Lecture 1: Some illustrative optimal control problems

Enrique FERNÁNDEZ-CARA
Dpto. E.D.A.N. - Univ. of Sevilla

Four interesting optimal control problems
Results and open questions
They lead to new theoretical results . . .
They are connected to applications . . .

The state equation:

$$\begin{cases} \mathcal{A}(y) = \mathcal{B}(v) \\ + \dots \end{cases} \qquad (S)$$

The cost:

$$(v, y) \mapsto J(v, y)$$

The constraints:

$$v \in \mathcal{V}_{ad}, \quad y \in \mathcal{Y}_{ad}$$

The optimal control problem:

Minimize $J(v, y)$
Subject to $v \in \mathcal{V}_{ad}, \quad y \in \mathcal{Y}_{ad}, \ (v, y)$ satisfies (S)

Main questions: ∃/uniqueness/multiplicity, characterization, computation, . . .

Outline

1. Optimal control of a capacitor
   - The problem
   - The main results and their proofs
2. Control on the coefficients, homogenization, optimal materials
   - The original problem
   - The relaxed problem
3. Optimal design for Navier-Stokes flow
   - The problem
   - An optimality result
4. Optimal control oriented to therapies for tumor growth models
   - The problem
   - The results

$\Omega \subset \mathbb{R}^N$ bounded, regular, connected, open; $\Gamma = \partial\Omega$.
$\omega \subset\subset \Omega$ non-empty, open; 1_ω: the characteristic function
The state system:

$$\begin{cases} -\Delta y = 1_\omega u & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1}$$

$y = y(x)$: electric potential; the density of charge is $1_\omega u$;
$E = -\nabla y$ is the associated electric field
Question: How to choose u to have y as good as possible?
For instance, for given $a, b > 0$, $y_d \in L^2(\Omega)$ and $\mathcal{U}_{\text{ad}} \subset L^2(\omega)$:

$$\begin{cases} \text{Minimize} \quad J(u) = \dfrac{a}{2}\displaystyle\int_\Omega |y - y_d|^2\,dx + \dfrac{b}{2}\int_\omega |u|^2\,dx \\ \text{Subject to} \quad u \in \mathcal{U}_{\text{ad}},\ (1) \end{cases} \tag{2}$$

where $a, b > 0$.

Theorem 1: existence, uniqueness

Assume: $\mathcal{U}_{\rm ad} \subset L^2(\omega)$ is non-empty, closed, convex.
Then: $\exists!$ optimal $\hat{u}$

Theorem 2: characterization (optimality)

Same hypotheses. Then: $\exists \hat{y}, \hat{p}$ with

$$\begin{cases} -\Delta \hat{y} = \hat{u} 1_\omega & \text{in} \quad \Omega \\ \hat{y} = 0 & \text{on} \quad \Gamma \end{cases} \tag{3}$$

$$\begin{cases} -\Delta \hat{p} = \hat{y} - y_d & \text{in} \quad \Omega \\ \hat{p} = 0 & \text{on} \quad \Gamma \end{cases} \tag{4}$$

$$\int_\omega (a\hat{p} + b\hat{u})(u - \hat{u})\, dx \geq 0 \quad \forall u \in \mathcal{U}_{\rm ad} \tag{5}$$

PROOF OF THEOREM 1:

Recall: $J(u) = \frac{a}{2}\int_\Omega |y - y_d|^2\,dx + \frac{b}{2}\int_\omega |u|^2\,dx \;\forall u \in \mathcal{U}_{\rm ad}$

$u \mapsto J(u)$ is strictly convex, coercive and continuous (hence weakly lsc) in $L^2(\omega)$

$\mathcal{U}_{\rm ad}$ is closed and convex

Hence . . . $\square$

QUESTIONS: $a = 0$? $b = 0$? Interpretations?

PROOF OF THEOREM 2:

Try to write that $\langle J'(\hat u), u - \hat u\rangle \geq 0 \; u \in \mathcal{U}_{\rm ad}$, with $\hat u \in \mathcal{U}_{\rm ad}$

If $\hat p$ solves (4), then $\langle J'(\hat u), u - \hat u\rangle = \int_\omega (a\hat p + b\hat u)\,(u - \hat u)\,dx$ Consequently, . . . $\square$

Remark

In this case, the reciprocal holds: if (3) – (5) holds, then $\hat u$ is the optimal control

Remark

From the previous argument: $\langle J'(u), v\rangle = \displaystyle\int_\omega (ap + bu)\, v\, dx$,
with $-\Delta p = y - y_d$ in Ω, $p = 0$ on Γ (the adjoint state) Useful!

