WORKSHOP ON PDES AND CONTROL 2025 (PKM-60)

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On Keller–Segel models with positive total flux: analytic and modeling perspectives

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Nonlinear differential problems with applications to real phenomena

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Main aim of the presentation

A)
$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & v_t = \Delta v - v + u \qquad \text{in } \Omega \times (0, T_{max}), \\ \boxed{u_\nu = v_\nu = 0 \text{ on } \partial\Omega \times (0, T_{max})}, & u(x,0) = u_0(x), \ v(x,0) = v_0(x) \qquad x \in \bar{\Omega}. \end{cases}$$

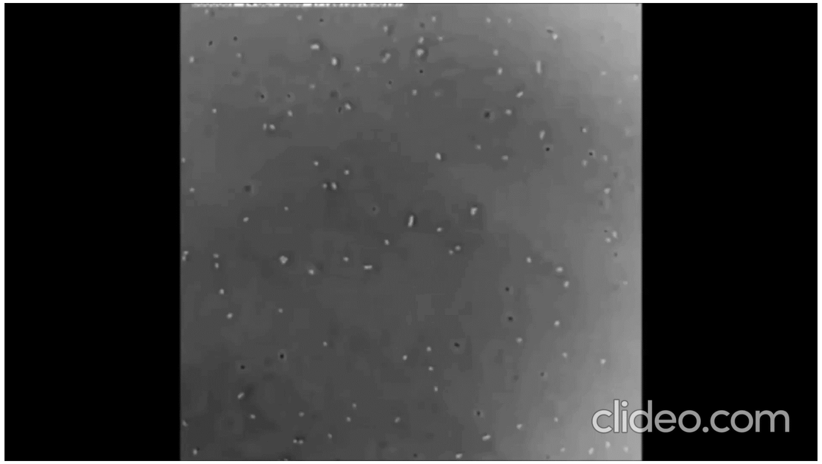
B)
$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & v_t = \Delta v - v + u \qquad \text{in } \Omega \times (0, T_{max}), \\ \text{Robin-type boundary confitions}, & u(x,0) = u_0(x), \ v(x,0) = v_0(x) \qquad x \in \bar{\Omega}. \end{cases}$$

- The Keller-Segel system with Neumann boundary conditions
- Main results and discussions on A)
- Some natural extensions on A)
- Introduction of B). The Robin boundary conditions
- Comparison and main differences between A) and B)

Neumann Boundary Conditions
VS.
Robin Boundary Conditions

How far do standard methods work?

CHEMOTAXIS: movement of an organism or entity in response to a chemical stimulus



$$\begin{cases} u_t = \Delta u - \boxed{\chi} \nabla \cdot (u \nabla v) \\ v_t = \Delta v - v + \boxed{u} \end{cases}$$

$(x,t) \in \Omega \times (0, T_{max})$

u=u(x,t) cells' density, *v=v(x,t)* chemical signal. Chemosensitivity $\chi > 0$

Taxis term χ>0 has a gathering effect on *u*. The signal *v* has attractive effect on *u,* which produces *v*

Keller EF, Segel LA. Initiation of slime mold aggregation viewed as an instability. *Journal of Theoretical Biology* 1970; **26**:399–415

Idealization of the motion of the cells, inside a domain and initially distributed accordingly to the law of *u(x,0)* and *v(x,0)*. Ω smooth and bounded *n*-dimensional domain, T_{max} lifespan of the solutions. The evolution is influenced by the competition between the aggregation impact, increasing for larger size of χ>0. The more *u* increases the more *v* increases. The Laplacian operator provides diffusion to the system

The Keller-Segel model: first indications

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \qquad v_t = \Delta v - v + u & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = 0 \ \text{ on } \partial\Omega \times (0, T_{max}), \ u(x,0) = u_0(x), \ v(x,0) = v_0(x) & x \in \bar{\Omega}. \end{cases}$$

