Boundary observability for the Grushin equation on a multi-dimensional domain

Mathilda Trabut
Supervisors: Franck Boyer and Morgan Morancey
Joint work with: Jérémi Dardé

Institut de Mathématiques de Toulouse

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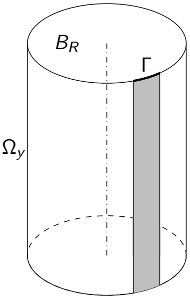
The Grushin equation

The equation

B_R ball of radius R in $\mathbb{R}^{d_x}$, Ω_y bounded open subset of $\mathbb{R}^{d_y}$, with $d_x \geq 2$, $d_y \geq 1$.
$\Omega = B_R \times \Omega_y$.

$$\begin{cases} \partial_t z - \Delta_x z - \|x\|^2 \Delta_y z = 0 & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{in } (0,T) \times \partial\Omega, \\ z(0) = z_0 & \text{in } \Omega \end{cases}$$



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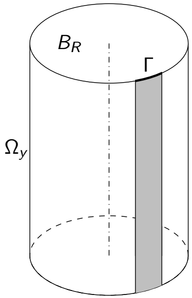

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Goal: Prove an observability inequality

$$\|z(T)\|^2_{H^1_0(\Omega)} \leq C \int_0^T \int_{\Gamma\times\Omega_y} \left|\frac{\partial z(t)}{\partial n}\right|^2 d\sigma dt$$

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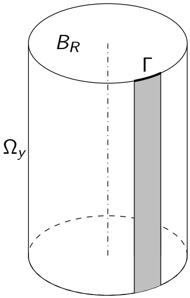

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Main issue: Degeneracy at the center of the ball

The result

Result when $\Gamma = \partial B_R$: [Beauchard, Dardé, Ervedoza, 2020] there exists a time $T^* = R^2/2d_x$ such that

- if $T > T^*$ the observability inequality holds
- if $T < T^*$ the observability inequality does not hold !

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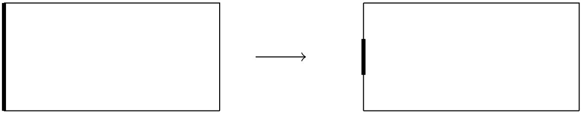
- if $T < T^*$ the observability inequality does not hold !

Theorem [T, Dardé, 2025]

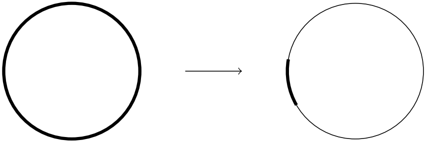
The same result holds for Γ nonempty open subset of ∂B_R, with the same T^*.

Examples

On a coupled heat system [Benabdallah, Boyer, González-Burgos, Olive, 2014]



[T, 2025]



Lebeau-Robbiano strategy on observability inequalities

Theorem [L. Miller, 2010]

Let $-A^*$ be a generator of a semigroup, and B_0^* and B^* be observation operators. If we have

- Observability for B_0^*

$$\left\| e^{-\tau A^*} z \right\|^2 \leq C_{obs} e^{\frac{2b}{\tau^\beta}} \int_0^\tau \left\| B_0^* e^{-tA^*} z \right\|^2 dt, \; z \in \mathcal{D}(A^*), \tau \in (0, T),$$

and there exists a family of subspaces $(\mathcal{E}_\Lambda)_{\Lambda>0}$ such that

- Spectral inequality

$$\left\| B_0^* z \right\|^2 \leq C_{spec} e^{2a\Lambda^\delta} \left\| B^* z \right\|^2, \quad \Lambda > 0, \quad z \in \mathcal{E}_\Lambda,$$

- Dissipation

$$\left\| e^{-tA^*} z \right\|^2 \leq C_{dissip} e^{-2c\Lambda t} \left\| z \right\|^2, \quad \Lambda > 0, \quad z \in \mathcal{E}_\Lambda^\perp,$$

then observability in time T holds for B^* with cost $e^{\frac{C}{T^\beta}}$.

Idea of proof (explanation on the controllability point of view)

[G.Lebeau, L.Robbiano, 1995]

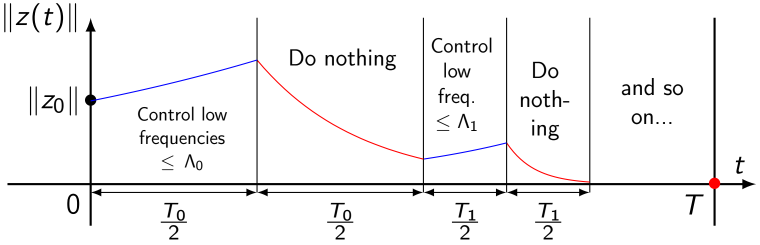


Figure. 1: The Lebeau-Robbiano method

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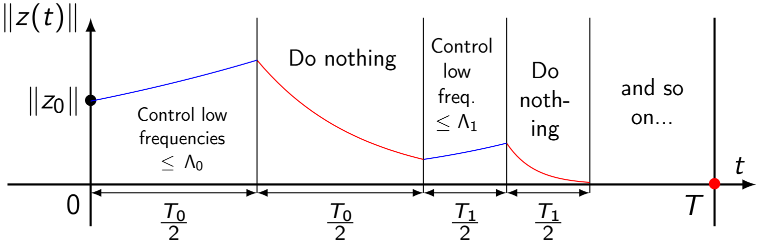

Figure. 1: The Lebeau-Robbiano method

In our problem it fails because of the minimal time for observability on $\partial B_R \times \Omega_y$.