QUESTIONS: First suggested iterative method:
$-\Delta y^n = u^{n-1} 1_\omega$ in Ω, $y^n = 0$ on Γ
$-\Delta p^n = y^n - y_d$ in Ω, $p^n = 0$ on Γ
$\int_\omega (ap^n + bu^n)(u - u^n)\, dx \geq 0\ \forall u \in \mathcal{U}_{\rm ad}$, $u^n \in \mathcal{U}_{\rm ad}$
Convergence? Other iterates based on gradient computation?

Many possible generalizations . . .
QUESTIONS: Similar optimal control problems for other PDEs?
$y_t - \Delta y = u 1_\omega$ in $\Omega \times (0, T)$ + . . .
or $y_{tt} - \Delta y = u 1_\omega$, $i y_t + \Delta y = u 1_\omega$ . . .
similar nonlinear PDEs, etc.
Existence/uniqueness? Characterization?
Convergent algorithms?

Assume: in Ω we find two different dielectric materials, with permeability coefficients α and β $(0 < \alpha < \beta)$. How can we determine an optimal distribution?

The electrostatic potential $y = y(x)$ for a partition $\{G_1, G_2\}$ of Ω:
$-\nabla \cdot (a(x)\nabla y) = f(x)$ in Ω, $\quad y = 0$ on Γ
where $a(x) = \alpha$ in G_1, $a(x) = \beta$ in G_2
(f is given; a is the control and y is the state)

Question: How to choose a to have y as good as possible?
For instance, for given $y_d \in L^2(\Omega)$:

$$\begin{aligned} &\text{Minimize } \; j(a) = \frac{1}{2}\int_\Omega |y - y_d|^2\,dx \\ &\text{Subject to } \; a \in \mathcal{A}_{\rm ad} = \{\, a \in L^\infty(\Omega) : \; a(x) \in \{\alpha, \beta\} \text{ a.e.} \,\} \end{aligned} \tag{6}$$

Even beter:

$$\begin{aligned} &\text{Minimize } \; j(a) = \frac{1}{2}\int_\Omega |y - y_d|^2\,dx \\ &\text{Subject to } \; a \in \mathcal{A}_{\rm ad}, \; \int_\Omega a\,dx \leq I \end{aligned} \tag{7}$$

We assume $N = 2$ (for simplicity)
IN GENERAL, $\nexists$ SOLUTION:
Minimizing $\{a^n\}$, $a^n \to a^0$ weakly-$*$, $y^n \to y$ weakly, but ...
(typical for control on the coefficients)

Notation: $\mathcal{A}(\alpha, \beta)$ is the family of 2×2 matrices A such that
$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad (A(x))^{-1}\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad x$ a.e. in Ω

If $A^n, A^0 \in \mathcal{A}(\alpha, \beta)$, A^n H-converges to A^0 if
$\forall \mathcal{O} \subset \Omega$, $\forall g$ the corresponding solutions satisfy
$y^n \to y^0$ weakly in H_0^1 and $A^n \nabla y^n \to A^0 \nabla y^0$ weakly in L^2
[Murat and Tartar, 1978 ...]

Theorem 3: compactness

The family $\mathcal{A}(\alpha, \beta)$ is compact for the H-convergence

The key point: we can have $A^n = a^n I$ $\forall n$ and non-diagonal A^0
Explicit examples; thus, no solution for (6)

What can be done? Relaxation:
(Q) is the relaxed problem of (P) if

(a) $\exists$ solutions to (Q)
(b) Solutions to (Q) $\equiv$ weak limits of minimizing sequences of (P)

Notation: $\tilde{\mathcal{A}}_{\rm ad}$ is the family of all symmetric $A \in \mathcal{A}(\alpha,\beta)$ with

$$\alpha \leq \lambda_1(x) \leq \lambda_2(x) \leq \beta, \quad \frac{\alpha\beta}{\alpha+\beta-\lambda_2(x)} \leq \lambda_1(x) \text{ a.e. in } \Omega$$

A new problem:

$$\begin{cases} \text{Minimize } \; j(A) := \frac{1}{2}\int_\Omega |Y - y_d|^2\,dx \\ \text{Subject to } \; A \in \tilde{\mathcal{A}}_{\rm ad}, \quad -\nabla\cdot(A(x)\nabla Y) = f(x) \text{ in } \Omega, \ \dots \end{cases} \tag{8}$$

Theorem 4: relaxation

$A \in \tilde{\mathcal{A}}_{\rm ad} \Leftrightarrow A$ is the H-limit of some $a^n I$, with $a^n \in \mathcal{A}_{\rm ad}$
Hence, the relaxed problem of (6) is (8)

Physical interpretation: a composite anisotropic material

QUESTIONS: Optimality systems for (6) and (8)? Convergent iterates? Numerics?