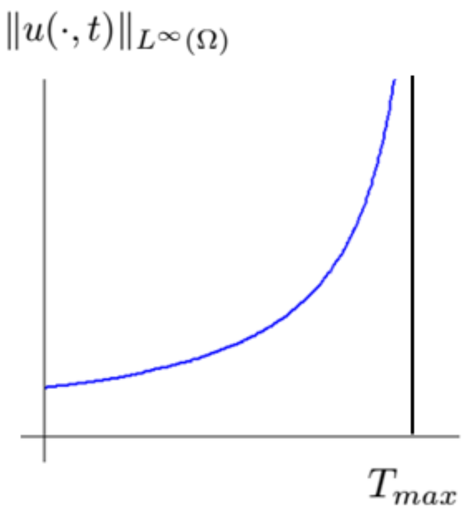
The model may admit global bounded solutions, for which $T_{max} = \infty$, and unbounded ones blowing up in finite time (T_{max} finite, and δ-formations at points of the domain emerge. For $n = 1$ all solutions are uniformly bounded in time (diffusion dominates self-attraction), whereas for $n \geq 2$ self-attraction might overcome diffusion and blow-up may appear.

Higher dimensions "enforce" blow-up
χm large Blow – Up and Instability

Dichotomy criterion

$$\limsup_{t \to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = +\infty$$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad t \in (0,\infty)$$



Nagai, Winkler, Herrero, Velázquez, Tao,Lankeit, Fuest, Marras, Frassu, Columbu....Acosta-Soba, <u>*Guillén-González*</u>, Rodríguez Galván, Rodríguez Bellido .... *Sorry if I forgot any of you!*

THE PROBLEM: crucial property for the zero-flux Keller-Segel model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \qquad v_t = \Delta v - v + u & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = 0 \ \text{ on } \partial\Omega \times (0, T_{max}), \ u(x,0) = u_0(x), \ v(x,0) = v_0(x) & x \in \bar{\Omega}. \end{cases}$$

Let (u,v) be a positive classical solution in $\Omega \times (0, T_{max})$

$$\int_\Omega u_t = \frac{d}{dt}\int_\Omega u = \int_\Omega \nabla \cdot (\nabla u - \chi u \nabla v) = 0 \quad t \in (0, T_{max})$$

The Divergence Theorem

$$\frac{d}{dt}\int_\Omega u = 0 \Rightarrow \int_\Omega u = \int_\Omega u_0(x)dx \quad \text{for all } t \in (0, T_{max})$$

THE MASS IS PRESERVED

MASS CONSERVATION

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad t \in (0, T_{max})$$

On some variants of the Keller-Segel model with Neumann BC

$$u_t = \nabla \cdot (S(u,v)\nabla u - T(u,v)\nabla v) + h(u), \qquad v_t = \Delta v + g(u,v) \quad \text{in} \quad \Omega \times (0, T_{max}).$$

$S(u,v)$	$T(u,v)$	$h(u)$	$g(u,v)$
Diffusion	**Chemoattractant**	**Growth/Death rate for *u***	**Growth/Death rate for *v***

$S(u,v)$–Diffusion	$T(u,v)$–(Chemo)sensitivity	$h(u)$–Logistic	$g(u,v)$–Chemical growth
u^{m_1}	u^{m_2}	$u^\alpha - u^\beta$	$-v + u^\gamma$
1	u		$-uv$
$u/\sqrt{u^2 + \lvert\nabla u\rvert^2}$	$u/\sqrt{1 + \lvert\nabla v\rvert^2}$	$\beta > \alpha$	$-v + u$
1	u/v		$-u^\gamma v$
$v^{-\alpha} u$	u		$-v + u$
u^{m_1}	u^{m_2}	$u^\alpha(1 - \int_\Omega u^\beta)$	$-v + u$

Common denominator $u_\nu = v_\nu = 0 \;\text{ on }\; \partial\Omega \times (0, T_{\max})$ $\Rightarrow$ $\|u(\cdot,t)\|_{L^1(\Omega)}$ **is bdd**

- ➢ *S(u,v)* smoothing effects, *T(u,v)* instability/aggregation actions, *h(u)* an external source
- ➢ *g(u,v)=-v+u*, cells produces chemoattractant; *g(u,v)=-uv*, cells consume chemoattractant

