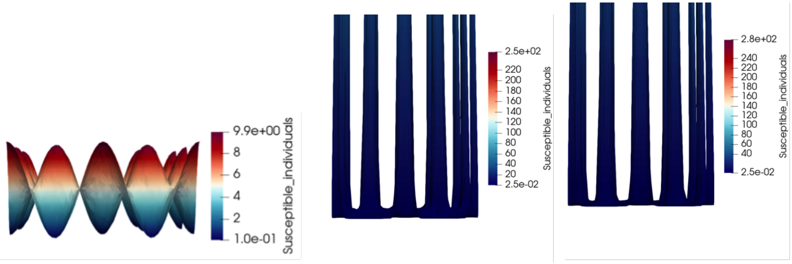
$$\int_\Omega |v_h^n|\, dx \le \int_\Omega \Pi^h |v_h^n|\, dx = \int_\Omega v_h^n\, dx - 2\int_\Omega \Pi^h([v_h^n]_-)\, dx \le C.$$

- **Existence of solution for v-scheme:** Using a fixed point argument.
- **Well-posedness of u-scheme:** Lax-Milgram Lemma.

Simulations using the regularized scheme



(a) A (b) B (c) C

Figura: No spurious oscillations

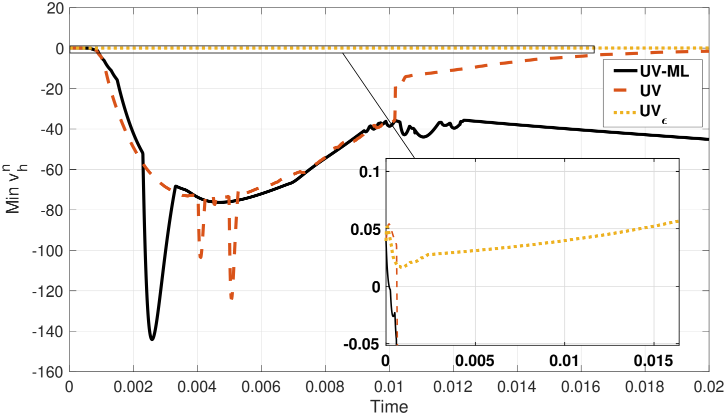


Figura: Minimum of v_h^n.

Convergence rates

$m \times m$	$\|u(t_n) - u_h^n\|_{l^\infty(L^2)}$	Order	$\|u(t_n) - u_h^n\|_{l^2(H^1)}$	Order
36×36	4,9473 e-03	-	1,2841 e-01	-
44×44	3,3225 e-03	1.9840	1,0517 e-01	0.9951
52×52	2,3847 e-03	1.9851	8,9042 e-02	0.9966
60×60	1,7952 e-03	1.9845	7,7198 e-02	0.9974

Cuadro: Convergence rates in space for u.

$m \times m$	$\|v(t_n) - v_h^n\|_{l^\infty(L^2)}$	Order	$\|v(t_n) - v_h^n\|_{l^2(H^1)}$	Order
36×36	5,9491 e-03	-	1,1678 e-01	-
44×44	3,9829 e-03	1.9994	9,5542 e-02	1.0005
52×52	2,8439 e-03	2.0164	8,0837 e-02	1.0004
60×60	2,1262 e-03	2.0324	7,0056 e-02	1.0003

Cuadro: Convergence rates in space for v.

References

- S. Wu, J. Wang, J. Shi, Dynamics and pattern formation of a diffusive predator-prey model with predator-taxis. Math. Mod. and Meth. in Appl. Sciences, Vol. 28, No. 11 (2018), 2275-2312.
- J. W. Barrett and J. F. Blowey, Finite element approximation of a nonlinear cross-diffusion population model, Numer. Math. 98 (2004), no. 2, 195-221.
- F. Guillén-González, M.A. Rodríguez-Bellido and D.A. Rueda-Gómez, A finite element scheme for a diffusive SIS epidemic model with repulsive infected-taxis. (Preprint, 2025).

¡Thank you very much!