QUESTIONS: The H-closure of $\mathcal{A}_{\rm ad}$ for N-dimensional problems ($N \geq 3$)?
Similar results for parabolic and hyperbolic PDEs? Nonlinearities?

In view of the difficulty: periodic structures
Many results under these conditions for many related problems

Assume: Ω is filled with a Navier-Stokes fluid
We try to find the optimal shape of a body travelling in Ω:

$$\begin{array}{l}\text{Minimize } \; T(B,y) := 2\nu \displaystyle\int_{\Omega\setminus B} |Dy|^2 \, dx \\ \text{Subject to } \; B \in \mathcal{B}_{\rm ad}, \;\; (y,\pi) \text{ solves NS in } \Omega\setminus B\end{array} \tag{9}$$

$$\left\{\begin{array}{lll} -\nu\Delta y + (y\cdot\nabla)y + \nabla\pi = 0, \quad \nabla\cdot y = 0 & \text{in} & \Omega\setminus B \\ y = y_\infty & \text{on} & \Gamma \\ y = 0 & \text{on} & \partial B \end{array}\right. \tag{10}$$

$\mathcal{B}_{\rm ad}$ is the family of admissible bodies
For instance: $B \in \mathcal{B}_{\rm ad} \Leftrightarrow B = \overline{\mathcal{O}}$ for some connected open $\mathcal{O}$ with $D_0 \subset \mathcal{O} \subset D_1$, $\partial\mathcal{O} \in W^{1,\infty}$

We are minimizing the drag, subject to $B \in \mathcal{B}_{\rm ad}$, since

$$T(B,y) = -C_0 \int_\Gamma y_\infty \cdot (\sigma(y,\pi)\cdot n) \, d\Gamma$$

In general:
NO WAY TO PROVE $\exists$, unless ARTIFICIAL CONDITIONS ARE IMPOSED TO $\mathcal{B}_{\rm ad}$
(typical for optimal design)

Explanation: a minimizing sequence $\{B^n, y^n\}$. Then:

- $\|y^n\|_{H^1}$ is uniformly bounded, whence $y^n \to y$ weakly in H^1
- $B^n \to B^0$ in the Haussdorf distance sense

But: there is no reason to have $y = y^0$!

This would be the case if all $B \in \mathcal{B}_{\rm ad}$ are uniformly $W^{1,\infty}$.
But . . .

QUESTIONS: Minimal uniform regularity hypotheses for existence?
A "natural" condition on $\mathcal{B}_{\rm ad}$ ensuring that $y = y^0$?

Assume $\exists$. We look for a "body variations" formula:

$$D(\hat{B}+u) = D(\hat{B}) + D'(\hat{B};u) + o(u), \quad \hat{B}+u = \{\, x = (I+u)(\xi) : \xi \in \hat{B} \,\}$$

(differentiating $u \mapsto D(\hat{B}+u)$; $\hat{B}$ is a reference body shape)

Theorem 5: optimality

Assume: $\partial\hat{B}, \Gamma \in W^{2,\infty}$ and $u \in W^{2,\infty}$. Then:

$$D'(\hat{B};u) = \int_{\partial\hat{B}} \left(\frac{\partial w}{\partial n} - \frac{\partial y}{\partial n} \right) \cdot \frac{\partial y}{\partial n} \, (u \cdot n) \, d\sigma,$$

where (w,q) is the associated adjoint state:

$$\begin{cases} -\nu\Delta w_i + \sum_j \partial_i y_j \, w_j - \sum_j y_j \, \partial_j w_i + \partial_i q = -2\nu \, \Delta y_i \\ \nabla \cdot w = 0, \quad \text{etc.} \end{cases}$$

Again very useful!

QUESTIONS: A sequence $\{B^n\}$ "converging" to a solution?
Second-order derivatives and applications?