On some variants of the Keller-Segel model with different BC

$$u_t = \nabla \cdot (S(u,v)\nabla u - T(u,v)\nabla v) + f(u), \qquad v_t = \Delta v + g(u,v) \quad \text{in} \quad \Omega \times (0, T_{max}).$$

$\|u(\cdot,t)\|_{L^1(\Omega)}$ **is bdd**

$S(u,v)$	$T(u,v)$	$f(u)$	$g(u,v)$	Boundary condition for v
1	u		u	$\boxed{v_{\|\partial\Omega\times(0,T_{\max})} = 0}$
1	u		$-v+u$	$\boxed{v_{\|\partial\Omega\times(0,T_{\max})} = 0}$
1	u		$-vu$	$\boxed{v_{\|\partial\Omega\times(0,T_{\max})} = v^*}$
1	u	$u-u^2$	$-vu$	$v_{\nu\|\partial\Omega\times(0,T_{\max})} = 1-v$
1	u	$u-u^2$	$-vu$	$v_{\|\partial\Omega} = v^*$
u^m	u		$-vu$	$\boxed{v_{\nu\|\partial\Omega\times(0,T_{\max})} = 1-v}$
u^m	$-u/v$		$-vu$	$v_{\|\partial\Omega\times(0,T_{\max})} = v^*$
u^m	$-u$		$-vu$	$v_{\|\partial\Omega\times(0,T_{\max})} = v^*$

Dirichlet

Robin

Common denominator $S(u,v)u_\nu - T(u,v)v_\nu = 0 \quad \text{on} \quad \partial\Omega \times (0, T_{\max})$

$$\frac{d}{dt}\int_\Omega u = \cancel{\int_{\partial\Omega} (S(u,v)u_\nu - T(u,v)v_\nu)} + \int_\Omega f(u) = \int_\Omega f(u) \ \text{on} \ (0, T_{\max})$$

Interplay between derivative of *u* and *v*

$\boxed{S(u,v)u_\nu - T(u,v)v_\nu}$ **TOTAL FLUX**

Blow-up despite logistics. Preventing gathering with gradient terms

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^{\kappa}, & 0 = \Delta v - v + u \quad \text{in}\, \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = 0 \text{ on } \partial\Omega \times (0, T_{max}), \; u(x,0) = u_0(x), & x \in \bar{\Omega}. \end{cases}$$

$$\kappa < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\} \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5 \end{cases}$$

BLOW-UP for *c=0*
Winkler, 2018, *Z. Angew. Math. Phys.*

$$\lambda \boxed{u} - \mu \boxed{u^{\kappa}} - c \boxed{|\nabla u|^{\gamma}} \qquad \gamma > \frac{2n}{n+1}$$

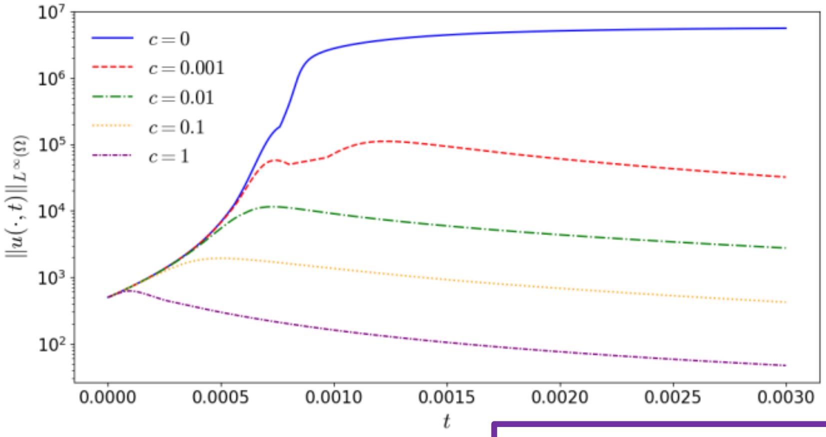
BLOW-UP IS PREVENTED for *c>0*
Ishida, Lankeit, Viglialoro, 2024
Discrete Continuous Dyn. Syst. Ser. B