We look at the proof of the result on $\partial B_R \times \Omega_y$

Fourier decomposition in y variable: our problem is equivalent to observability through Γ uniformly in p of

$$\begin{cases} \partial_t z_p - \Delta z_p + \lambda_p^2 ||x||^2 z_p = 0 & \text{in } (0,T) \times B_R, \\ z_p = 0 & \text{in } (0,T) \times \partial B_R, \\ z_p(0) = z_{p,0} & \text{in } B_R, \end{cases} \qquad (H_p)$$

where $(\lambda_p^2)_{p \in \mathbb{N}}$ denotes the eigenvalues of the Dirichlet-Laplacian operator on Ω_y.

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Goal: prove

$$\|z_p(T)\|^2_{H^1_0(B_R)} \leq K(T,p,\Gamma) \int_0^T \int_\Gamma \left| \frac{\partial z_p(t)}{\partial n} \right|^2 d\sigma dt$$

with a good estimate on $K(T,p,\Gamma)$.

What is known

Lemma [Beauchard, Dardé, Ervedoza, 2020]

For any $T > 0$, $p \in \mathbb{N}$ there exists $C > 0$ such that

$$K(T, p, \partial B_R) \leq C \left(\frac{1}{T} + \lambda_p \right) \exp(\lambda_p R^2 \coth(2\lambda_p T)).$$

Lemma [Zhu, Zhuge, 2024]

For any $T > 0$, $p \in \mathbb{N}$ there exists $C, c > 0$ such that

$$K(T, p, \partial B_R) \leq C \exp\left(c(1 + 1/T + \lambda_p)\right)$$

Good news: observability on the whole boundary holds for any $T > 0$.

Challenges: get the good dependance in T and p.

Good cost on ∂B_R

Lemma

There exists a polynomial P and a constant $a > 0$ such that

$$K(T, p, \partial B_R) \leq P(\lambda_p) \exp\left(\lambda_p R^2 \coth\left(2a\lambda_p^{1/2}\right)\right) \exp(C/T^2).$$

Idea of proof:

- Use [Zhu, Zhuge] in the regime $T\lambda_p \leq a\lambda_p^{1/2}$ and get a cost in $\exp(C/T^2)$.
- Use [Beauchard, Dardé, Ervedoza] in the regime $T\lambda_p > a\lambda_p^{1/2}$ and get a cost in $P(\lambda_p) \exp\left(\lambda_p R^2 \coth\left(2a\lambda_p^{1/2}\right)\right)$.

Good cost on Γ

Lemma

For any $\gamma > 0$, there exists a polynomial Q and a constant $s > 0$ such that for any $T > 0$, $p \in \mathbb{N}$,

$$K(T, p, \Gamma) \leq Q(\lambda_p) \exp(1/(sT)^2) \exp\left((1+\gamma)\lambda_p R^2 \coth(a\lambda_p^{1/2})\right).$$

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Idea of proof:

→ We adapt Miller's result.

- His goal was to get the best short time cost (smallest c in $\exp(c/T^\beta)$)
- Our goal is to get a cost not worse than $\exp\left(\lambda_p R^2\right)$

→ How ? We follow his proof and play differently with the parameters.

About spectral inequality and dissipation

Denote $(Y_{mk})_{m\in\mathbb{N}}$ an orthonormal basis of $L^2(\partial B_R)$ of eigenfunctions of the Laplace-Beltrami operator on ∂B_R. Define

$$\mathcal{E}_\Lambda = \left\{ \sum_{m<\sqrt{\Lambda}} \langle z, Y_{mk} \rangle_{L^2(\partial B_R)} Y_{mk} \quad \middle| \quad z \in H^1_0(B_R) \right\}$$

Spectral inequality: [Jerison, Lebeau, 1999], finite sum of eigenfunctions of the Laplace-Beltrami operator.

Dissipation: eigenvalues of $-\Delta + \lambda_p^2 \|x\|^2$ greater than Λ on $\mathcal{E}_\Lambda^\perp$.

Conclusion

Recall that $T^* = R^2/2d_x$, let $\varepsilon > 0$, $\gamma > 0$ such that $T - \varepsilon > T^*(1+\gamma)$.

Dissipation during $(T-\varepsilon)$

$$\|z_p(T)\|^2_{H^1_0(B_R)} \lesssim \exp(-2d_x(T-\varepsilon)\lambda_p)\,\|z_p(\varepsilon)\|^2_{H^1_0(B_R)}$$

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Observability of $z_p(\varepsilon)$

$$\|z_p(\varepsilon)\|^2_{H^1_0(B_R)} \leq Q(\lambda_p)\exp(1/(s\varepsilon)^2)\exp\left((1+\gamma)R^2\lambda_p\coth(a\lambda_p^{1/2})\right) \times \int_0^T\int_\Gamma \left|\frac{\partial z_p(t)}{\partial n}\right|^2 d\sigma\, dt$$

$\to$ uniform estimate in p.

Perspectives

- Restriction of the observation zone in the direction Ω_y, 2D result [Koenig, 2017].

- Kolmogorov equation

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Thanks for your attention !