$\Omega \subset \mathbb{R}^N$: organ (the brain), $N = 2$ or $N = 3$
$T > 0$: final time
c, β: cancer cells and inhibitors populations (fonctions of (x, t); the state)
v: a therapy, acting on $\omega \subset \Omega$ (the control)
Glioblastoma + radiotherapy [Swanson et al., 1990 . . .]
The state system:

$$\begin{cases} c_t - \nabla \cdot (D(x)\nabla c) = f(c) - F(c,\beta) & \text{in } \Omega \times (0,T) \\ \beta_t - \mu \Delta \beta = -h(\beta) - H(c,\beta) + v 1_\omega & \text{in } \Omega \times (0,T) \\ c(0) = c^0, \quad \beta(0) = 0 & \text{in } \Omega, \quad \text{etc.} \end{cases}$$

f and h give the proliferation and dissipation laws of c and β
F and H determine how c and β interact

Simplest choice: $f(c) = \rho c, \quad h(\beta) = m\beta, \quad F(c,\beta) = Rc\beta, \quad H(c,\beta) = Mc\beta$
Assumed in the sequel

Constraints on v (to be realistic):

$$v \in \mathcal{V}_{\rm ad} = \{ \, v : 0 \le v \le A, \ \int_0^T v \, dt \le B, \ v = 0 \text{ for } t \notin \mathcal{I} \, \}$$

($\mathcal{I}$ is the set of times for therapy application)

Question: how to choose v to have c as good as possible?

$$\begin{array}{l} \text{Minimize} \ \ K(v, c, \beta) = \frac{a}{2} \int_\Omega |c(x, T)|^2 + \frac{b}{2} \int_{\omega \times (0,T)} |v|^2 \\ \text{Subject to} \ \ v \in \mathcal{V}_{\rm ad}, \ \ (c, \beta) \ \text{ satisfies } \ (12) \end{array} \tag{11}$$

$$\left\{ \begin{array}{ll} c_t - \nabla \cdot (D(x) \nabla c) = \rho c - R c \beta & \text{in } \Omega \times (0, T) \\ \beta_t - \mu \Delta \beta = -m\beta - M c \beta + v 1_\omega & \text{in } \Omega \times (0, T) \\ c(0) = c^0, \ \ \beta(0) = 0 & \text{in } \Omega, \ \ \text{etc.} \end{array} \right. \tag{12}$$

Theorem 6: existence

Assume: $\mathcal{V}_{\rm ad} \subset L^2(\omega \times (0,T))$ is as before
Then: $\exists$ optimal control-state $(\hat{u}, \hat{c}, \hat{\beta})$

Theorem 7: characterization (optimality)

Same hypotheses, $(\hat{u}, \hat{c}, \hat{\beta})$ is optimal
Then: $\exists (\hat{p}, \hat{\eta})$ such that one has (12),

$$\begin{cases} -\hat{p}_t - \nabla \cdot (D(x) \nabla \hat{p}) = \rho \hat{p} - R\hat{\beta}\hat{p} - M\hat{\beta}\hat{\eta} \\ -\hat{\eta}_t - \mu \Delta \hat{\eta} = -m\hat{\eta} - R\hat{c}\hat{p} - M\hat{c}\hat{\eta} \\ \hat{p}(T) = \hat{c}(T), \;\; \hat{\eta}(T) = 0 \;\; \text{etc.} \end{cases} \tag{13}$$

$$\iint_{\omega \times (0,T)} (a\hat{p} + b\hat{u})(u - \hat{u})\, dx\, dt \geq 0 \quad \forall u \in \mathcal{V}_{\rm ad}, \;\; \hat{u} \in \mathcal{V}_{\rm ad} \tag{14}$$

The arguments are similar to those above . . .

QUESTIONS: Detailed argument for existence? For optimality?

Also: $\langle J'(u), v\rangle = \iint_{\omega\times(0,T)} (ap + bu)\, v,$ where

$$\begin{cases} -p_t - \nabla\cdot(D(x)\nabla p) = \rho p - R\beta p - M\beta\eta & \text{in } \Omega\times(0,T) \\ -\eta_t - \mu\Delta\eta = -m\eta - Rcp - Mc\eta & \text{in } \Omega\times(0,T) \\ p(T) = c(T), \ \ \eta(T) = 0 & \text{in } \Omega, \ \ \text{etc.} \end{cases}$$

(the adjoint state associate to u)
Once more: useful

QUESTIONS: Uniqueness of optimal state-control?
The reciprocal of the optimality result?

QUESTIONS: Iterative methods for the computation of $\hat{u}$?
Convergence?
[Echevarria et al., 2007]

THANK YOU VERY MUCH ...