growth **natural death** **accidental death**

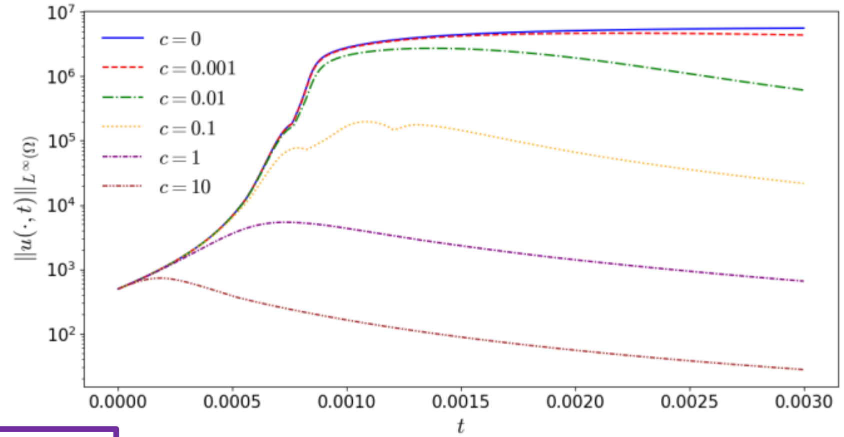
Biologically, gradient terms appear as additional decay terms depending on the size of the gradient of the population density. (Souplet *Math. Methods Appl. Sci.*, 1996.)

Blow-up despite strong logistics (*n=3*)?

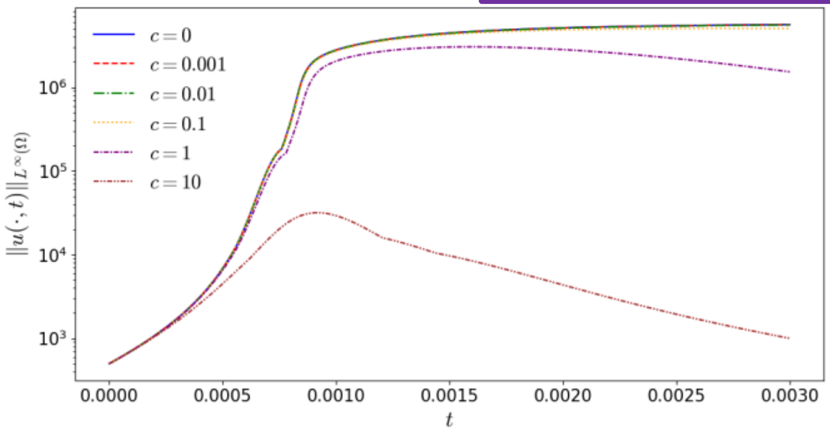
Li, Acosta-Soba, Columbu, Viglialoro. *Stud. Appl. Math* 2025



(A) $\gamma = 1.75$ ***>1.5 ->boundedness, c>0***



(B) $\gamma = 1.4.$



(C) $\gamma = 1.1.$

In (B) and (C) for small c explosions may appear

$$\lambda u - \mu u^\kappa - c|\nabla u|^\gamma$$

$$\Downarrow$$

1.5= $\gamma < \dfrac{2n}{n+1}$ ***c>0* ??**

$k = 1.1 < 7/6 \approx 1.167 \Rightarrow$ ***c=0*, Bow-up!**

Outward flows for *u* and *v* and positive total flux

$$v_\nu = -hv \quad \text{on} \quad \partial\Omega \times (0, T_{\max}) \qquad u_\nu = (\alpha - 1)\chi huv \quad \alpha \in (0,1]$$

Robin boundary condition

outward flows for *u* and *v*

Total flux

$$\boxed{u_\nu - \chi u v_\nu = \alpha \chi huv \quad \text{on} \quad \partial\Omega \times (0, T_{\max})}$$

Zero-Flux (α=0)!!

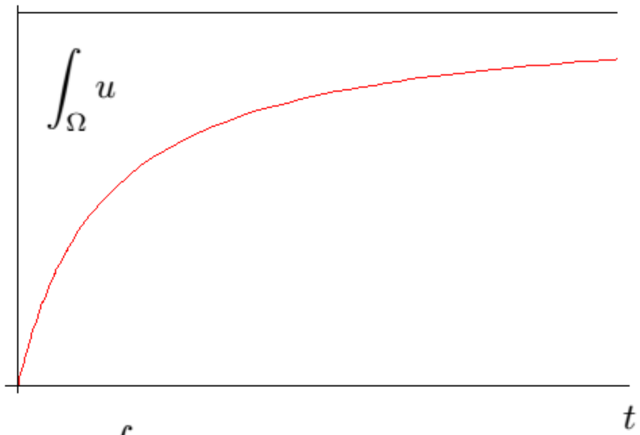
The outward flux of the chemoattractant *v* transports the cells across the boundary toward the interior of the domain itself; the result is a positive (inward) total flux

Even for appropriate outward flux of the cells' configuration, the taxis-driven effect of the outward flow of the chemoattractant can yet keep producing a positive total flux

Positive total flux and behaviour of the mass

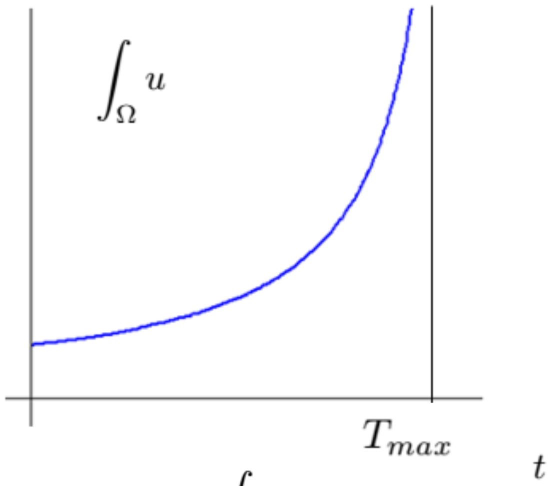
$$u_t = \nabla u - \chi \nabla \cdot (u \nabla v)$$
$$u_\nu - \chi u v_\nu = \alpha \chi h u v$$

$$\frac{d}{dt}\int_\Omega u = \alpha\chi h \int_{\partial\Omega} uv \geq 0 \quad \text{for all } t \in (0, T_{\max})$$



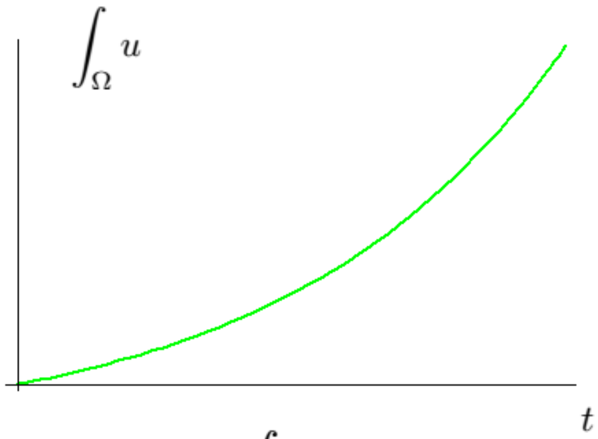
$$\int_\Omega u \leq C \quad \text{on } (0, T_{max})$$

RELATED ANALYSIS: Blowup, boundendess?



$$\limsup_{t \to T_{max}} \int_\Omega u(\cdot, t) = +\infty$$

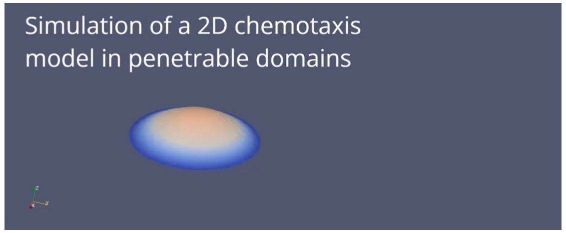
$$\limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty$$



$$\limsup_{t \to +\infty} \int_\Omega u(\cdot, t) = +\infty$$

$$T_{max} \leq \infty$$

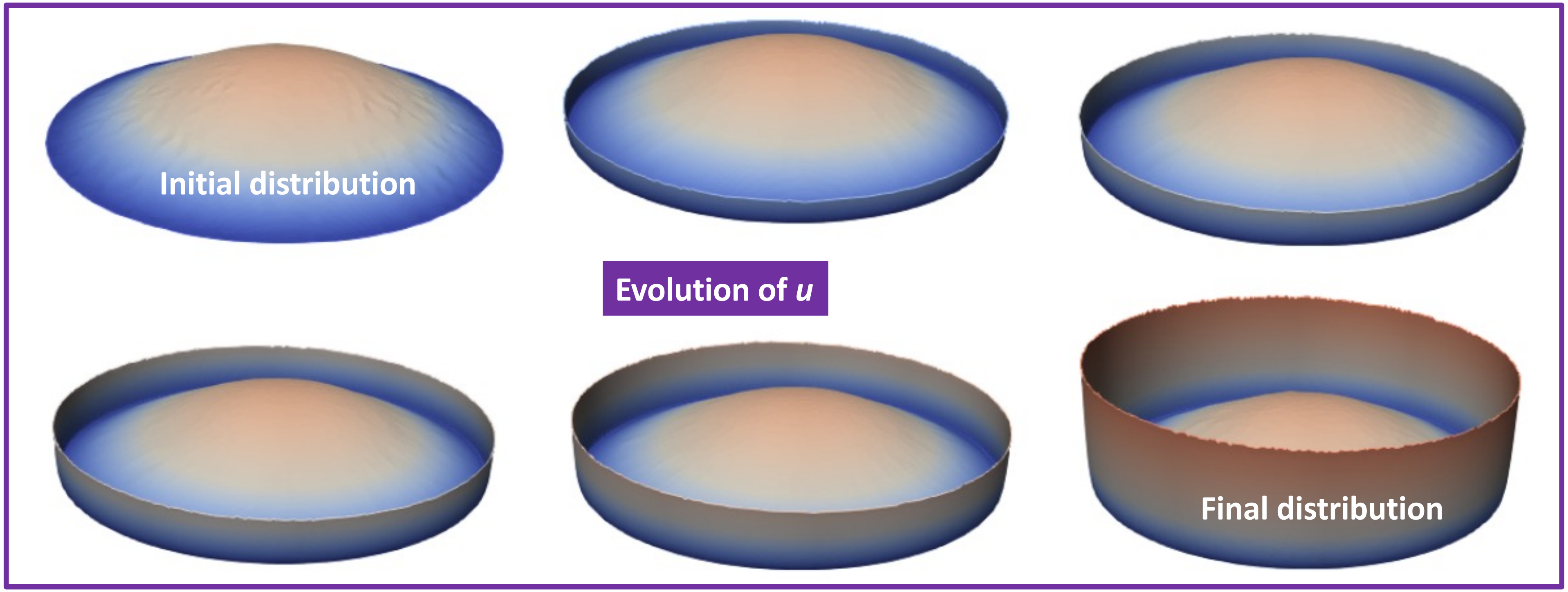
Numerical simulations and questions in 2D for the case $\alpha=1$



$$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$
$$u_0(x,y,0) = v_0(x,y,0) = 13e^{-x^2-y^2}, \chi = 0.14, h = 60$$

$$u_\nu - \chi u v_\nu = \chi h u v$$

Data: 2960 triangles, $\Delta t=10^{-6}$

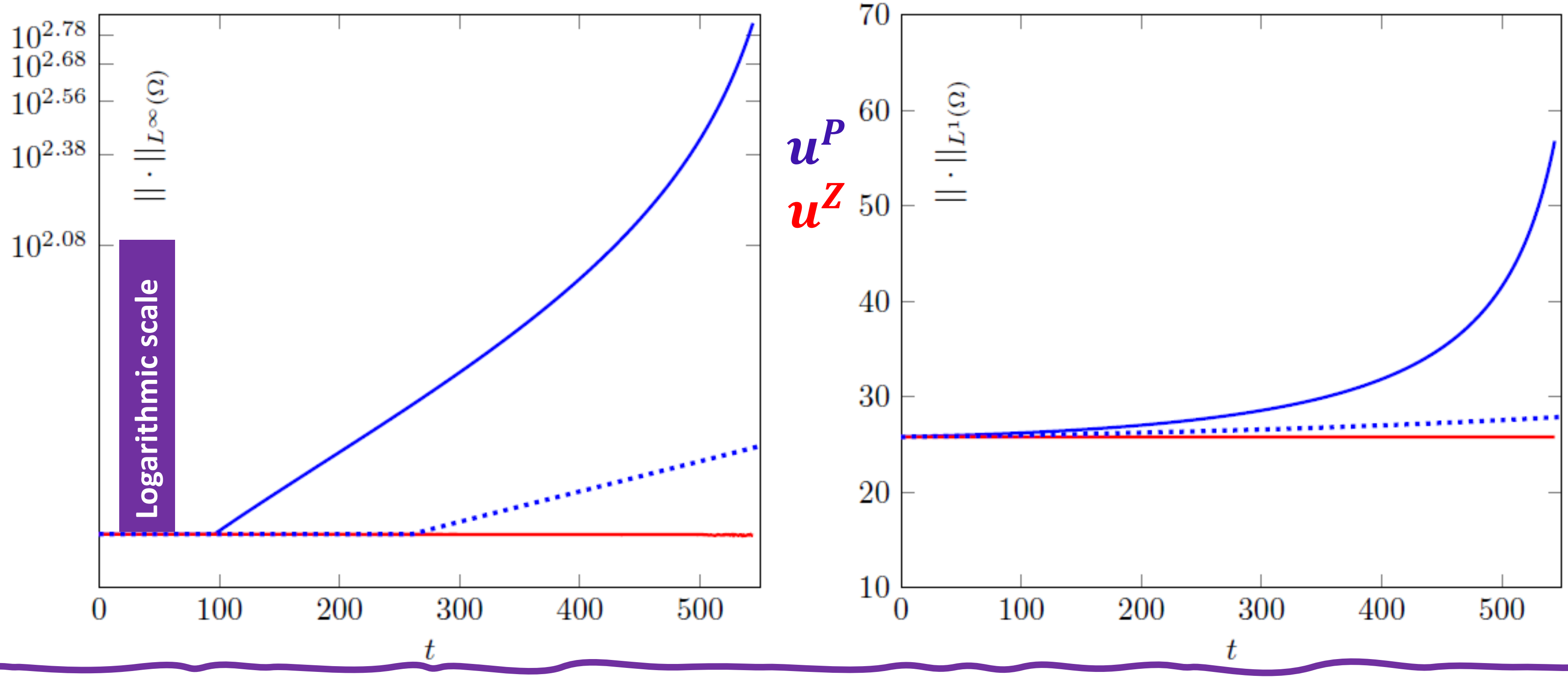


Numerical simulations: nil flux VS. positive flux

$$\boxed{u_\nu - \chi u v_\nu = \alpha \chi h u v}$$

Blue/Red line <-> positive/Zero flux: u^P / u^Z

α = 1 /α = 0.7 <->continuous/dotted line



CONJECTURE: $\max(u^P) \geq \max(u^Z)$ **for all** $t \in (0, T_{\max})$

(fixed the same other data; domain and initial distributions)

Positve flux: How to control the mass?

$$\frac{d}{dt}\int_\Omega u = \boxed{\alpha\chi h \int_{\partial\Omega} uv} \geq 0 \quad \text{for all } t \in (0, T_{\max})$$

Trace Embeddings

$$\int_{\partial\Omega} \psi \leq C_{\partial\Omega} \int_\Omega \psi + D_{\partial\Omega} \int_\Omega |\nabla \psi| \qquad C_{\partial\Omega}(n,\Omega) \text{ and } D_{\partial\Omega}(n,\Omega)$$

Young's Inequality
Trace Embedding

$$\frac{d}{dt}\int_\Omega u \leq c_1 \int_\Omega u^2 + c_2 \int_\Omega |\nabla u|^2 + c_3 \int_\Omega v^2 + c_4 \int_\Omega |\nabla v|^2$$

Manipulating the equation $\tau v_t = \Delta v - v + u$

$$\frac{d}{dt}\int_\Omega u \leq c_1 \int_\Omega u^2 + c_2 \int_\Omega |\nabla u|^2 \quad \text{for all } t \in (0, T_{max})$$

- SOURCE WITH BIOLOGICAL INTERPRETATIONS
- NOTE THE HIGH POWER, 2

STRONG LOGISTIC TO HAVE BOUNDEDNESS OF THE MASS

$$\frac{d}{dt}\int_\Omega u \leq c_1 \int_\Omega u^2 + c_2 \int_\Omega |\nabla u|^2 + \int_\Omega h(u, |\nabla u|)$$

$$h(u, |\nabla u|) = au - bu^2 - c|\nabla u|^2$$

We need "negative" terms to control the sum on the r.h.s.

Boundedness result

K. Baghaei, S. Frassu, Y. Tanaka, G. Viglialoro, To what extent does the consideration of positive total flux influence the dynamics of Keller–Segel-type models? *Submitted.*

$$(\mathcal{P}_\tau)\begin{cases} u_t = \Delta u - \chi\nabla(u\nabla v) + au - bu^2 - c|\nabla u|^2 & \text{in } \Omega\times(0,T_{\max}),\\ \tau v_t = \Delta v - v + u & \text{in } \Omega\times(0,T_{\max}),\\ u_\nu = (\alpha-1)\chi huv,\ v_\nu = -hv & \text{on } \partial\Omega\times(0,T_{\max}),\\ u(x,0)=u_0(x), \tau v(x,0)=\tau v_0(x) & x\in\bar\Omega.\end{cases}$$

Theorem *Let* $\Omega\subset\mathbb{R}^n$ bounded of class $C^{2+\delta}$, $\delta\in(0,1)$, $\tau\in\{0,1\}$, $\chi,a,c,h>0$ *and* $\alpha\in[0,1]$. *Then there exist* $C_{\partial\Omega}=C_{\partial\Omega}(\Omega,n)$ *and* $D_{\partial\Omega}=D_{\partial\Omega}(\Omega,n)$ *such that for every*

$$u_0,v_0:\bar\Omega\to\mathbb{R}^+,\ \textit{with } u_0,v_0\in C^{2+\delta}(\bar\Omega)\quad \textit{complying with}\quad u_{0\nu}=(\alpha-1)\chi hu_0v_0\quad \textit{and}\quad v_{0\nu}=-hv_0\quad \textit{on } \partial\Omega$$

whenever

$$\boxed{b>\frac{(\chi\alpha D_{\partial\Omega})^2h}{16cC_{\partial\Omega}}+\frac{\chi\alpha}{2}\sqrt{hC_{\partial\Omega}}}\quad \textit{problem } (\mathcal{P}_\tau) \textit{ admits a unique solution}$$

$(u,v)\in C^{2+\delta,1+\frac{\delta}{2}}(\bar\Omega\times[0,\infty))\times C^{2+\delta,\tau+\frac{\delta}{2}}(\bar\Omega\times[0,\infty))\times C^{2+\delta,\tau+\frac{\delta}{2}}(\bar\Omega\times[0,\infty))$ *such that* $0\le u,v\in L^\infty(\Omega\times(0,\infty))$.

✓ **For α=0 it is sufficient *b>0***

✓ ***b* increases with α, *h* and χ (responsible of gathering effects)**

THANK YOU FOR YOUR ATTENTION

TANTI AUGURI

KISKO E MANOLO!